On a q-Difference Painlevé III Equation: I. Derivation, Symmetry and Riccati Type Solutions

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Abstract

A q-difference analogue of the Painlevé III equation is considered. Its derivations, affine Weyl group symmetry, and two kinds of special function type solutions are discussed.

1   Introduction

Importance of the Painlevé equations is widely recognized in mathematics and mathematical physics, since the solutions plays a role of nonlinear version of special functions. So much studies for the Painlevé equations have been done from various points of view, such as classical solutions, asymptotics, geometric or algebraic structures and so on [13, 2]. On the other hand, discrete analogue of the Painlevé equations have been studied extensively after discovery of the singularity confinement property [4], which is a discrete analogue of the Painlevé property. What matters in studies of the discrete Painlevé equations was that they were in some sense too rich and looked almost uncontrollable. For example, number of known discrete Painlevé equations is several ten's, while that for the Painlevé equations is only six.

The situation has been changed after Sakai's theory [22]. Sakai constructed "space of initial conditions", or defining manifold of a discrete Painlevé equation called q-$\mathrm{P_{VI}}$ by blowing up $\mathbb{P}^2$ at nine points. Based on this fact, Sakai gave a classification of discrete Painlevé equations with the aid of theory of rational surfaces, and discussed the symmetries. The Painlevé equations appear naturally from the degeneration of defining manifold which is also interpreted as continuous limit. This theory gives us clear view to both discrete and continuous Painlevé equations. It also suggests the list of discrete Painlevé equations to be studied further.

In this paper, we shall work with the following q-difference equation,

$$\overline{f_1} = \frac{q^{2N+1}c^2}{f_0 f_1} \frac{1 + a_0 q^n f_0}{a_0 q^n + f_0},$$
$$\underline{f_0} = \frac{q^{2N+1}c^2}{f_0 f_1} \frac{a_1 q^{-n+\nu} + f_1}{1 + a_1 q^{-n+\nu} f_1}, \tag{1.1}$$

where $f_i = f_i(n; \nu, N)$ $(i = 0, 1)$ are dependent variables, $n \in \mathbb{Z}$ is the independent variable, $\nu, N \in \mathbb{Z}$ are parameters, and q, a_0, a_1, c are constants. Moreover, $\overline{f_i}$ and $\underline{f_i}$ denote $f_i(n+1; \nu, N)$ and $f_i(n-1; \nu, N)$, respectively. We call equation (1.1) a q-difference Painlevé III equation(q-P$_{\text{III}}$). Since q-P$_{\text{III}}$ appears in "Mul.6" in Sakai's classification, it is expected that q-P$_{\text{III}}$ is a "good" equation.

The purpose of our work is to establish and to give explicit description for the following properties of q-P$_{\text{III}}$ (1.1):

- q-P$_{\text{III}}$ admits a continuous limit to the Painlevé III equation (P$_{\text{III}}$),

$$\frac{d^2 v}{dx^2} = \frac{1}{v}\left(\frac{dv}{dx}\right)^2 - \frac{1}{x}\frac{dv}{dx} + \frac{1}{x}\left(\alpha v^2 + \beta\right) + \gamma v^3 + \frac{\delta}{v}, \qquad \gamma = -\delta = 4, \tag{1.2}$$

  where α, β are parameters.

- q-P$_{\text{III}}$ is derived from the discrete-time Relativistic Toda lattice equation [23], while P$_{\text{III}}$ is derived from the relativistic Toda lattice equation [21].

- q-P$_{\text{III}}$ can be regarded as a discrete dynamical system on the root lattice of type $A_2 \times A_1$ [22, 9]. As a consequence, it admits symmetry of (extended) affine Weyl group of type $A_1^{(1)} \times A_1^{(1)}$ as the group of Bäcklund transformations.

- q-P$_{\text{III}}$ admits two classes of classical solutions, namely, the solutions which are reducible to discrete Riccati equation, and rational solutions. There are two kinds of Riccati type solutions, one of which is expressed by the Jackson's (modified) q-Bessel functions, while the Riccati type solution of P$_{\text{III}}$ is expressed by the (modified) Bessel functions [17].

- The rational solutions of P$_{\text{III}}$ (1.2) are expressed as ratio of some special polynomials (Umemura polynomials). They admit determinant formula of Jacobi–Trudi type whose entries are given by the Laguerre polynomials [10]. The rational solutions of q-P$_{\text{III}}$ are characterized by similar special polynomials which also admit determinant formula.

In this paper, being the first half of our work, we discuss the derivation, symmetry, and particular solutions of Riccati type. Rational solutions will be discussed in the next paper.

This paper is organized as follows. In Section 2 we derive P$_{\text{III}}$ and q-P$_{\text{III}}$ from the relativistic Toda lattice and discrete-time relativistic Toda lattice, respectively. Moreover, we also derive q-P$_{\text{III}}$ from the birational representation of affine Weyl group of type $A_2^{(1)} \times A_1^{(1)}$ which was presented in [9]. In Section 3, we construct two kinds of particular solutions of Riccati type, one of which is expressed by Jackson's modified q-Bessel functions. We also construct determinant formula for these solutions.

