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q-Newton Binomial: From Euler To Gauss

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Abstract

A counter-intuitive result of Gauss (formulae (1.6), (1.7) below) is made less mysterious by virtue of being generalized through the introduction of an additional parameter.

1 A formula of Gauss revisited

Consider the Newton binomial for a positive integer N:

$$(1-x)^N = \sum_{\ell=0}^{N} \binom{N}{\ell} (1-x)^\ell. \tag{1.1}$$

Substituting $x = 1$ into this formula, we get

$$\sum_{\ell=0}^{N} \binom{N}{\ell} (-1)^\ell = 0. \tag{1.2}$$

What happens with these two equalities in the q-mathematics framework? Newton's formula (1) becomes Euler's formula

$$(1 \dot{-} x)^N = (1-x)(1-qx)...(1-q^{N-1}x) = \sum_{\ell=0}^{N} \begin{bmatrix} N \\ \ell \end{bmatrix} (-x)^\ell q^{\binom{\ell}{2}}, \tag{1.3}$$

where $\begin{bmatrix} N \\ \ell \end{bmatrix} = \begin{bmatrix} N \\ \ell \end{bmatrix}_q$ are the Gaussian polynomials, or q-binomial coefficients:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!} = \frac{[n]...[n-k+1]}{[k]!}, \quad k \in \mathbf{Z}_+, \tag{1.4}$$

$$[k]! = [k]_q! = [1][2]...[k], \quad [0]! = 1, \quad [n] = [n]_q = (1-q^n)(1-q).$$

Substituting $x = 1$ into the Euler formula (1.3), we find

$$\sum_{\ell=0}^{N} \begin{bmatrix} N \\ \ell \end{bmatrix} (-1)^{\ell} q^{\binom{\ell}{2}} = 0. \tag{1.5}$$

This does not look exactly as a q-analogue of formula (1.2).

How about the sum $\sum_{\ell=0}^{N} \begin{bmatrix} N \\ \ell \end{bmatrix} (-1)^{\ell}$?

The answer is quite surprising. Denote

$$s_{N|0} = (-1)^N \sum_{\ell=0}^{N} \begin{bmatrix} N \\ \ell \end{bmatrix} (-1)^{\ell}. \tag{1.6}$$

Gauss found that

$$s_{2m+1|0} = 0, \quad m \in \mathbf{Z}_+, \tag{1.7a}$$
$$s_{2m+2|0} = (1-q)(1-q^3)...(1-q^{2m+1}), \quad m \in \mathbf{Z}_+. \tag{1.7b}$$

These formulae are easy to prove, but they are nevertheless mystifying: there is no hint in the definition (1.6) that some sort of 2-periodicity is involved. In addition, formula (1.2) may claim the following sums as proper q-analogues:

$$s_{N|1} = (-1)^N \sum_{\ell=0}^{N} \begin{bmatrix} N \\ \ell \end{bmatrix} (-q)^{\ell}, \tag{1.8}$$

or even

$$s_{N|r} = (-1)^N \sum_{\ell=0}^{N} \begin{bmatrix} N \\ \ell \end{bmatrix} (-q^r)^{\ell}. \tag{1.9}$$

Indeed, we shall verify later on that

$$s_{2m+1|1} = -(1-q^{2m+1}) s_{2m|0} = -\sqcap_{t=0}^{m} (1-q^{2t+1}), \tag{1.10a}$$
$$s_{2m|1} = s_{2m|0} = \sqcap_{t=1}^{m} (1-q^{2t-1}). \tag{1.10b}$$

Similar but more complex formulae can be derived for other values of r, not just for $r = 0$ and $r = 1$. We shall abstain from such derivations, as they are superseded by the general formulae (1.12) below.

What seems to be happening here is that the functions

$$S_N(x) = (-1)^N \sum_{\ell=0}^{N} \begin{bmatrix} N \\ \ell \end{bmatrix} (-x)^{\ell} \tag{1.11}$$

possess some interesting properties worthy of attention; and once the decision to pay attention has been made, one quickly conjectures the formulae

$$S_{2m+1}(x) = -\sum_{\ell=0}^{2m+1} \begin{bmatrix} 2m+1 \\ \ell \end{bmatrix} (-x)^\ell$$
$$= \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_{q^2} (x \dot{-} 1)^{2m+1-2k} (q^{2m+1}; q^{-2})_k, \tag{1.12a}$$

$$S_{2m+2}(x) = \sum_{\ell=0}^{2m+2} \begin{bmatrix} 2m+2 \\ \ell \end{bmatrix} (-x)^\ell$$
$$= \sum_{k=0}^{m+1} \begin{bmatrix} m+1 \\ k \end{bmatrix}_{q^2} (x \dot{-} 1)^{2m+2-2k} (q^{2m+1}; q^{-2})_k. \tag{1.12b}$$

The additional notations employed above are to be understood as

$$(u \dot{+} v)^\ell = \sqcap_{k=0}^{\ell-1} (u + q^k v), \quad \ell \in \mathbf{N}; \quad (u \dot{+} v)^0 = 1, \tag{1.13}$$

and

$$(a; Q)_\ell = \sqcap_{k=0}^{\ell-1} (1 - Q^k a), \quad \ell \in \mathbf{N}; \quad (a; Q)_0 = 1. \tag{1.14}$$

If we define

$$\epsilon(N) = \begin{cases} 1, & N \text{ is even} \\ 0, & N \text{ is odd} \end{cases} = \left\lfloor \frac{N+2}{2} \right\rfloor - \left\lfloor \frac{N+1}{2} \right\rfloor, \tag{1.15}$$

then formulae (1.12) can be rewritten as

$$S_N(x) = (-1)^N \sum_{\ell=0}^{N} \begin{bmatrix} N \\ \ell \end{bmatrix} (-x)^\ell$$
$$= \sum_{k=0}^{\lfloor N/2 \rfloor} \begin{bmatrix} \lfloor N/2 \rfloor \\ k \end{bmatrix}_{q^2} (x \dot{-} 1)^{N-2k} (q^{N-\epsilon(N)}; q^{-2})_k. \tag{1.16}$$

Substituting $x = 1$ into formulae (1.12) we recover Gauss' formulae (1.7).

Let us now prove formulae (1.12). Denote the RHS of formulae (1.16) by $\tilde{S}_N(x)$. To show that

$$S_N(x) = \tilde{S}_N(x), \tag{1.17}$$

we shall verify, first, that

$$\frac{dS_N}{d_q x} = [N] S_{N-1}, \tag{1.18a}$$

$$\frac{d\tilde{S}_N}{d_q x} = [N] \tilde{S}_{N-1}(x), \tag{1.18b}$$

and second, that

$$S_N(1) = \tilde{S}_N(1); \tag{1.19}$$

here

$$\frac{df(x)}{d_q x} = \frac{f(qx) - f(x)}{qx - x} \tag{1.20}$$

is the q-derivative. Since $S_1 = \tilde{S}_1 = x - 1$, these verifications would suffice.

