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On the Self-Similar Solutions of Generalized Hydrodynamics Equations and Nonlinear Wave Patterns

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Abstract

Solutions of the system of dynamical equations of state and equations of the balance of mass and momentum are studied. The system possesses families of periodic, quasiperiodic and soliton-like invariant solutions. Self-similar solutions of this generalized hydrodynamic system are studied. Various complicated regimes, arising as a result with terms desribing relaxing and dissipative properties of the medium are described.

1 Introduction

The problem of constructing the condensed media model adequately describing an influence of their internal structure, manifested in high-rate processes (combustion and detonation waves propagation, earthquakes, etc.), actually is far from being solved. The difficulties arising when one constructs the equation of state are associated with lack of knowledge on the mechanism controlling the process of relaxation and also on the class and form of functions approximating correctly the experimental results. In this situation the deviation of the system from the state of thermodynamical equilibrium is expedient to describe as chemical reactions and to regard the corresponding degrees of reaction completeness as internal variables [1]. Generally speaking, this approach does not give any advantage, for the reactions mechanism usually remains unknown, yet it becomes helpful when the process under consideration is not far from equilibrium. To obtain the governing (constituent) equation in this case, phenomenological nonequilibrium thermodynamics methods may be employed [1,2], enabling to express coefficients of the governing equations as functions of measurable physical parameters, regardless of the detailed mechanism of relaxing process.

Following this way, the dynamical equation of state has been obtained [1], aimed at describing high-rate, high-intensive processes in multicomponent relaxing media. Together with the balance of mass and momentum equations, taken in the hydrodynamical approximation, it forms a closed system of the following form:

$$\rho\frac{du^i}{dt}+\frac{\partial p}{\partial x_i}=\Im^i, \qquad \frac{d\rho}{dt}+\rho\frac{\partial u^i}{\partial x_i}=0, \tag{1}$$

$$\tau\frac{dp}{dt}-\chi\frac{d\rho}{dt}=\kappa\rho^n-p+A\left[\frac{2}{\rho}\left(\frac{d\rho}{dt}\right)^2-\frac{d^2\rho}{dt^2}\right]+E\frac{d^2p}{dt^2},$$

where u^i are mass velocity components, ρ is density, p is pressure, $d/dt = \partial/\partial t + u^i\partial/\partial x_i$, $\Im$ is mass force τ, χ, κ, A and E are parameters, which completely describe medium behavior in the long-wave approximation [1].

In the following sections we shall investigate the features of invariant solutions of system (1) which, in contrast to classical hydrodynamical systems, posesses families of periodic, quasiperiodic and soliton-like invariant solutions.

2 On the periodic self-similar solutions of the system of balance equations closed by the first-order governing equation

By straightforward calculation one can determine that system (1) is invariant under the Galilei group $G(n)$. It admits an extra one-parameter group generated by the operator $M = \rho\partial/\partial\rho + p\partial/\partial p$. if $\Im = \rho\gamma$ and $n = 1$. So in the case of one spatial variable, the ansatz

$$u = D + U(\omega), \qquad \rho = exp[\xi t + S(\omega)], \qquad p = \rho Z(\omega), \qquad \omega = x - Dt, \tag{2}$$

connected in the standard way with a symmetry group generated by the operator $\Re = \partial/\partial t + D\partial/\partial x + \xi M$, enables to go from the initial PDE system to a subsequent system of ODE. It is obvious that expression (2) describes a travelling wave moving with the constant velocity D. The parameter ξ appears to be connected with the spatial inhomogeneity ahead of the wave front.