Remark 1. 1. Originally q-$\mathrm{P_{III}}$ (1.1) was derived in [12] as "asymmetric generalization" of q-difference Painlevé II equation (q-$\mathrm{P_{II}}$) [18],

$$x_{n+1}x_n x_{n-1} = \xi \frac{x_n + \zeta\lambda^n}{1 + \zeta\lambda^n x_n}, \tag{1.3}$$

where ζ, ξ are parameters, and λ is a constant. In fact, putting

$$q = \lambda^{-2}, \qquad \nu = 0, \qquad a_1 = \frac{1}{a_0\lambda},$$
$$x_{2n} = f_1(n;0,N), \qquad x_{2n+1} = f_0(n;0,N), \tag{1.4}$$

q-$\mathrm{P_{III}}$ (1.1) is reduced to q-$\mathrm{P_{II}}$ (1.3) with

$$\xi = \lambda^{-4N+2}c^2, \qquad \zeta = \frac{1}{a_0\lambda}. \tag{1.5}$$

2. The following q-difference equation,

$$w_{n+1}w_{n-1} = \frac{cd(w_n - aq^n)(w_n - bq^n)}{(w_n - c)(w_n - d)}, \tag{1.6}$$

is also identified as a q-difference Painlevé III equation [18, 19]. Here, q is a constant and a, b, c, d are parameters. A class of particular solutions in terms of Jackson's q-Bessel functions is constructed in [11]. It looks that equation (1.6) has a different nature compared to our q-$\mathrm{P_{III}}$ (1.1), and it may be natural to regard equation (1.6) as a degenerate case of the q-difference Painlevé VI equation (q-$\mathrm{P_{VI}}$) [7, 22],

$$y_n y_{n+1} = \frac{a_3 a_4 (z_{n+1} - b_1 q^n)(z_{n+1} - b_2 q^n)}{(z_{n+1} - b_3)(z_{n+1} - b_4)},$$
$$z_n z_{n+1} = \frac{b_3 b_4 (y_n - a_1 q^n)(y_n - a_2 q^n)}{(y_n - a_3)(y_n - a_4)}, \qquad \frac{b_1 b_2}{b_3 b_4} = q\frac{a_1 a_2}{a_3 a_4}, \tag{1.7}$$

where a_i and b_i $(i = 1,2,3,4)$ are parameters.

2　Derivations of q-$\mathrm{P_{III}}$

2.1　Derivation of $\mathrm{P_{III}}$ from relativistic Toda lattice

In this section we derive $\mathrm{P_{III}}$ (1.2) from the relativistic Toda lattice equation [21, 23],

$$\frac{d}{dt}d_n = d_n(c_n - c_{n-1}),$$
$$\frac{d}{dt}c_n = c_n(d_{n+1} + c_{n+1} - d_n - c_{n-1}), \qquad n \in \mathbb{Z}. \tag{2.1}$$

We introduce the variables V_n and K_n by

$$c_n = -\frac{V_n K_n}{K_{n-1}}, \qquad d_n = -K_{n-1} + \frac{V_{n-1}K_{n-1}}{K_{n-2}}, \tag{2.2}$$

which yields

$$\frac{d}{dt} \log V_n = (K_{n-1} + V_{n-1}) - (K_n + V_n),$$

$$\frac{d}{dt} K_n = K_n V_n - K_{n+1} V_{n+1}, \qquad n \in \mathbb{Z}. \tag{2.3}$$

We next impose two-periodicity on equation (2.3). Then we have,

$$\frac{d}{dt} \log V_0 = (K_1 + V_1) - (K_0 + V_0),$$

$$\frac{d}{dt} \log V_1 = (K_0 + V_0) - (K_1 + V_1),$$

$$\frac{d}{dt} K_0 = K_0 V_0 - K_1 V_1,$$

$$\frac{d}{dt} K_1 = K_1 V_1 - K_0 V_0. \tag{2.4}$$

In order to derive P$_{\mathrm{III}}$ (1.2), we introduce additional constants α_n $(n = 0, 1)$ in such a way that

$$\frac{d}{dt} \log V_0 = (K_1 + V_1) - (K_0 + V_0) + \alpha_0,$$

$$\frac{d}{dt} \log V_1 = (K_0 + V_0) - (K_1 + V_1) + \alpha_1,$$

$$\frac{d}{dt} K_0 = K_0 V_0 - K_1 V_1,$$

$$\frac{d}{dt} K_1 = K_1 V_1 - K_0 V_0. \tag{2.5}$$

We note that integrability is still kept under this generalization in a sense that it admits the Lax pair. It is easy to see from equation (2.5) that we have $K_0 + K_1 = $ const and $\frac{d}{dt} \log(V_0 V_1) = \alpha_0 + \alpha_1$. We normalize them so that

$$\alpha_0 + \alpha_1 = 1, \qquad V_0 V_1 = \mathrm{e}^t, \qquad K_0 + K_1 = \beta_0. \tag{2.6}$$