We start with formula (1.18a). We have:

$$\begin{aligned}
\frac{dS_N}{d_q x} &= \frac{d}{d_q x}\left((-1)^N \sum_{\ell=0}^{N} \begin{bmatrix} N \\ \ell \end{bmatrix} (-1)^\ell x^\ell\right) = (-1)^N \sum_{\ell=1}^{N} \begin{bmatrix} N \\ \ell \end{bmatrix} [\ell](-1)^\ell x^{\ell-1} \quad \text{[by (1.22)]} \\
&= (-1)^N [N] \sum_{\ell=1}^{N} \begin{bmatrix} N-1 \\ \ell-1 \end{bmatrix} (-x)^{\ell-1}(-1) = [N](-1)^{N-1} \sum_{\ell=0}^{N-1} \begin{bmatrix} N-1 \\ \ell \end{bmatrix} (-x)^\ell \\
&= [N] S_{N-1},
\end{aligned} \tag{1.21}$$

where we used the obvious formula

$$\begin{bmatrix} w \\ \ell \end{bmatrix} [\ell] = [w] \begin{bmatrix} w-1 \\ \ell-1 \end{bmatrix}. \tag{1.22}$$

Next, formula (1.18b), which we shall check separately for odd and even N, making use of the easy verifiable relation

$$\frac{d(x \dot{+} v)^\alpha}{d_q x} = [\alpha](x \dot{+} v)^{\alpha-1}. \tag{1.23}$$

So, for N odd, we have

$$\frac{d\tilde{S}_1}{d_q x} = \frac{d}{d_q x}(x-1) = 1 = \tilde{S}_0, \tag{1.24}$$

and then

$$\begin{aligned}
\frac{d\tilde{S}_{2m+3}}{d_q x} &= \sum_{k=0}^{m+1} \begin{bmatrix} m+1 \\ k \end{bmatrix}_{q^2} [2m+3-2k](x \dot{-} 1)^{2m+2-2k} (q^{2m+3}; q^{-2})_k \\
&= [2m+3] \sum_{k=0}^{m+1} \begin{bmatrix} m+1 \\ k \end{bmatrix}_{q^2} (x \dot{-} 1)^{2m+2-2k} (q^{2m+1}; q^{-2})_k = [2m+3]\tilde{S}_{2m+2},
\end{aligned}$$

because

$$[2m+3-2k](q^{2m+3}; q^{-2})_k = [2m+3](q^{2m+1}; q^{-2})_k \ ; \tag{1.25}$$

for N even, we find

$$\begin{aligned}
\frac{d\tilde{S}_{2m+2}}{d_q x} &= \sum_{k=0}^{m+1} \begin{bmatrix} m+1 \\ k \end{bmatrix}_{q^2} [2m+2-2k](x \dot{-} 1)^{2m+1-2k} (q^{2m+1}; q^{-2})_k \\
&= [2m+2] \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_{q^2} (x \dot{-} 1)^{2m+1-2k} (q^{2m+1}; q^{-2})_k = [2m+2]\tilde{S}_{2m+1},
\end{aligned}$$

because

$$\begin{bmatrix} m+1 \\ k \end{bmatrix}_{q^2} [2m+2-2k] = [2m+2] \begin{bmatrix} m \\ k \end{bmatrix}_{q^2}, \tag{1.26}$$

which is true in view of the obvious relation

$$[u]_{q^2} = [2u]_q/[2]_q. \tag{1.27}$$

It remains to verify formula (1.19), which is nothing but the Gauss formula (1.7). We shall verify the latter in 4 easy steps.
<u>1^{st} Step</u> is formula (1.7a):

$$\begin{aligned} s_{2m+1|0} &= -\sum_{\ell \geq 0} \begin{bmatrix} 2m+1 \\ \ell \end{bmatrix} (-1)^\ell = \sum_{\ell \geq 0} \begin{bmatrix} 2m+1 \\ 2m+1-\ell \end{bmatrix} (-1)^{\ell-1} \\ &= \sum_{L \geq 0} \begin{bmatrix} 2m+1 \\ L \end{bmatrix} (-1)^{2m-L} = -s_{2m+1|0}, \end{aligned}$$

so that $s_{2m+1|0} = 0$;
<u>2^{nd} Step</u> is formula (1.10b):

$$s_{2m|1} = \sum_{\ell \geq 0} \begin{bmatrix} 2m \\ \ell \end{bmatrix} (-q)^\ell = \sum_{\ell \geq 0} \begin{bmatrix} 2m \\ \ell \end{bmatrix} (-1)^\ell = s_{2m|0}. \tag{1.28}$$

Indeed,

$$\frac{s_{2m|1} - s_{2m|0}}{q-1} = \sum_{\ell \geq 0} \begin{bmatrix} 2m \\ \ell \end{bmatrix} (-1)^\ell [\ell] \text{ [by (1.22)]} = -[2m] \sum_{\ell \geq 1} \begin{bmatrix} 2m-1 \\ \ell-1 \end{bmatrix} (-1)^{\ell-1}$$

[by (1.7a)] $= 0$;
<u>3^{rd} Step</u> is formula (1.10a):

$$\sum_{\ell \geq 0} \begin{bmatrix} 2m+1 \\ \ell \end{bmatrix} (-q)^\ell = (1-q^{2m+1}) \sum_{\ell \geq 0} \begin{bmatrix} 2m \\ \ell \end{bmatrix} (-1)^\ell. \tag{1.29}$$

Indeed, since

$$\begin{bmatrix} 2m+1 \\ \ell \end{bmatrix} = \begin{bmatrix} 2m \\ \ell \end{bmatrix} + q^{2m+1-\ell} \begin{bmatrix} 2m \\ \ell-1 \end{bmatrix}, \tag{1.30}$$

we have:

$$\begin{aligned} \sum_{\ell \geq 0} \begin{bmatrix} 2m+1 \\ \ell \end{bmatrix} (-q)^\ell &= \sum_{\ell \geq 0} (-q)^\ell \begin{bmatrix} 2m \\ \ell \end{bmatrix} + \sum_{\ell \geq 1} (-q)^\ell q^{2m+1-\ell} \begin{bmatrix} 2m \\ \ell-1 \end{bmatrix} \text{ [by (1.28)]} \\ &= s_{2m|0} - q^{2m+1} \sum \begin{bmatrix} 2m \\ \ell-1 \end{bmatrix} (-1)^{\ell-1} = (1-q^{2m+1}) s_{2m|0}; \end{aligned}$$