Till the end of this section we shall assume that $A = E = 0$. Using the ansatz (2) we obtain an ODE system that does not contain S variable in explicit form. Functions U and Z satisfy the equations

$$U\Delta\frac{dU}{d\omega} = U\left(\tau\gamma U + \sigma Z - \kappa\right) = U\phi, \tag{3}$$

$$U\Delta\frac{dZ}{d\omega} = \left(Z - U^2\right)\phi + \left(\gamma U + Z\xi\right)\Delta,$$

where $\Delta = \tau U^2 - \chi, \sigma = 1 + \xi\tau$. It is not difficult to see that the only critical point of system (3) belonging to the physical parameter range (i.e. laying in the half-plane $Z > 0$ beyond the manifold $U\Delta = 0$) is the point A having the coordinates $U_0 = -\kappa\xi/\gamma, Z_0 = \kappa$. Our goal is to state the conditions that guarantee the existence of periodic solutions in the vicinity of the critical point A.

Putting aside for a while the analysis of the ODE system, let us formulate a boundary-value problem for the initial system of PDE. So, we look for the conditions leading to the existence of the self-similar solutions describing shock wave propagation The initial-value problem happens to be self-similar provided that both states of the medium ahead and behind the shock front are expressed by the formula (2). Assuming the state ahead of the front to be independent of time, we obtain:

$$\rho_1 = \rho_0 \exp(\xi x/D), \qquad p_1 = Z_0\rho_1. \tag{4}$$

These functions will satisfy the initial system if $Z_0 = \kappa$ and $\gamma = \kappa\xi/D$. We immediately conclude from the last expression that $U_0 = -D$, hence the critical point A represents the spatially inhomogeneous state ahead of the wave front. Note that the slope of inhomogeneity is defined by the parameter ξ provided that D is fixed.

Let us rewrite system (3) in variables $x = U - U_0, y = Z - \kappa$:

$$U\Delta \frac{d}{d\omega}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} -\kappa\xi\tau, & -\sigma D \\ (\Delta_0 + \tau\theta)\gamma, & \xi\Delta_0 + \sigma\theta \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x(\tau\gamma x + \sigma y) \\ y[\tau\gamma x + \sigma y - x(x + 2U_0)] \end{pmatrix}. \quad (5)$$

Here $\Delta_0 = \tau D^2 - \chi, \Theta = \kappa - D^2$. In order to apply the Hopf theorem, we have to require that the linearization matrix of system (5) posesses a pair of pure imaginary eigenvalues. This is so provided that $\kappa = D^2 + \xi\chi$ and, under the assumption that $\xi < 0$, the following inequalities hold:

$$\kappa < D^2 < \chi/\tau. \quad (6)$$

To study the limit cycle creation conditions as well as its stability properties it is convenient to go to the canonical Poincaré representation [3,4]. Omitting this standard procedure, let us formulate the result obtained.

Theorem 1. *Let $\xi < 0$ and inequalities (6) hold. Then there exists an open interval $J \subset R^1$ in the vicinity of the critical value $D_{cr} = \sqrt{\kappa - \xi\chi}$ such that the system (5) posesses a family of stable periodic solutions whenever $D \in J$.*

The results of qualitative investigation of the self-similar solution of system (1) with the first-order governing equation have been supported by the direct numerical simulation. A piston problem with the inhomogeneous boundary conditions given by the formula (4) was solved using the Godunov numerical scheme [3].

The piston velocity V_p was varied near the critical value $V_{cr} = \sqrt{\kappa - \xi\chi}$, while the rest of parameters were chosen in accordance with the statements of Theorem 1. The effect of structures' formation behind the front of the shock wave created by the piston movement has been observed for $V_p \in (V_1, V_2)$, where $V_1 < V_{cr} < V_2$, while for $V_p < V_1$ and $V_p > V_2$ the patterns formation did not take place.

The results of the numerical experiments enable one to expect the self-similar periodic solutions, predicted by Theorem 1, to be asymptotically stable, though, to back this hypothesis, extra investigations are needed.