Under this normalization, one can write down the equations for V_0 and K_1. Eliminating K_1, we obtain

$$\frac{d^2 V_0}{dt^2} = \frac{1}{V_0} \left(\frac{dV_0}{dt} \right)^2 + V_0^3 - (\alpha_0 + \beta_0)V_0^2 + (\beta_0 - \alpha_0 + 1)\mathrm{e}^t - \frac{\mathrm{e}^{2t}}{V_0}, \tag{2.7}$$

or

$$\frac{d^2 V_0}{ds^2} = \frac{1}{V_0} \left(\frac{dV_0}{ds} \right)^2 - \frac{1}{s} \frac{dV_0}{ds} + \frac{V_0^2}{s^2} (V_0 - (\alpha_0 + \beta_0)) + \frac{\beta_0 - \alpha_0 + 1}{s} - \frac{1}{V_0}, \tag{2.8}$$

where $s = \mathrm{e}^t$, which is a version of P$_{\mathrm{III}}$ (named as "P$_{\mathrm{III}'}$" in [17]). Equation (2.8) is rewritten into the ordinary form of P$_{\mathrm{III}}$ (1.2) with $\alpha = -4(\alpha_0 + \beta_0)$, $\beta = 4(-\alpha_0 + \beta_0 + 1)$, $\gamma = -\delta = 4$ by putting $s = x^2$ and $V_0 = xv$.

Remark 2. Without introducing additional constants α_n, we obtain the equation for V_0 as

$$\frac{d^2V_0}{dt^2} = \frac{1}{V_0}\left(\frac{dV_0}{dt}\right)^2 + V_0^3 - \beta_0 V_0^2 + \beta_0 - \frac{1}{V_0}. \tag{2.9}$$

Equation (2.9) is classified as "XII" in Gambier's classification [6], and integrated in terms of the elliptic functions. Therefore, such generalization of the relativistic Toda lattice equation is crucial in order to derive P_{III}. Such situation can be seen for other Painlevé equations, where they are derived by the similar generalization of proper soliton equations [1, 15, 16, 14].

2.2 Derivation of q-P_{III} from discrete-time relativistic Toda lattice

Let us next consider the discrete case. We start from the discrete-time relativistic Toda lattice equation [23] in the following form,

$$\frac{\overline{d}_n}{d_n} = \frac{a_{n-1} - d_{n-1}}{a_n - d_n}, \qquad \frac{\overline{c}_n}{c_n} = \frac{a_{n-1} + c_{n-1}}{a_n + c_n},$$
$$a_n = 1 + d_n + \frac{c_{n+1}}{a_{n+1}}, \qquad n \in \mathbb{Z}, \tag{2.10}$$

where $a_n = a_n(t)$, $c_n = c_n(t)$, $d_n = d_n(t)$ and $\overline{}$ denotes the value at $t+1$. We note that for convenience we have inverted the space direction n of the original equation in [23]. We introduce the variables $K_n = K_n(t)$, $V_n = V_n(t)$ by

$$\frac{c_n}{a_n} = \overline{V}_n, \qquad a_n = K_n. \tag{2.11}$$

Then equation (2.10) is rewritten as

$$\frac{\overline{V}_n}{V_n} = \frac{K_{n-1} + V_{n-1}}{K_n + V_n},$$
$$\frac{\overline{K}_{n+1}}{K_n} = \frac{1 + \overline{V}_n}{1 + \overline{V}_{n+1}}\frac{\overline{K}_{n+1} + \overline{V}_{n+1}}{\overline{K}_n + \overline{V}_n}. \tag{2.12}$$

We next impose two-periodicity on equation (2.12), which yields

$$\frac{\overline{V}_0}{V_0} = \frac{K_1 + V_1}{K_0 + V_0}, \qquad \frac{\overline{V}_1}{V_1} = \frac{K_0 + V_0}{K_1 + V_1},$$
$$\frac{\overline{K}_1}{K_0} = \frac{1 + \overline{V}_0}{1 + \overline{V}_1}\frac{\overline{K}_1 + \overline{V}_1}{\overline{K}_0 + \overline{V}_0}, \qquad \frac{\overline{K}_0}{K_1} = \frac{1 + \overline{V}_1}{1 + \overline{V}_0}\frac{\overline{K}_0 + \overline{V}_0}{\overline{K}_1 + \overline{V}_1}. \tag{2.13}$$

Similarly to the continuous case, we introduce additional constants a_n $(n = 0, 1)$ in such a way that

$$\frac{\overline{V}_0}{V_0} = \frac{K_1 + V_1}{K_0 + V_0}a_0, \qquad \frac{\overline{V}_1}{V_1} = \frac{K_0 + V_0}{K_1 + V_1}a_1,$$
$$\frac{\overline{K}_1}{K_0} = \frac{1 + \overline{V}_0}{1 + \overline{V}_1}\frac{\overline{K}_1 + \overline{V}_1}{\overline{K}_0 + \overline{V}_0}, \qquad \frac{\overline{K}_0}{K_1} = \frac{1 + \overline{V}_1}{1 + \overline{V}_0}\frac{\overline{K}_0 + \overline{V}_0}{\overline{K}_1 + \overline{V}_1}. \tag{2.14}$$