<u>4^{th} Step</u> is the last one: we prove that

$$s_{2m+2|0} = (1-q^{2m+1}) s_{2m|0}, \tag{1.31}$$

from which the Gauss formula (1.76) follows at once, since

$$s_{2|0} = 1 - [2] + 1 = 1 - (1+q) + 1 = 1 - q. \tag{1.32}$$

Now,

$$\begin{aligned} s_{2m+2|0} &= \sum_{\ell \geq 0} \begin{bmatrix} 2m+2 \\ \ell \end{bmatrix} (-1)^\ell \text{ [by (1.28)]} = \sum_{\ell \geq 0} \begin{bmatrix} 2m+2 \\ \ell \end{bmatrix} (-q)^\ell \quad \text{[by (1.30)]} \\ &= \sum_{\ell \geq 0} (-q)^\ell \begin{bmatrix} 2m+1 \\ \ell \end{bmatrix} + \sum_{\ell \geq 1} (-q)^\ell q^{2m+2-\ell} \begin{bmatrix} 2m+1 \\ \ell - 1 \end{bmatrix} \text{ [by (1.29)]} \\ &= (1 - q^{2m+1}) s_{2m|0} - q^{2m+2} \sum_{\ell \geq 1} \begin{bmatrix} 2m+1 \\ \ell - 1 \end{bmatrix} (-1)^{\ell - 1} \quad \text{[by (1.7a)]} \\ &= (1 - q^{2m+1}) s_{2m|0}. \end{aligned}$$

We are done. Formula (1.16) is thereby proven. Substituting into this formula $x = 0$, we get an interesting identity

$$\sum_{k=0}^{\lfloor N/2 \rfloor} \begin{bmatrix} \lfloor N/2 \rfloor \\ k \end{bmatrix}_{q^2} q^{\binom{N-2k}{2}} (q^{N - \epsilon(N)}; q^{-2})_k = 1. \tag{1.33}$$

2 A different proof

To prove *polynomial* identities (1.12) generalizing Gauss' formulae (1.7), we had to prove independently the Gauss result along the way. This is not entirely agreeable. One ought to prove formulae (1.12) directly, by-passing the verification of the original Gauss formulae.

Such a proof follows.

Let $R_N(x)$ stand for either $S_N(x)$ or $\tilde{S}_N(x)$. We shall verify that

$$R_{N+1}(x) = x R_N(x) - R_N(qx). \tag{2.1}$$

Since

$$S_0(x) = \tilde{S}_0(x) = 1, \quad S_1(x) = \tilde{S}_1(x) = x - 1, \tag{2.2}$$

such a verification will prove that $S_N(x) = \tilde{S}_N(x)$ for all N.

We start with $R_N(x) = S_N(x)$. Let's look for a relation of the form

$$S_{N+1}(x) = Bx S_N(bx) + A S_N(ax). \tag{2.3}$$

The x^0 - coefficients (recall that $S_n(x) = (-1)^n \sum_\ell \begin{bmatrix} n \\ \ell \end{bmatrix} (-x)^\ell)$ yield

$$A = -1; \tag{2.4}$$

The x^{N+1}- coefficients yield

$$B b^N = 1 \iff B = b^{-N}; \tag{2.5}$$

Finally, for $0 < r < N+1$, the x^r - coefficients provide

$$\begin{bmatrix} N+1 \\ r \end{bmatrix} = Bb^{r-1} \begin{bmatrix} N \\ r-1 \end{bmatrix} + a^r \begin{bmatrix} N \\ r \end{bmatrix}. \tag{2.6}$$

In view of the relation (2.5), formula (2.6) can be rewritten as

$$\begin{bmatrix} N+1 \\ r \end{bmatrix} = (b^{-1})^{N+1-r} \begin{bmatrix} N \\ r-1 \end{bmatrix} + a^r \begin{bmatrix} N \\ r \end{bmatrix}. \tag{2.7}$$

Now, since

$$\begin{bmatrix} N+1 \\ r \end{bmatrix} = \begin{bmatrix} N \\ r-1 \end{bmatrix} + q^r \begin{bmatrix} N \\ r \end{bmatrix} \tag{2.8a}$$

$$= q^{N+1-r} \begin{bmatrix} N \\ r-1 \end{bmatrix} + \begin{bmatrix} N \\ r \end{bmatrix}, \tag{2.8b}$$

equation (2.7) has two solutions:

$$b = 1, \quad a = q, \tag{2.9a}$$

$$b = q^{-1}, \quad a = 1. \tag{2.9b}$$

Thus,

$$S_{N+1}(x) = xS_N(x) - S_N(qx) \tag{2.10a}$$

$$= q^N xS_N(q^{-1}x) - S_N(x). \tag{2.10b}$$

(For $q = 1$, we get just *one* relation, $S_{N+1}(x) = (x-1)S_N(x)$.)

Denote by $\mathcal{O}$ the linear operator acting on functions of x by the rule:

$$\mathcal{O}(f(x)) = xf(x) - f(qx). \tag{2.11}$$

We need to check that

$$\mathcal{O}(\tilde{S}_N) = \tilde{S}_{N+1}. \tag{2.12}$$

We shall check separately the cases of even and odd N:

$$\tilde{S}_{2m+1}(x) = \sum_{k=0}^{m} (x \dot{-} 1)^{2m+1-2k} c_{m|k}, \tag{2.13}$$

$$c_{m|k} = \begin{bmatrix} m \\ k \end{bmatrix}_{q^2} (q^{2m+1}; q^{-2})_k, \tag{2.14}$$

$$\tilde{S}_{2m}(x) = \sum_{k=0}^{m} (x \dot{-} 1)^{2m-2k} d_{m|k}, \tag{2.15}$$

$$d_{m|k} = \begin{bmatrix} m \\ k \end{bmatrix}_{q^2} (q^{2m-1}; q^{-2})_k. \tag{2.16}$$

To proceed further, let's establish first that

$$\mathcal{O}((x \dot{-} 1)^s) = (x \dot{-} 1)^{s+1} + q^{s-1}(1-q^s)(x \dot{-} 1)^{s-1}. \tag{2.17}$$

Indeed,

$$\begin{aligned}\mathcal{O}((x \dot{-} 1)^s) &= x(x \dot{-} 1)^s - (qx \dot{-} 1)^s = ((x-q^s)+q^s)(x \dot{-} 1)^s - q^s(x \dot{-} 1q^{-1})^s \\ &= (x-q^s)(x \dot{-} 1)^s + q^s(x \dot{-} 1)^{s-1}(x-q^{s-1}) - q^s(x-q^{-1})(x \dot{-} 1)^{s-1} \\ &= (x \dot{-} 1)^{s+1} + q^s(x \dot{-} 1)^{s-1}((x-q^{s-1})-(x-q^{-1})) \\ &= (x \dot{-} 1)^{s+1} + q^s(x \dot{-} 1)^{s-1}q^{-1}(1-q^s).\end{aligned}$$