3 Complicated self-similar solutions arising in the case of a second-order governing equation

Now, let us consider system (1) with $A \neq 0$ and $E \neq 0$. The symmetry of this system will not be changed if we assume that $\kappa = B(p/\rho)$. Inserting ansatz (2) into (1), we obtain an ODE system in this case, cyclic with respect to S. Variables U, Z and $W = dU/d\omega$ may be shown to satisfy the following system:

$$\begin{aligned} U' &= UW, \\ Z' &= \gamma U + \xi Z + W(Z - U^2) \equiv \Phi, \\ W' &= (A - EU^2)^{-1}[M\Phi + G(Z) + W(\chi - MZ)] - W^2 \end{aligned} \quad (7)$$

where $M = \tau - E\xi, G(Z) = Z - B(Z), (\cdot)' = Ud(\cdot)/d\omega$.

One easily verifies that a point of phase space having the coordinates $U = -\xi Z_0/\gamma$, $Z = Z_0 > 0, W = 0$ will be a critical point belonging to the physical parameter range if the function G has the following expansion:

$$G(Z) = g_1(Z - Z_0) + g_2(Z - Z_0)^2 + g_3(Z - Z_0)^3.$$

Assuming that such a decomposition does take place, let us rewrite system (7) in the coordinates $x = U - U_0 \equiv U + \xi Z_0/\gamma, y = Z - Z_0$:

$$\begin{pmatrix} X \\ Y \\ W \end{pmatrix}' = \begin{pmatrix} 0, & 0, & U_0 \\ \gamma, & \xi, & \Delta \\ L\gamma, & L\xi + G_1, & \sigma \end{pmatrix} \begin{pmatrix} X \\ Y \\ W \end{pmatrix} + \begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix}, \tag{8}$$

where $L = M/K, G_i = g_i/K, K = A - EU_0^2, \sigma = (\chi - MU_0^2)/K, \Delta = \kappa - U_0^2$,

$$H_1 = UW, \qquad H_2 = W[Y - X(2U_0 + X)],$$

$$H_3 = G_2Y^2 - (2U_0LX + W)W + 2U_0EX[L\gamma X + L\xi Y + \sigma W + G_2Y^2 -$$

$$2U_0LWX]/K + G_3Y^3 - LWX^2 + E(1 + 4U_0^2E/K)\times$$

$$X^2(L\gamma X + L\xi Y + \sigma W)/K + O(|X, Y, W|^3).$$

We are going to state the conditions that guarantee the existence of periodic and quasiperiodic solutions of system (8). This may be done by the analytical means provided that its linearization matrix has one zero and two pure imaginary eigenvalues. The above requirement will be fulfilled if the following relations hold:

$$\chi = MU_0^2 - \xi K > 0, \qquad g_1 = 0, \tag{9}$$

$$\Omega^2 = \xi\chi/K > 0. \tag{10}$$

In order that the condition (10) be satisfied, we shall assume that $\xi < 0$ and $K < 0$.

A general analysis of the $(0, \pm i\Omega)$ bifurcation was given by Guckenheimer and Holmes [4]. To take advantage of their results, we should go to the coordinate system in which the linearization matrix $\hat{M}$ becomes quasidiagonal and next find the normal form of the system obtained by a local asymptotic transformation followed by the averaging procedure. The linearization matrix will be quasidiagonal in the coordinate system defined by the expression

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{U_0\Omega^2} \begin{pmatrix} -\xi\gamma, & -\xi^2, & \xi U_0^2 \\ -\gamma\Omega, & -\xi\Omega, & 0 \\ U_0(\Omega^2 + \gamma LU_0), & L\xi U_0^2, & -\xi U_0^2 \end{pmatrix} \begin{pmatrix} X \\ Y \\ W \end{pmatrix}. \tag{11}$$

In new variables the system (8) is as follows:

$$x_1' = -\Omega x_2 + \sum_{i \le j} A_{ij}x_ix_j + \sum_{i \le j \le k} A_{ijk}x_ix_jx_k + \cdots$$

$$x_2' = \Omega x_1 + \sum_{i \le j} B_{ij}x_ix_j + \sum_{i \le j \le k} B_{ijk}x_ix_jx_k + \cdots \tag{12}$$

$$x_3' = \sum_{i\le j} C_{ij}x_ix_j + \sum_{i\le j\le k} C_{ijk}x_ix_jx_k + \cdots$$

We do not write down the coefficients A_{ij}, B_{ij}, C_{ij} in explicit form for they are very cumbersome.