We remark equation (2.14) is still integrable in a sense that it passes the singularity confinement test [4]. It is easy to see from equation (2.14) that we have $\overline{K}_0 \overline{K}_1 = K_0 K_1 = $ const and $\overline{V}_0 \overline{V}_1 = a_0 a_1 V_0 V_1$. We normalize as

$$a_0 a_1 = q^2, \qquad V_0 V_1 = c_0^2 q^{2t}, \qquad K_0 K_1 = b_0^2, \tag{2.15}$$

where b_0 and c_0 are constants. Under this normalization, we obtain the equation,

$$\overline{X}YX = \frac{a_0}{q} \frac{1 + c_0 q q^t Y}{Y + c_0 q q^t}, \qquad YX\underline{Y} = \frac{a_0}{q} \frac{X + \frac{b_0}{c_0} q^{-t}}{1 + \frac{b_0}{c_0} q^{-t} X}, \tag{2.16}$$

where the variables $X = X(t)$ and $Y = Y(t)$ are defined by

$$X = -\frac{a_0 c_0}{b_0} q^t \frac{K_1}{\overline{V}_0}, \qquad Y = \frac{1}{c_0 q} q^{-t} \overline{V}_0, \tag{2.17}$$

respectively, and $\underline{\ }$ denotes the value at $t - 1$. Equation (2.16) is essentially the same as q-P$_{\text{III}}$ (1.1).

We finally comment on the continuous limit of q-P$_{\text{III}}$ to P$_{\text{III}}$. The simplest way is to take the limit on the level of equation (2.14); replacing t, K_i, V_i and α_i by t/h, $1 + hK_i$, hV_i and $1 + \alpha_i h$ $(i = 0, 1)$, respectively, and taking the limit $h \to 0$, we obtain equation (2.5).

2.3 Birational representation of affine Weyl group $\widetilde{W}(A_1^{(1)} \times A_2^{(1)})$

In [9], the following birational representation of (extended) affine Weyl group $\widetilde{W}(A_1^{(1)} \times A_2^{(1)})$ is presented:

Theorem 1 ([9]). *Let $\boldsymbol{C}(a_i, f_i$ $(i = 0, 1, 2))$ be the field of rational functions in a_i and f_i. We define the transformations s_i $(i = 0, 1, 2)$ and π acting on $\boldsymbol{C}(a_i, f_i$ $(i = 0, 1, 2))$ by*

$$s_i(a_j) = a_j a_i^{-a_{ij}}, \qquad s_i(f_j) = f_j \left(\frac{a_i + f_i}{1 + a_i f_i} \right)^{u_{ij}} \qquad (i, j = 0, 1, 2),$$

$$\pi(x_i) = x_{i+1}, \qquad (x = a, f), \qquad i \in \mathbb{Z}/3\mathbb{Z}, \tag{2.18}$$

respectively, where $A = (a_{ij})_{i,j=0}^2$ is the generalized Cartan matrix of type $A_2^{(1)}$ and $U = (u_{ij})_{i,j=0}^2$ is an orientation matrix of the corresponding Dynkin diagram:

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \qquad U = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}. \tag{2.19}$$

Moreover, we define the transformations w_0, w_1 and r by

$$w_0(f_i) = \frac{a_i a_{i+1}(a_{i-1} a_i + a_{i-1} f_i + f_{i-1} f_i)}{f_{i-1}(a_i a_{i+1} + a_i f_{i+1} + f_i f_{i+1})},$$

$$w_1(f_i) = \frac{1 + a_i f_i + a_i a_{i+1} f_i f_{i+1}}{a_i a_{i+1} f_{i+1}(1 + a_{i-1} f_{i-1} + a_{i-1} a_i f_{i-1} f_i)},$$

$$r(f_i) = \frac{1}{f_i}, \qquad w_0(a_i) = w_1(a_i) = r(a_i) = a_i, \qquad (i \in \mathbb{Z}/3\mathbb{Z}). \tag{2.20}$$

Then, the transformations $\langle s_0, s_1, s_2, \pi, w_0, w_1, r \rangle$ generate the extended affine Weyl group of type $A_1^{(1)} \times A_2^{(1)}$. Namely, these transformations satisfy the following fundamental relations,

$$s_i^2 = 1, \qquad (s_i s_{i+1})^3 = 1, \qquad \pi^3 = 1, \qquad \pi s_i = s_{i+1} \pi \qquad (i = 0, 1, 2), \tag{2.21}$$

$$w_0^2 = w_1^2 = r^2 = 1, \qquad r w_0 = w_1 r, \tag{2.22}$$

and the actions of $\langle s_0, s_1, s_2, \pi \rangle$ and $\langle w_0, w_1, r \rangle$ commute with each other.