Now,

$$\begin{aligned}\mathcal{O}(\tilde{S}_{2m+1}) &= \sum_{k=0}^{m} c_{m|k}\mathcal{O}((x \dot{-} 1)^{2m+1-2k}) \\ &= \sum_{k=0}^{m} c_{m|k}((x \dot{-} 1)^{2m+2-2k} + q^{2m-2k}(1-q^{2m+1-2k})(x \dot{-} x1)^{2m-2k}) \\ &= \sum_{k=0}^{m+1}(x \dot{-} 1)^{2m+2-2k}(c_{m|k} + c_{m|k-1}q^{2m+2-2k}(1-q^{2m+3-2k})), \qquad (2.18\ell)\end{aligned}$$

while

$$\tilde{S}_{2m+2} = \sum_{k=0}^{m+1}(x \dot{-} 1)^{2m+2-2k}d_{m+1|k}, \tag{2.18r}$$

so we need to verify that

$$d_{m+1|k} = c_{m|k} + c_{m|k-1}q^{2m+2-2k}(1-q^{2m+3-2k}), \tag{2.19}$$

which is

$$\begin{aligned}&= \begin{bmatrix} m+1 \\ k \end{bmatrix}_{q^2}(q^{2m+1};q^{-2})_k \\ &= \begin{bmatrix} m \\ k \end{bmatrix}_{q^2}(q^{2m+1};q^{-2})_k + \begin{bmatrix} m \\ k-1 \end{bmatrix}_{q^2}(q^{2m+1};q^{-2})_{k-1}q^{2m+2-2k}(1-q^{2m+3-2k}),\end{aligned} \tag{2.20}$$

which is equivalent to

$$\begin{bmatrix} m+1 \\ k \end{bmatrix}_{q^2} = \begin{bmatrix} m \\ k \end{bmatrix}_{q^2} + \begin{bmatrix} m \\ k-1 \end{bmatrix}_{q^2}(1-q^{2m+1-2(k-1)})^{-1}q^{2m+2-2k}(1-q^{2m+3-2k}),$$

which is finally

$$\begin{bmatrix} m+1 \\ k \end{bmatrix}_{q^2} = \begin{bmatrix} m \\ k \end{bmatrix}_{q^2} + \begin{bmatrix} m \\ k-1 \end{bmatrix}_{q^2}(q^2)^{m+1-k},$$

and this is so by formula (2.8b).

Next,

$$\begin{aligned}\mathcal{O}(\tilde{S}_{2m}) &= \sum_{k=0}^{m} d_{m|k}\mathcal{O}((x \dot{-} 1)^{2m-2k}) \\ &= \sum_{k=0}^{m} d_{m|k}((x \dot{-} 1)^{2m+1-2k} + q^{2m-2k-1}(1-q^{2m-2k})(x \dot{-} 1)^{2m-1-2k}) \\ &= \sum_{k=0}^{m} (x \dot{-} 1)^{2m+1-2k}(d_{m|k} + d_{m|k-1}q^{2m+1-2k}(1-q^{2m+2-2k})), \end{aligned} \tag{2.21ℓ}$$

while

$$\tilde{S}_{2m+1} = \sum_{k=0}^{m} (x \dot{-} 1)^{2m+1-2k} c_{m|k}, \tag{2.21r}$$

so we need to check that

$$c_{m|k} = d_{m|k} + d_{m|k-1}q^{2m+1-2k}(1-q^{2m+2-2k}), \tag{2.22}$$

which is

$$\begin{aligned}\begin{bmatrix} m \\ k \end{bmatrix}_{q^2} (q^{2m+1}; q^{-2})_k &= \begin{bmatrix} m \\ k \end{bmatrix}_{q^2} (q^{2m-1}; q^{-2})_k \\ &+ \begin{bmatrix} m \\ k-1 \end{bmatrix}_{q^2} (q^{2m-1}; q^{-2})_{k-1} q^{2m+1-2k}(1-q^{2m+2-2k}), \end{aligned} \tag{2.23}$$

which is equivalent to

$$\begin{bmatrix} m \\ k \end{bmatrix}_{q^2} (1-q^{2m+1}) = \begin{bmatrix} m \\ k \end{bmatrix}_{q^2} (1-q^{2m-1-2(k-1)}) + \begin{bmatrix} m \\ k-1 \end{bmatrix}_{q^2} q^{2m+1-2k}(1-q^{2m+2-2k}),$$

which can be rewritten as

$$\begin{bmatrix} m \\ k \end{bmatrix}_{q^2} q^{2m+1-2k}(1-q^{2k}) = \begin{bmatrix} m \\ k-1 \end{bmatrix}_{q^2} q^{2m+1-2k}(1-q^{2m+2-2k}),$$

which is equivalent to

$$\begin{bmatrix} m \\ k \end{bmatrix} [k] = \begin{bmatrix} m \\ k-1 \end{bmatrix} [m+1-k],$$

which is obvious.

Remark 2.24. Set

$$\tilde{S}_N(x) = \sum_{k} e_{N|k}(x \dot{-} 1)^{N-2k}, \tag{2.25}$$

so that

$$c_{m|k} = e_{2m+1|k}, \quad d_{m|k} = e_{2m|k}. \tag{2.26}$$

Then the pair of equalities (2.19) and (2.22) can be rewritten as the single one:

$$e_{N+1|k} = e_{N|k} + e_{N|k-1} q^{N+1-2k}(1 - q^{N+2-2k}), \tag{2.27}$$

equivalent to the relation

$$\tilde{S}_{N+1} = \mathcal{O}(\tilde{S}_N).$$

3 The Taylor expansions point of view

Formula (1.16) (or (2.25)) is reminiscent of the Taylor expansion:

$$f(x) = \sum_{k \geq 0} \frac{f^{(k)}(a)}{k!}(x-a)^k, \tag{3.1}$$

where

$$f^{(i)}(x) = \left(\frac{d}{dx}\right)^i (f(x)). \tag{3.2}$$

There exist many different q-versions of the classical Taylor expansion. We shall make use below of the following particular one:

$$f(x) = \sum_{k \geq 0} \frac{f^{(k)}(a)}{[k]!}(x \dot{-} a)^k, \tag{3.3}$$

where now

$$f^{(k)}(x) = \left(\frac{d}{d_q x}\right)^k (f(x)). \tag{3.4}$$

We shall prove formula (3.3) for f being polynomial in x. It's enough to consider the case $f(x) = x^n$, so that

$$f^{(k)}(x) = [k]! \begin{bmatrix} n \\ k \end{bmatrix} x^{n-k}, \tag{3.5}$$

and we thus have to check that

$$x^n = \sum_k \begin{bmatrix} n \\ k \end{bmatrix} a^{n-k}(x \dot{-} a)^k. \tag{3.6}$$

This can be verified either directly, or deduced from the identity (formula (2.10) in [5], p. 75)

$$\sum_k \begin{bmatrix} n \\ k \end{bmatrix} a^{n-k}(x \dot{+} b)^k = \sum_k \begin{bmatrix} n \\ k \end{bmatrix} x^{n-k}(a \dot{+} b)^k. \tag{3.7}$$

for $b = -a$.