Next we employ the local asymptotic transformation

$$x^i = y^i + \sum_{j\le k} P^i_{jk} y^i y^k \tag{13}$$

choosing the coefficients P^i_{jk} in such a way that variables y^i up to $O(|y|^2)$ satisfy the following system:

$$\begin{aligned} y_1' &= -\Omega y_2 + y_3(M_1y_1 + S_1y_2) + O(|y|^2),\\ y_2' &= \Omega y_2 + y_3(S_2y_1 + M_2y_2) + O(|y|^2),\\ y_3' &= N_1(y_1^2 + y_2^2) + N_2y_3^2. \end{aligned} \tag{14}$$

It occurs that parameters M_2, S_2 can be chosen arbitrarily, while the remaining ones are as follows:

$$\begin{aligned} &M_1 = A_{13} + B_{23} - M_2, \qquad S_1 = -B_{13} + A_{23} + S_2,\\ &N_1 = (C_{11} + C_{22})/2, \qquad N_2 = C_{33}. \end{aligned} \tag{15}$$

(for details see [3], Appendix 2).

Going to the coordinates $\bar r = \sqrt{y_1^2 + y_2^2}$, $\Theta = \arcsin\,(y_2/\sqrt{y_1^2 + y_2^2})$, and averaging over the "fast" variable Θ, we obtain after the rescaling $r = -\sqrt{|N_1N_2|}\bar r, z = -N_2y_3$ a canonical system

$$r' = arz, \qquad z' = br^2 - z^2, \tag{16}$$

where

$$a = -(A_{13} + B_{23})/(2C_{33}), \qquad b = -sgn[C_{33}(C_{11} + C_{22})]. \tag{17}$$

It is evident that the canonical system is even more degenerate than the basic one (8) for it does not contain any linear term. In order to "unfold" this degeneracy, a two-parameter set of small perturbations should be introduced:

$$r' = \mu_1 r + arz \qquad z' = \mu_2 + br^2 - z^2 \tag{18}$$

The possible regimes arising after the unfolding occur to depend on signs of the coefficients a and b. We shall consider the case $a > 0,\ b = -1$ in more detail. It is easy to see that the system (18) posesses a critical point B in the half-plane $r < 0$ provided that $\mu_2 > (\mu_1/a)^2$. This point corresponds to a periodic solution of system (8) since the canonical system (18) has been obtained via the averaging over the angular variable.

The critical point B is a stable focus when $\mu_1 < 0$ and is an unstable focus otherwise. When $\mu_1 = 0$ and $\mu_2 > 0$ it becomes a center, which evidently disappears when $\mu_1 \neq 0$, giving rise to another regimes which are structurally stable.

In fact, the case $0 < |\mu_1| << \mu_2 << 1$ requires a rather delicate treatment. E.g., when $\mu_1 \neq 0$ a limit cycle creation may be shown to take place. Using the arguments that have been attached earlier, one easily gets convinced of that this corresponds to the creation of a quasiperiodic solution of the initial system. In addition to the abovementioned regimes there exists a set of homoclinic loops when $a = 2$ and $b = -1$.

The stability of these solutions depends in essential way on the coefficients standing at third-order monomials, which, for the sake of simplicity, have been consequently omitted starting from the equation (14). As was shown in [4], the canonical form may be presented up to $O(|r, z|^3)$ as

$$\begin{cases} r' = \mu_1 r + arz, \\ z' = \mu_2 + br^2 - z^2 + fz^3 \end{cases} \tag{19}$$

with a and b still defined by the formula (17). When $a > 0, b = -1$ and $f < 0$ the initial system occurs to possess stable periodic and quasiperiodic solutions, and when, in addition, $a = 2$, a stable homoclinic loop does exist [4] in the vicinity of the manifold $\mu_2 = -4f\mu_1/3$.