One can construct translations T_i $(i = 1, 2, 3, 4)$ by

$$T_1 = \pi s_2 s_1, \qquad T_2 = s_1 \pi s_2, \qquad T_3 = s_2 s_1 \pi, \qquad T_4 = r w_0. \tag{2.23}$$

We note that these translations satisfy $T_i T_j = T_j T_i$ $(i, j = 1, 2, 3, 4)$ and $T_1 T_2 T_3 = 1$. The actions of T_i on a_i and c are given by

$$
\begin{aligned}
&T_1(a_0) = q a_0, && T_1(a_1) = q^{-1} a_1, && T_1(a_2) = a_2, && T_1(c) = c, \\
&T_2(a_0) = a_0, && T_2(a_1) = q a_1, && T_2(a_2) = q^{-1} a_2, && T_2(c) = c, \\
&T_4(a_0) = a_0, && T_4(a_1) = a_1, && T_4(a_2) = a_2, && T_4(c) = q c, \tag{2.24}
\end{aligned}
$$

respectively. We also remark that $a_0 a_1 a_2 = q$ and $f_0 f_1 f_2 = q c^2$ are invariant with respect to the actions of $\widetilde{W}(A_1^{(1)} \times A_2^{(1)})$ and $\widetilde{W}(A_2^{(1)})$, respectively.

The above affine Weyl group actions determine three birational, commutative, discrete flows on root lattice of type $A_1 \times A_2$. Regarding one of the directions as the "time evolution", other directions can be considered as those of Bäcklund (Schlesinger) transformations.

In [9], T_4 is chosen as the time evolution, whose explicit action is given as

$$T_4(f_0) = a_0 a_1 f_1 \frac{1 + a_2 f_2 + a_2 a_0 f_2 f_0}{1 + a_0 f_0 + a_0 a_1 f_0 f_1}, \qquad T_4(f_1) = a_1 a_2 f_2 \frac{1 + a_0 f_0 + a_0 a_1 f_0 f_1}{1 + a_1 f_1 + a_1 a_2 f_1 f_2},$$

$$T_4(f_2) = a_2 a_0 f_0 \frac{1 + a_1 f_1 + a_1 a_2 f_1 f_2}{1 + a_2 f_2 + a_2 a_0 f_2 f_0}. \tag{2.25}$$

Choosing one of the A_2 direction, for example T_1, as the time evolution, we have

$$T_1(f_1) = f_2 \frac{1 + a_0 f_0}{a_0 + f_0} = \frac{q c^2}{f_0 f_1} \frac{1 + a_0 f_0}{a_0 + f_0},$$

$$T_1^{-1}(f_0) = f_2 \frac{a_1 + f_1}{1 + a_1 f_1} = \frac{q c^2}{f_0 f_1} \frac{a_1 + f_1}{1 + a_1 f_1}. \tag{2.26}$$

Operating $T_1^n T_2^\nu T_4^N$ $(n, \nu, N \in \mathbb{Z})$ on equations (2.25) and (2.26), denoting $T_1^n T_2^\nu T_4^N(f_i) = f_i(n; \nu, N)$ $(i = 0, 1, 2)$, we obtain q-P_{IV},

$$\tilde{f}_0 = a_0 a_1 q^\nu f_1 \frac{1 + a_2 q^{-\nu} f_2 + a_2 a_0 q^{n-\nu} f_2 f_0}{1 + a_0 q^n f_0 + a_0 a_1 q^\nu f_0 f_1},$$

$$\tilde{f}_1 = a_1 a_2 q^{-n} f_2 \frac{1 + a_0 q^n f_0 + a_0 a_1 q^\nu f_0 f_1}{1 + a_1 q^{-n+\nu} f_1 + a_1 a_2 q^{-n} f_1 f_2},$$

$$\tilde{f}_2 = a_2 a_0 q^{n-\nu} f_0 \frac{1 + a_1 q^n f_1 + a_1 a_2 q^{-n} f_1 f_2}{1 + a_2 q^{-\nu} f_2 + a_2 a_0 q^{n-\nu} f_2 f_0}, \tag{2.27}$$

and q-P_{III} (1.1), respectively. In equation (2.27), $\tilde{f}_i$ denote $\tilde{f}_i = f_i(n; \nu, N+1)$ $(i = 0, 1, 2)$.

Remark 3. By construction, *q*-P$_{\text{III}}$ (1.1) describes one of the Bäcklund transformations of *q*-P$_{\text{IV}}$ (2.27), and vice versa. Moreover, under the specialization (1.4), *q*-P$_{\text{IV}}$ (2.27) coincides with the Bäcklund transformation of *q*-P$_{\text{II}}$ (1.3) proposed by Joshi et al [8].

q-P$_{\text{III}}$ admits the symmetry described by the affine Weyl group of type $A_1^{(1)} \times A_1^{(1)}$. In fact, we can easily verify the following fact:

Proposition 1. *We define* w_0', w_1' *by*

$$w_0' = s_0 s_1 s_0, \qquad w_1' = s_2. \tag{2.28}$$

Then, w_0' *and* w_1' *satisfy the fundamental relations of* $W(A_1^{(1)})$,

$$(w_0')^2 = (w_1')^2 = 1. \tag{2.29}$$

Moreover, the actions of $\langle w_0', w_1' \rangle$ *commute with the actions of* T_1 *and* $\langle w_0, w_1, r \rangle$.