Thus, formula (3.3) is proven. Taking $f(x)$ to be $S_N(x)$,

$$S_N(x) = (-1)^N \sum_k \begin{bmatrix} N \\ k \end{bmatrix} (-x)^k, \tag{3.8}$$

where, by formula (1.18a),

$$S_N^{(k)}(x) = [k]! \begin{bmatrix} N \\ k \end{bmatrix} S_{N-k}(x), \tag{3.9}$$

we get

$$S_N(x) = \sum_k \begin{bmatrix} N \\ k \end{bmatrix} G_{N-k}(x \dot{-} 1)^k = \sum_k \begin{bmatrix} N \\ k \end{bmatrix} (x \dot{-} 1)^{N-k} G_k, \tag{3.10}$$

where, by the Gauss formula (1.7),

$$G_k = S_k(1) = \begin{cases} 0, & k \text{ odd}, \\ (q^{k-1}; q^{-2})_{\lfloor k/2 \rfloor}, & k \text{ even}. \end{cases} \tag{3.11}$$

Thus,

$$S_N(x) = \sum_k \begin{bmatrix} N \\ 2k \end{bmatrix} (x \dot{-} 1)^{N-2k} (q^{2k-1}; q^{-2})_k. \tag{3.12}$$

Comparing formulae (1.12) and (3.12), we see that we must have

$$\begin{bmatrix} N \\ 2k \end{bmatrix}_q (q^{2k-1}; q^{-2})_k = \begin{cases} \begin{bmatrix} m \\ k \end{bmatrix}_{q^2} (q^{2m+1}; q^{-2})_k, & N = 2m+1 \\ \begin{bmatrix} m \\ k \end{bmatrix}_{q^2} (q^{2m-1}; q^{-2})_k, & N = 2m \end{cases} \tag{3.13}$$

and these relations can be easily verified. Thus,

$$(-1)^N \sum_{k=0}^{N} \begin{bmatrix} N \\ k \end{bmatrix} (-x)^k = \sum_{k=0}^{\lfloor N/2 \rfloor} \begin{bmatrix} N \\ 2k \end{bmatrix} (x \dot{-} 1)^{N-2k} (q^{2k-1}; q^{-2})_k. \tag{3.14}$$

Remark 3.15. Euler's formula (1.13) suggests that one should consider more general family of polynomials:

$$P_N(x) = \sum_{\ell=0}^{N} \begin{bmatrix} N \\ \ell \end{bmatrix} x^\ell q^{\alpha \ell^2}, \tag{3.16}$$

with $\alpha = 0$ corresponding to the Gauss case, $\alpha = 1/2$ corresponding to the Euler case, and $\alpha = 1$ corresponding to the Szegö case [1,7]. Applying the arguments used above, we find:

$$\frac{dP_N(x)}{d_q x} = [N]q^\alpha P_{N-1}(q^{2\alpha}x), \tag{3.17}$$

$$P_{N+1}(x) = q^\alpha x P_N(q^{2\alpha}x) + P_N(qx) \tag{3.18a}$$

$$= q^{N+\alpha} x P_N(q^{2\alpha-1}x) + P_N(x), \tag{3.18b}$$

$$P_N(x) = \sum_{k=0}^{N} \begin{bmatrix} N \\ k \end{bmatrix} \rho_{N-k}(x)\theta_k, \tag{3.19}$$

where

$$\rho_n(x) = q^{(1-2\alpha)\binom{n}{2}}(-q^{(2n-1)\alpha}x; q^{-1})_n \tag{3.20}$$

satisfies the same q-differential equation (3.17) as $P_n(x)$:

$$\frac{d\rho_n(x)}{d_q x} = [n]q^\alpha \rho_{n-1}(q^{2\alpha}x), \tag{3.21}$$

and θ_k's are some x-independent connection coefficients. Unfortunately, I haven't been able to find a compact expression for the coefficients $\theta_k = \theta_k(q;\alpha)$.

4 The geometric progressions point of view

Formula (1.2)

$$\sum_{\ell=0}^{N} \binom{N}{\ell}(-1)^\ell = \delta_0^N, \quad N \in \mathbf{Z}_+, \tag{4.1}$$

can be equivalently put into the following interesting form:

$$\sum_{\ell=0}^{\infty} \frac{t^\ell}{(1+t)^{\ell+1}} = 1. \tag{4.2}$$

(We treat all series as formal power series, and so don't have to pay attention to questions of convergence. The series (4.2) converges for real $t > -1/2$.) Indeed, multiply equality (4.1) by $(-t)^N$ and then sum on all $N \in \mathbf{Z}_+$:

$$1 = \sum_{N,\ell}(-t)^N \binom{N}{\ell}(-1)^\ell = \sum_{s,\ell}(-t)^{s+\ell}\binom{s+\ell}{\ell}(-1)^\ell = \sum_{\ell\geq 0} t^\ell \sum_{s\geq 0}\binom{s+\ell}{\ell}(-t)^s$$
$$= \sum_{\ell\geq 0}\frac{t^\ell}{(1+t)^{\ell+1}},$$

where we used the following version of the Newton's binomial

$$\frac{1}{(1-t)^{N+1}} = \sum_{s\geq 0}\binom{N+s}{s} t^s. \tag{4.3}$$

We can perform similar conversion upon the formula (1.5), an Euler-type q-analogue of formula (4.1). Multiply the equality

$$\sum_{\ell=0}^{N} \begin{bmatrix} N \\ \ell \end{bmatrix} (-1)^{\ell} q^{\binom{\ell}{2}} = \delta_0^N, \quad N \in \mathbf{Z}_+, \tag{4.4}$$

by $(-t)^N$ and sum over all $N \in \mathbf{Z}_+$:

$$\begin{aligned} 1 &= \sum_{N,\ell} (-t)^N \begin{bmatrix} N \\ \ell \end{bmatrix} (-1)^{\ell} q^{\binom{\ell}{2}} = \sum_{s,\ell \geq 0} (-t)^{s+\ell} \begin{bmatrix} s+\ell \\ \ell \end{bmatrix} (-1)^{\ell} q^{\binom{\ell}{2}} \\ &= \sum_{\ell} t^{\ell} q^{\binom{\ell}{2}} \sum_{s} \begin{bmatrix} s+\ell \\ \ell \end{bmatrix} (-t)^s \ [\text{by } (4.6)] = \sum_{\ell \geq 0} \frac{t^{\ell} q^{\binom{\ell}{2}}}{(1 \dotplus t)^{\ell+1}} \ . \end{aligned}$$