Employing the equations (11), (15) and (17) one is able to express a and b as functions of the initial system's parameters:

$$a = 1 - [\Omega^2(\Omega^2 + \xi^2) + (\Omega\xi U)^2 E/K]/[(\gamma U_0)^2 G_2], \tag{20}$$

$$b = sgn[G_2(2\Omega^2(\xi^2 + \Omega^2)(1 + EU_0^2/K) - G_2((\Omega\Delta)^2 + (\gamma U_0)^2))] \tag{21}$$

When $a = 2$ the index G_2 is as follows:

$$G_2 = -[\Omega^2(\Omega^2 + \xi^2) + (\Omega\xi U)^2 E/K]/(\gamma U_0)^2. \tag{22}$$

The condition $b < 0$ will have the simplest form if we go to the coordinates $x = A|\xi|/\tau$ and $y = E|\xi|/\tau$:

$$2\kappa^2 y(x - D^2/2) < (x - D^2/2)(\Delta^2 - 3\kappa^2) + 3D^2(\Delta^2 - \kappa^2)/2. \tag{23}$$

The solution of the inequality (23), essentially depending on signs of $\Delta^2 - 3\kappa^2$ and $\Delta^2 - \kappa^2$, may be easily handled.

In order to state the stability of periodic and quasiperiodic regimes created after the unfolding degeneracy, it would be desired to estimate the sign of f. Including third-order terms into the formula (14) and taking advantage of the transformation (7.4.25) from [4], leading to the representation (19), one finally obtains the following expression:

$$f = \frac{h}{N_2^2} - \frac{1}{6}\left\{\frac{2e}{|N_1 N_2|} + \frac{6c}{|N_1 N_2|} - \frac{2d}{N_2^2}\right\}$$

where $h = C_{333} + (A_{33}C_{23} - B_{33}C_{13})/\Omega, C = (L_c + 3P_c)/8$,

$$L_c = A_{122} + B_{112} + P_{12}^2(2A_{22} + B_{12}) + P_{12}^1(2B_{11} + A_{12}) + 2(A_{11}P_{22}^1 + B_{22}P_{11}^2) +$$

$$A_{12}P_{22}^2 + B_{12}P_{11}^1 + P_{12}^3(B_{13} + A_{23}) + (A_{13}C_{12} + 2N_1 B_{13})/(2\Omega),$$

$$P_c = A_{111} + B_{222} + 2(A_{11}P_{11}^1 + B_{22}P_{22}^2) + A_{12}P_{11}^2 + B_{12}P_{22}^1 +$$

$$(B_{23}C_{12} + 2N_1B_{13})/(2\Omega),$$

$$2d = B_{13}N_2/\Omega + A_{133} + B_{233} + 2(A_{33}P^3_{13} + B_{33}P^3_{23})+$$

$$2(A_{11}P^1_{33} + B_{22}P^2_{33}) + +A_{12}P^2_{33} + B_{12}P^1_{33},$$

$$2e = C_{113} + C_{233} + C_{13}(P^3_{13} + P^1_{22}) + C_{13}P^1_{11} + C_{23}(P^2_{22} + P^2_{11} + P^3_{23})-$$

$$P^3_{12}(A_{23} - B_{13}) - [C_{12}(B_{23} - C_{33}) + 2C_{11}B_{13}]/\Omega.$$

So the index f is rather complicated and its direct estimation meets essential difficulties. Yet the abovementioned regimes do exist regardless of the sign of f. To make sure of this, let us note that transformation $t' = -t, r' = r, z' = -z, \mu'_1 = -\mu_1, \mu'_2 = -\mu_2$ gives the system, which differs from (19) only in the sign of coefficient f. So it transforms stable regimes into unstable ones and vice versa. The results obtained may be presented as follows.

Theorem 2. *If parameters ξ and K are negative, g_2 is expressed by the formula (22) and relations (9), (10), (23) hold, then system (7) possesses periodic and quasi-periodic solutions as well as a set of homoclinic loops.*

Let us stress that stability of the regimes created after the unfolding degeneracy remains unidentified. Nevertheless, it may be studied either by numerical evaluation of the coefficient f or by numerical solution of the initial–value problem for the system (7).