3 Particular solutions of Riccati type

3.1 Solutions in terms of Jackson's modified *q*-Bessel functions

Many of Painlevé and discrete Painlevé equations are known to admit 1-parameter family of particular solutions expressible in terms of (*q*-)classical special functions. Such solutions are obtained by finding the special cases where the Painlevé or discrete Painlevé equations are reduced to the (discrete) Riccati equations.

For example, *q*-P$_{\text{IV}}$ (2.27) admits the following Riccati type solutions:

Proposition 2 ([9]). *q*-P$_{\text{IV}}$ *(2.27) admits particular solutions given by*

$$f_0 = a_0 c^2 q^{n+2N+1} \frac{G_n(N)}{G_n(N+1)}, \qquad f_1 = -\frac{q}{a_0 q^n} \frac{G_n(N+1)}{G_n(N)}, \qquad f_2 = -1 \tag{3.1}$$

for $\nu = 0$, *where* $G_n(N)$ *is a function satisfying the contiguity relations*

$$
\begin{aligned}
G_{n+1}(N+1) &= G_n(N+1) + a_0^2 c^2 q^{2n+2N+1} G_n(N), \\
G_{n+1}(N) &= G_n(N+1) + c^2 q^{2N+1} G_n(N).
\end{aligned}
\tag{3.2}
$$

In particular, equations (3.2) admit polynomial solutions in $q^{N+1/2}c$ for $a_0 = q$, and the polynomials $H_n(x) = (q^{N+1/2}c)^{-n} G_n(N)$, $x = (q^{N+1/2}c + q^{-N-1/2}c^{-1})/2$ are identified as *the continuous q-Hermite polynomials* [3].

In the above solutions, the variables N and n play a role of independent variable and parameter, respectively. Since *q*-P$_{\text{III}}$ describes a Bäcklund transformation of *q*-P$_{\text{IV}}$, the above solutions of *q*-P$_{\text{IV}}$ can be also regarded as those for *q*-P$_{\text{III}}$ only by exchanging the role of N and n. For notational simplicity, we introduce a parameter μ and a function $I_\mu(n)$ by

$$q^{N+1/2}c = q^\mu, \qquad I_\mu(n) = (q^{N+1/2}c)^{-n} G_n(N). \tag{3.3}$$

Then equations (3.2) are rewritten as

$$I_{\mu+1}(n+1) = q^{-\mu}I_{\mu+1}(n) + a_0^2 q^{2n+\mu}I_\mu(n),$$
$$I_\mu(n+1) = q^{-\mu}I_{\mu+1}(n) + q^\mu I_\mu(n), \tag{3.4}$$

respectively. We note that the three-term relation of $I_\mu(n)$ with respect to n,

$$I_\mu(n+2) = \left(q^\mu + q^{-\mu}\right)I_\mu(n+1) - \left(1 - a_0^2 q^{2n}\right)I_\mu(n), \tag{3.5}$$

is nothing but Jackson's q-modified Bessel equation [3].

Proposition 3. *Let $I_\mu(n)$ be a function satisfying equation (3.4). Then*

$$f_0 = q^{2\mu}a_0 \frac{I_\mu(n)}{I_{\mu+1}(n)}, \qquad f_1 = -\frac{q}{a_0}\frac{I_{\mu+1}(n)}{I_\mu(n)}, \tag{3.6}$$

satisfy q-P_{III} (1.1) with $\nu = 0$ and $q^{N+1/2}c = q^\mu$.

We remark here that essentially the same solution is also constructed in [5, 20]. By successive application of the Bäcklund transformation T_2 on these solutions, we obtain "higher-order" solutions for $\nu \in \mathbb{Z}_{>0}$. Moreover, such solutions admit the following determinant formula.

Theorem 2. *Let $I_\mu(n)$ be a function satisfying equation (3.4). For each $\nu \in \mathbb{Z}_{\geq 0}$, we define a $\nu \times \nu$ determinant $\varphi_\nu^\mu(n)$ by*

$$\varphi_\nu^\mu(n) = \det\left(I_\mu(n-i+j)\right)_{1\leq i,j\leq\nu} \quad (\nu \in \mathbb{Z}_{>0}), \qquad \varphi_0^\mu(n) = 1. \tag{3.7}$$

Then,

$$f_0 = a_0 q^{2\mu}\frac{\varphi_\nu^{\mu+1}(n)\varphi_{\nu+1}^\mu(n)}{\varphi_{\nu+1}^{\mu+1}(n)\varphi_\nu^\mu(n)}, \qquad f_1 = -\frac{q}{a_0}\frac{\varphi_{\nu+1}^{\mu+1}(n)\varphi_\nu^\mu(n-1)}{\varphi_\nu^{\mu+1}(n-1)\varphi_{\nu+1}^\mu(n)}, \tag{3.8}$$

satisfy q-P_{III} (1.1) with $\nu \in \mathbb{Z}_{\geq 0}$ and $q^{N+1/2}c = q^\mu$.

Since Theorem 2 is obtained by the interpretation of the case of q-P_{IV} [9], we omit the proof. We also note that it is possible to construct similar determinant formula for negative ν [9].