Thus,

$$\sum_{\ell=0}^{\infty} \frac{t^{\ell} q^{\binom{\ell}{2}}}{(1 \dotplus t)^{\ell+1}} = 1; \tag{4.5}$$

we used in the calculation above the following Euler version of formula (4.3):

$$\frac{1}{(1 \dot{-} t)^{N+1}} = \sum_{s \geq 0} \begin{bmatrix} N+s \\ s \end{bmatrix} t^s. \tag{4.6}$$

Let us now apply the same conversion device to the Gauss result (1.7):

$$G_N = \sum_{k=0}^{N} \begin{bmatrix} N \\ k \end{bmatrix} (-1)^k = \begin{cases} 0, & N \text{ odd}, \\ (q^{N-1}; q^{-2})_{\lfloor N/2 \rfloor}, & N \text{ even}. \end{cases} \tag{4.7}$$

Multiplying by $(-t)^N$ and summing on N we find:

$$\begin{aligned} \sum_{N} (-t)^N G_N &= \sum_{m} t^{2m} (q^{2m-1}; q^{-2})_m = 1 + \sum_{m=1}^{\infty} (1-q)...(1-q^{2m-1}) t^{2m} \\ &= \sum_{N} (-t)^N \sum_{k} \begin{bmatrix} N \\ k \end{bmatrix} (-1)^k = \sum_{s,k} (-t)^{k+s} \begin{bmatrix} k+s \\ k \end{bmatrix} (-1)^k \\ &= \sum_{k} t^k \sum_{s} \begin{bmatrix} k+s \\ k \end{bmatrix} (-t)^s = \sum_{k \geq 0} \frac{t^k}{(1 \dotplus t)^{k+1}}. \end{aligned}$$

Thus,

$$\sum_{k \geq 0} \frac{t^k}{(1 \dotplus t)^{k+1}} = 1 + \sum_{m=1}^{\infty} (1-q)...(1-q^{2m-1}) t^{2m}. \tag{4.8}$$

This formula is the first from a pair found by Carlitz in [3]. The second formula in that pair is the case $\{r = 1\}$ of the following general relation

$$\sum_{\ell=0}^{\infty} \frac{(q^r t)^{\ell} q^{\binom{\ell}{2}}}{(1 \dotplus t)^{\ell+1}} = \sum_{N \geq 0} (1 \dot{-} q^r)^N (-t)^N, \tag{4.9}$$

which can be proven as follows:

$$\sum_{\ell=0}^{\infty} \frac{(q^r t)^\ell q^{\binom{\ell}{2}}}{(1\dot{+}t)^{\ell+1}} = \sum_{\ell} q^{r\ell} t^\ell q^{\binom{\ell}{2}} \sum_{s} \begin{bmatrix} \ell+s \\ \ell \end{bmatrix} (-t)^s = \sum_{N\geq 0} t^N \sum_{\ell=0}^{N} (-1)^{N-\ell} \begin{bmatrix} N \\ \ell \end{bmatrix} q^{\binom{\ell}{2}} (q^r)^\ell$$
$$= \sum_{N} (-t)^N \sum_{\ell=0}^{N} \begin{bmatrix} N \\ \ell \end{bmatrix} q^{\binom{\ell}{2}} (-q^r)^\ell \text{ [by (1.3)]} = \sum_{N} (-t)^N (1 \dot{-} q^r)^N.$$

For $r = 0$, formula (4.9) becomes formula (4.5). Since r is arbitrary, replacing in formula (4.9) tq^r by another variable z, we get

$$\sum_{\ell=0}^{\infty} \frac{z^\ell q^{\binom{\ell}{2}}}{(1\dot{+}t)^{\ell+1}} = \sum_{N\geq 0} (-1)^N (t \dot{-} z)^N, \tag{4.10}$$

a q-analogue of the geometric progression formula

$$\frac{1}{1+t} \sum_{\ell=0}^{\infty} \left(\frac{z}{1+t} \right)^\ell = \sum_{N=0}^{\infty} (z-t)^N. \tag{4.11}$$

5 Gauss-like non-alternating sums

For $x = -1$, Newton's formula (1.1) yields

$$\sum_{\ell=0}^{N} \binom{N}{\ell} = 2^N. \tag{5.1}$$

Similarly, the Euler binomial (1.3) for $x = -q$ provides

$$\sum_{\ell=0}^{N} \begin{bmatrix} N \\ \ell \end{bmatrix} q^{\binom{\ell+1}{2}} = (1\dot{+}q)^N. \tag{5.2}$$

If we apply to these two banalities Gauss-like ansatz, we should look at the sums of the form

$$\sum_{\ell=0}^{N} \begin{bmatrix} N \\ \ell \end{bmatrix} (q^r)^\ell. \tag{5.3}$$

Not much is known about such sums, at least as far as I can tell. (See Remark 6.12.) However, we shall see below that for $r = 1/2$,

$$\sum_{\ell=0}^{N} \begin{bmatrix} N \\ \ell \end{bmatrix} q^{\ell/2} = (-q^{1/2}; q^{1/2})_N. \tag{5.4}$$

Changing q into q^2, this formula may be rewritten in the form

$$\sigma_N = \sum_{\ell=0}^{N} \begin{bmatrix} N \\ \ell \end{bmatrix}_{q^2} q^\ell = (1\dot{+}q)^N. \tag{5.5}$$

Let's prove it. This formula is obviously true for $N = 0, 1$. Using induction on N and observing that

$$\sigma_N = \sum_{\ell=0}^{N} \begin{bmatrix} N \\ \ell \end{bmatrix}_{q^2} q^{\ell} = \sum_{\ell=0}^{N} \begin{bmatrix} N \\ N-\ell \end{bmatrix}_{q^2} q^{\ell} = \sum_{\ell=0}^{N} \begin{bmatrix} N \\ \ell \end{bmatrix}_{q^2} q^{N-\ell}, \tag{5.6}$$

we find:

$$\begin{aligned} \sigma_{N+1} &= \sum_{\ell \geq 0} \begin{bmatrix} N+1 \\ \ell \end{bmatrix}_{q^2} q^{\ell} \text{ [by (2.8b)]} = \sum_{\ell=0} \left(\begin{bmatrix} N \\ \ell \end{bmatrix}_{q^2} + \begin{bmatrix} N \\ \ell-1 \end{bmatrix}_{q^2} q^{2N+2-2\ell} \right) q^{\ell} \\ &= \sigma_N + \sum_{\ell \geq 0} \begin{bmatrix} N \\ \ell \end{bmatrix}_{q^2} q^{2N+1-\ell} \text{ [by (5.6)]} = \sigma_N + q^{N+1}\sigma_N = (1+q^{N+1})\sigma_N. \end{aligned} \tag{5.7}$$

Thus,

$$\sigma_{N+1} = (1+q^{N+1})\sigma_N, \tag{5.8}$$

and since $\sigma_0 = 1$, formula (5.5) follows.