We have solved system (7) by means of the Runge-Kutta method. To unfold the degeneracy, the two–parameter set of small perturbations was introduced:

$$-D \equiv U_0 \to -(D + \varepsilon), \qquad \gamma \to \gamma(1 - \gamma\varepsilon/D),$$

$$G \to (1 + g_1y)\delta + g_2y^2 + g_3g^3.$$

Imposing in addition requirements $g_1 = -2D^2E/(K\kappa)$, one obtains the system corresponding to the canonical form (19) with $\mu_1 = -\varepsilon D\tau/K$, and $\mu_2 = \xi\tau\delta[D/(\Omega K)]^2$.

Numerical simulation has been carried out with the following values of the parameters obeying the requirements of Theorem 2:

I. $A = 1.25, D = \sqrt{2}, E = 2.1, Z_0 = 1, \xi = -1, g_3 = -5.2, \tau = 1.5$;

II. $A = 1, E = 0, D = 2, \gamma = 1.15, g_3 = 8, Z_0 = 1, \xi = 2, \tau = 1$.

In the first case, depending on ε, δ, stable periodic and quasiperiodic solutions have been obtained as well as homoclinic loops. In the second case these patterns were found to be unstable.

Concluding remarks

The analytical and numerical studying of self-similar solutions of the generalized hydrodynamics system enables one to state the existence of various complicated regimes arising as a result of interaction of nonlinear terms with the terms describing relaxing and dissipative properties of the medium. Of special interest is the fact that oscillatory solutions of the dynamical systems considered here correspond to the self-similar initial-value problem of the PDE system (1) if the stationary state ahead of the wave front is spatially inhomogeneous. So we have obtained the evidence of the creative role of inhomogeneities in the process of patterns formation. For technical reasons, the spatial inhomogeneity in this paper was connected with the external force, but generally speaking it may be attributed to any other source (e.g., wave moving in the opposite direction).

In the case of the system closed by the first–order governing equation, analytical results were compared with the numerical solutions of the piston problem, revealing the wave patterns development behind the front of the shock wave for the values of the parameters that correspond to the Hopf bifurcation conditions stated by Theorem 1. Note that the effect of the shock wave fragmentation observed in numerical experiments disappears beyond some critical values of the piston velocity.

On analyzing the balance of mass and momentum in the system closed by the first- and second-order governing equations, we had specified parameters in such a way that system (1) admitted a continuous group generated by the operator $\partial/\partial t + D\partial/\partial x + \xi(\rho\partial/\partial\rho + p\partial/\partial p)$. This gave us possibility to employ the ansatz (2) in both cases and to compare the travelling wave solutions corresponding to the different governing equations. On the basis of the qualitative analysis and numerical simulations, we are able to state that in the case of the second-order governing equation the system (1) possesses not only periodic self-similar solutions but also quasiperiodic solutions corresponding to the toroidal attractors as well as solitary wave solutions corresponding to the homoclinic loops. We should stress that existence of the last two types of solutions is directly linked with the inclusion of higher-order terms into the governing equation so the patterns become more and more complicated as we proceed away from the equilibrium.

Note that when studying the system (8) we have obtained the expression for the indices of the canonical Poincaré form of an arbitrary three-dimensional dynamical system with $(0, \pm i\Omega)$ degeneracy of the linear part (cf. equations (17) and (24)) as a by-product.

We would like to remark in conclusion that the canonical forms technique [4], employed to classify the regimes arising after the degeneracy has been removed, is very similar to the method put forward by W.I. Fushchych in thev early 70s in order to investigate a non-Lie symmetry of linear systems of PDE. The procedure of obtaining the canonical form corresponding to system (8) is evidently based on a "hidden" symmetry of its linear part, yet employment of the asymptotic methods together with averaging procedure enables one to analyze nonlinear problems (cf also [5])[1].

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