3.2 q-P_{III} specific solutions

In the previous section, we have discussed such solutions that n, ν and N (or μ) played a role of independent variable, determinant size and parameter of the solution, respectively. It is possible to construct another Riccati type solution in which the roles of ν and N (or μ) are exchanged.

We first look for the special values of N and c which admit decoupling of q-P_{III} (1.1) into discrete Riccati equation. In fact, it is easy to verify the following proposition:

Proposition 4. *q-P*$_{\mathrm{III}}$ *(1.1) is decoupled into the discrete Riccati equation,*

$$\overline{f}_1 = -\frac{q^{-\nu+1}}{a_0 a_1}\frac{1+a_0 q^n f_0}{f_0}, \qquad f_1 = -a_0 a_1 q^\nu \frac{1}{a_0 q^n + f_0}, \tag{3.9}$$

when $N = 0$ and $c = 1$.

Eliminating f_1 from equation (3.9), we obtain a discrete Riccati equation for f_0,

$$\overline{f}_0 = \frac{-a_0 q^{n+1} + \left(a_0^2 a_1^2 q^{2\nu-1} - a_0^2 q^{2n+1}\right) f_0}{1 + a_0 q^n f_0}. \tag{3.10}$$

Standard linearization of equation (3.10) gives the following result:

Proposition 5. *Let $H_\nu(n)$ be a function satisfying the following linear difference equation,*

$$H_\nu(n+1) = \left[1 + a_0^2 a_1^2 q^{2\nu} - a_0^2 q^{2n}\right] H_\nu(n) - a_0^2 a_1^2 q^{2\nu} H_\nu(n-1). \tag{3.11}$$

Then,

$$f_0 = \frac{1}{a_0 q^n}\frac{H_\nu(n+1) - H_\nu(n)}{H_\nu(n)},$$
$$f_1 = -a_0^2 a_1 q^{n+\nu}\frac{H_\nu(n)}{H_\nu(n+1) - \left(1 - a_0^2 q^{2n}\right) H_\nu(n)}, \tag{3.12}$$

satisfy q-P$_{\mathrm{III}}$ *(1.1) for $N = 0$ and $c = 1$.*

One can investigate the evolution with respect to ν by substituting the above expression into the Bäcklund transformations of *q*-P$_{\mathrm{III}}$ described in Section 2.2. After some manipulation we arrive at the following result:

Theorem 3. *Let $H_\nu(n)$ be a function satisfying the following contiguity relations,*

$$H_\nu(n+1) - H_\nu(n) = -a_0^2 q^{2n} H_{\nu-1}(n), \tag{3.13}$$
$$H_{\nu+1}(n) - H_\nu(n) = -a_0^2 a_1^2 q^{2\nu} H_\nu(n-1). \tag{3.14}$$

Then,

$$f_0 = -a_0 q^n \frac{H_{\nu-1}(n)}{H_\nu(n)}, \qquad f_1 = \frac{1}{a_0^2 a_1 q^{n+\nu-2}}\frac{H_\nu(n)}{H_{\nu-1}(n-1)}, \tag{3.15}$$

satisfy q-P$_{\mathrm{III}}$ *(1.1) with $N = 0$ and $c = 1$.*

Remark 4. As is expected from the symmetry described in Section 2.2, the contiguity relations (3.13) and (3.14) for $H_\nu(n)$ are symmetric with respect to n and ν. In particular, the three-term relation with respect to ν is obtained as

$$H_{\nu+1}(n) = \left[1 + a_0^2 q^{2n} - a_0^2 a_1^2 q^{2\nu}\right] H_\nu(n) - a_0^2 q^{2n} H_{\nu-1}(n). \tag{3.16}$$

We obtain higher-order solutions in the form of rational functions in $H_\nu(n)$ by applying the Bäcklund transformation T_4 to the above solution. Furthermore, these solutions admit the following determinant formula:

Theorem 4. *For each $N \in \mathbb{Z}_{\geq 0}$, we define an $N \times N$ determinant $\phi_\nu^N(n)$ by*

$$\phi_\nu^N(n) = \det\left(H_{\nu+i-1}(n+j-1)\right)_{1 \leq i,j \leq N} \quad (N \in \mathbb{Z}_{>0}), \qquad \phi_\nu^0(n) = 1. \tag{3.17}$$

Then,

$$
\begin{aligned}
f_0 &= -a_0 q^{n+2N} \, \frac{\phi_{\nu-1}^{N+1}(n)\phi_\nu^N(n)}{\phi_\nu^{N+1}(n)\phi_{\nu-1}^N(n)}, \\[2mm]
f_1 &= \frac{1}{a_0^2 a_1 q^{n+\nu+2N-2}} \, \frac{\phi_\nu^{N+1}(n)\phi_{\nu-1}^N(n-1)}{\phi_{\nu-1}^{N+1}(n-1)\phi_\nu^N(n)},
\end{aligned}
\tag{3.18}
$$

satisfy q-$\mathrm{P}_{\mathrm{III}}$ (1.1) with $c = 1$, $N \in \mathbb{Z}_{\geq 0}$.

We next prove Theorem 4. It is a direct consequence of the following "multiplicative formula" with respect to ϕ.