The *derivation* of formula (5.7) above suggests consideration of more general sums

$$\sigma_N(\gamma) = \sum_{k=0}^{N} \begin{bmatrix} N \\ k \end{bmatrix}_{q^2} q^{\gamma k}. \tag{5.9}$$

Since

$$\sum_{k=0}^{N} \begin{bmatrix} N \\ k \end{bmatrix}_{q^2} q^{\gamma k} = \sum_{k=0}^{N} \begin{bmatrix} N \\ N-k \end{bmatrix}_{q^2} q^{\gamma k} = \sum_{k=0}^{N} \begin{bmatrix} N \\ k \end{bmatrix}_{q^2} q^{\gamma(N-k)} = q^{\gamma N}\sigma_N(-\gamma),$$

we find that

$$\sigma_N(-\gamma) = q^{-\gamma N}\sigma_N(\gamma). \tag{5.10}$$

Further,

$$\begin{aligned} \sigma_{N+1}(\gamma) &= \sum_{k=0}^{N} \begin{bmatrix} N+1 \\ k \end{bmatrix}_{q^2} q^{\gamma k} \text{ [by (2.8a)]} = \sum_{k=0}^{N} (q^{2k} \begin{bmatrix} N \\ k \end{bmatrix}_{q^2} + \begin{bmatrix} N \\ k-1 \end{bmatrix}_{q^2}) q^{\gamma k} \\ &= \sigma_N(\gamma+2) + q^{\gamma}\sigma_N(\gamma), \end{aligned}$$

so that

$$\sigma_N(\gamma+2) = \sigma_{N+1}(\gamma) - q^{\gamma}\sigma_N(\gamma). \tag{5.11}$$

Since we have already calculated $\sigma_N = \sigma_N(1)$ (5.5), formula (5.11) allows us to find $\sigma_N(\gamma)$ for arbitrary odd γ.

Setting

$$\sigma_N(2\ell+1) = \sigma_N(1) \sum_{s=0}^{\ell} c_{\ell|s} q^{\binom{s+1}{2}} Q^s, \quad Q = q^N, \quad \ell \in \mathbf{Z}_+, \tag{5.12}$$

we can translate the recurrence relation (5.11) into the form

$$c_{\ell+1|s} = (q^s - q^{2\ell+1})c_{\ell|s} + c_{\ell|s-1}, \tag{5.13}$$

with the understanding that

$$c_{\ell|s} = 0 \ \text{ unless } \ 0 \le s \le \ell. \tag{5.14}$$

Since

$$c_{0|0} = 1, \tag{5.15}$$

a little calculation shows that

$$c_{\ell|2r} = \begin{bmatrix} \ell - r \\ r \end{bmatrix}_{q^2} \frac{g_{\ell-r}}{g_r}, \tag{5.16a}$$

$$c_{\ell|2r+1} = \begin{bmatrix} \ell - r - 1 \\ r \end{bmatrix}_{q^2} \frac{g_{\ell-r}}{g_{r+1}}, \tag{5.16b}$$

where g_i's are the Gauss products:

$$g_i = \sqcap_{t \text{ odd } <2i}(1 - q^t), \ i \in \mathbf{N}; \quad g_0 = 1. \tag{5.17}$$

It's easy to verify that formulae (5.16) satisfy the recurrence relation (5.13) and the boundary condition (5.15). It's interesting to observe that formula (5.16) exhibits still another form of 2-periodicity.

The first few $\sigma_N(2\ell+1)$'s are written below:

$$\sigma_N(3)/\sigma_N(1) = (1-q) + qQ, \tag{5.18a}$$

$$\sigma_N(5)/\sigma_N(1) = (1-q)(1-q^3) + qQ(1-q^3) + q^3Q^2, \tag{5.18b}$$

$$\begin{aligned}\sigma_N(7)/\sigma_N(1) = {} & (1-q)(1-q^3)(1-q^5) + qQ(1-q^3)(1-q^5) \\ & + q^3Q^2(1-q^3)[2]_{q^2} + q^6Q^3,\end{aligned} \tag{5.18c}$$

$$\begin{aligned}\sigma_N(9)/\sigma_N(1) = {} & (1-q)(1-q^3)(1-q^5)(1-q^7) + qQ(1-q^3)(1-q^5)(1-q^7) \\ & + q^3Q^2(1-q^3)(1-q^5)[3]_{q^2} + q^6Q^3(1-q^5)[2]_{q^2} + q^{10}Q^4.\end{aligned} \tag{5.18d}$$

Passing to the limit $N \to \infty$ and considering $|q| < 1$, so that $Q = q^N \to 0$, we find:

$$\lim_{N\to\infty} \sigma_N(2\ell+1)/\sigma_N(1) = (1-q)(1-q^3)...(1-q^{2\ell-1}), \quad \ell \in \mathbf{N}. \tag{5.19}$$

Since

$$\sigma_\infty(\gamma) = \lim_{N\to\infty} \sigma_N(\gamma) = \sum_{k\ge 0} \begin{bmatrix} \infty \\ k \end{bmatrix}_{q^2} q^{\gamma k} = 1 + \sum_{k>0} \frac{q^{\gamma k}}{(1-q^2)...(1-q^{2k})}, \tag{5.20}$$

formula (5.19) can be rewritten as

$$\sum_{k\ge 0} \frac{q^{2\ell+1)k}}{(q^2;q^2)k} = (q;q^2)_\ell \sum_{k\ge 0} \frac{q^k}{(q^2;q^2)_k}. \tag{5.21}$$

Now

$$(a;\rho)_\ell = (a;\rho)_\infty/(\rho^\ell a;\rho)_\infty, \tag{5.22}$$

so that formula (5.21) can be rewritten as

$$\frac{1}{(q;q^2)_\infty}\sum_{k=0}^{\infty}\frac{z^k}{(q^2;q^2)_k} = \frac{1}{(z;q^2)_\infty}\sum_{k=0}^{\infty}\frac{q^k}{(q^2;q^2)_k}, \tag{5.23}$$

where we introduced

$$z = q^{2\ell+1}. \tag{5.24}$$

Formula (5.23) is true as it stands, for *arbitrary* z, because the difference of the LHS and the RHS of this formula is an analytic function of z for $|z| < 1$, vanishing for an infinite number of different values $z = q^{2\ell+1}$, $\ell \in \mathbf{Z}_+$, condensing to zero.

Remark 5.25. The *alternating* Gauss-like sums (1.9)

$$(-1)^N s_{N|r} = \sum_{\ell=0}^{N}\begin{bmatrix} N \\ \ell \end{bmatrix}(-1)^\ell (q^r)^\ell \tag{5.26}$$

have been effectively calculated in Section 1 for *integer* $r \in \mathbf{Z}$. The *non-alternating* sums (5.3)

$$\sum_{\ell=0}^{N}\begin{bmatrix} N \\ \ell \end{bmatrix}(q^r)^\ell \tag{5.27}$$

have been effectively calculated in this section for *half-integers* $r \in \frac{1}{2} + \mathbf{Z}$. There must be some underlying reasons for this dichotomy.

6 Remarks

Remark 6.1. The basic philosophy of q-language is *multiplicative* discretization of classical continuous mathematics. Interestingly enough, the formulae in this paper can be interpreted as statements in an *additive* discrete language, a certain q-analogue of the classical difference calculus. The latter can be summarized as follows.

Let $\boldsymbol{\theta} = (\theta(0), \theta(1), \ldots)$ be a fixed sequence. For every sequence $\{a_n\}$, define the q-difference sequences

$$(\Delta^0 a)_n = a_n, \tag{6.1a}$$

$$(\Delta^{k+1} a)_n = (\Delta^k a)_{n+1} - q^{\theta(k)}(\Delta^k a)_n, \quad k \in \mathbf{Z}_+. \tag{6.1b}$$

When the parameter $\boldsymbol{\theta}$ has the canonical form

$$\theta(k) = k, \quad k \in \mathbf{Z}_+, \tag{6.2}$$

the sequences $\{(\Delta^k a)_n | k, n \in \mathbf{Z}_+\}$ can be reconstructed from the boundary conditions

$$b_k = (\Delta^k a)_0, \quad k \in \mathbf{Z}_+, \tag{6.3}$$

by the easily verifiable formula

$$(\Delta^k a)_n = \sum_{s=0}^{n} b_{k+n-s} \begin{bmatrix} n \\ s \end{bmatrix} q^{ks}. \tag{6.4}$$

In particular, when $k = 0$ we get

$$a_n = (\Delta^0 a)_n = \sum_{s=0}^{n} b_{n-s} \begin{bmatrix} n \\ s \end{bmatrix} = \sum_{s=0}^{n} b_s \begin{bmatrix} n \\ s \end{bmatrix}. \tag{6.5}$$

Thus, evaluation of the sums (5.26) and (5.27):

$$\sum_{\ell=0}^{N} \begin{bmatrix} N \\ \ell \end{bmatrix} (\pm q^r)^\ell, \tag{6.6}$$

can be thought of as the process of reconstruction of the original sequence $\{a_N\}$ given the boundary q-difference sequence $\{(\Delta^n a)_0 = (\pm q^r)^n\}$.

In a superficially more general direction, say for the nonalternating case, if we fix $r, \rho \in \mathbf{Z}_+$ and set

$$b_s = \begin{bmatrix} s \\ \rho \end{bmatrix} q^{\alpha(s)} \quad , \quad \alpha(s) = (s - \rho)(r + \frac{1}{2}), \tag{6.7}$$

we find

$$\begin{aligned} a_n &= \sum_{s=0}^{n} \begin{bmatrix} n \\ s \end{bmatrix} b_s = \sum_{s=0}^{n} \begin{bmatrix} n \\ s \end{bmatrix} \begin{bmatrix} s \\ \rho \end{bmatrix} q^{\alpha(s)} = \begin{bmatrix} n \\ \rho \end{bmatrix} \sum_{s=\rho}^{n} \begin{bmatrix} n-\rho \\ s-\rho \end{bmatrix} q^{\alpha(s)} \\ &= \begin{bmatrix} n \\ \rho \end{bmatrix} \sum_{s=0}^{n-\rho} \begin{bmatrix} n-\rho \\ s \end{bmatrix} q^{s(r+\frac{1}{2})} = \begin{bmatrix} n \\ \rho \end{bmatrix} \tilde{\sigma}_{n-\rho}(2r+1), \end{aligned} \tag{6.8}$$

where

$$\tilde{\sigma}_N(\gamma; q) = \sigma_N(\gamma; q^{\frac{1}{2}}). \tag{6.9}$$

In particular, for $r = 0$ and $\rho = 1$, formula (6.8) yields:

$$a_n = [n](-q^{\frac{1}{2}}; q^{\frac{1}{2}})_{n-1}. \tag{6.10}$$

When $q = 1$, this becomes S. Rabinowitz's Crux 946 formula ([6], p. 194)

$$a_n = n \cdot 2^{n-1}, \quad b_n = n, \quad n \in \mathbf{Z}_+. \tag{6.11}$$

Remark 6.12. Many formulae in this paper can be found in the literature. The polynomials $(-1)^N S_N(-x)$ (1.11) are called by Andrews "Rogers-Szegö polynomials", and many of their interesting properties are listed on pp. 49-51 in [2]. Andrews also provides a very short proof of the Gauss formulae (1.7), on p. 37 in [2]. N. J. Fine has also studied these polynomials; formula (5.5) can be found on p. 29 of his book [4], as well as on p. 49 of the Andrews book [2].

Remark 6.13. The Gauss device can be thought of as chopping off the naturally occurring factors $q^{\binom{n}{2}}$ from the Euler q-analogue (1.32) of Newton's binomial (1.1). In the opposite spirit, one can ask about what happens when we *attach* these factors to a place that is naturally missing them, another Euler's form of Newton's binomial, formula (4.6):

$$V_N(t) = \sum_{s\geq 0} \begin{bmatrix} N+s \\ s \end{bmatrix} t^s q^{\binom{s}{2}}. \tag{6.14}$$

Since these objects are no longer polynomials but are in fact infinite series, we won't pursue this avenue here and leave it to the reader as an exercise. The numbers $v_N = V_N(q)$ can be found on p. 8 of Fine's book [4]:

$$v_{2k} = \frac{1}{(q^2;q^2)_k} \sum_{n\geq 0} q^{\binom{n+1}{2}} = \frac{1}{(q^2;q^2)_k} \sqcap_{n\geq 1} \left(\frac{1-q^{2n}}{1-q^{2n-1}} \right), \tag{6.15a}$$

$$v_{2k+1} = \frac{1}{(q;q^2)_k} = \frac{1}{(1-q)(1-q^3)...(1-q^{2k+1})}. \tag{6.15b}$$

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