Proposition 6. *The following formulas holds.*

$$1 + a_0 q^n f_0 = q^{-2N} \frac{\phi_\nu^{N+1}(n+1)\phi_{\nu-1}^N(n-1)}{\phi_\nu^{N+1}(n)\phi_{\nu-1}^N(n)}, \tag{3.19}$$

$$1 + \frac{f_0}{a_0 q^n} = -a_0^2 a_1^2 q^{2(\nu+N-1)} \frac{\phi_{\nu-1}^{N+1}(n-1)\phi_\nu^N(n+1)}{\phi_\nu^{N+1}(n)\phi_{\nu-1}^N(n)}, \tag{3.20}$$

$$1 + a_1 q^{-n+\nu} f_1 = \frac{1}{a_0^2 q^{2(n-1)}} \frac{\phi_\nu^{N+1}(n-1)\phi_{\nu-1}^N(n)}{\phi_{\nu-1}^{N+1}(n-1)\phi_\nu^N(n)}, \tag{3.21}$$

$$1 + \frac{f_1}{a_1 q^{-n+\nu}} = \frac{1}{a_0^2 a_1^2 q^{2(\nu-1)}} \frac{\phi_{\nu-1}^{N+1}(n)\phi_\nu^N(n-1)}{\phi_{\nu-1}^{N+1}(n-1)\phi_\nu^N(n)}. \tag{3.22}$$

In fact, it is an easy task to derive q-$\mathrm{P}_{\mathrm{III}}$ by using the above multiplicative formulas and equation (3.18). Furthermore, Proposition 6 follows from the following proposition.

Proposition 7. *$\phi_\nu^N(n)$ satisfies the following bilinear difference equations.*

$$
\begin{aligned}
\phi_\nu^{N+1}(n)\phi_{\nu-1}^N(n) &- a_0^2 q^{2(n+N)}\phi_{\nu-1}^{N+1}(n)\phi_\nu^N(n) \\
&= q^{-2N}\phi_\nu^{N+1}(n+1)\phi_{\nu-1}^N(n-1),
\end{aligned}
\tag{3.23}
$$

$$
\begin{aligned}
q^{2N}\phi_{\nu-1}^{N+1}(n)\phi_\nu^N(n) &- a_0^2 a_1^2 q^{2(\nu+N-1)}\phi_{\nu-1}^{N+1}(n-1)\phi_\nu^N(n+1) \\
&= \phi_\nu^{N+1}(n)\phi_{\nu-1}^N(n),
\end{aligned}
\tag{3.24}
$$

$$
\begin{aligned}
a_0^2 q^{2(n-1)}\phi_{\nu-1}^{N+1}(n-1)\phi_\nu^N(n) &- q^{-2N}\phi_\nu^{N+1}(n)\phi_{\nu-1}^N(n-1) \\
&= \phi_\nu^{N+1}(n-1)\phi_{\nu-1}^N(n),
\end{aligned}
\tag{3.25}
$$

$$
\begin{aligned}
q^{2(\nu-1)}\phi_{\nu-1}^{N+1}(n-1)\phi_\nu^N(n) &- q^{-2N}\phi_\nu^{N+1}(n)\phi_{\nu-1}^N(n-1) \\
&= \phi_{\nu-1}^{N+1}(n)\phi_\nu^N(n-1).
\end{aligned}
\tag{3.26}
$$

For example, equation (3.19) is derived from equation (3.23) by dividing both sides by $\phi_\nu^{N+1}(n)\phi_{\nu-1}^N(n)$. Therefore, the proof of Theorem 4 is now reduced to that of Proposition 7. We will give it in the next section.

3.3 Proof of Proposition 7

Our basic idea for proving Proposition 7 is the same as that was done in [9]. Bilinear difference equations are derived from the Plücker relations, which are quadratic identities among determinants whose columns are shifted. Therefore, we first construct such "difference formulas" that relate "shifted determinants" and $\phi_\nu^N(n)$, by using the contiguity relations of $H_\nu(n)$. We then derive bilinear difference equations with the aid of difference formulas from proper Plücker relations. We take equation (3.24) as an example to show this procedure explicitly. For other equations, see Appendix A.

We introduce a notation,

$$
\phi_\nu^N(n) = \begin{vmatrix} H_\nu(n) & H_{\nu+1}(n) & \cdots & H_{\nu+N-1}(n) \\ H_\nu(n+1) & H_{\nu+1}(n+1) & \cdots & H_{\nu+N-1}(n+1) \\ \vdots & \vdots & \ddots & \vdots \\ H_\nu(n+N-1) & H_{\nu+1}(n+N-1) & \cdots & H_{\nu+N-1}(n+N-1) \end{vmatrix}
$$
$$
= |\mathbf{0}_n, \mathbf{1}_n, \cdots, \mathbf{N-1}_n|, \tag{3.27}
$$

where $\mathbf{k}_n$ denotes a column vector,

$$
\mathbf{k}_n = \begin{pmatrix} H_{\nu+k}(n) \\ H_{\nu+k}(n+1) \\ \vdots \end{pmatrix}. \tag{3.28}
$$

We note that we abbreviate the suffix n when unnecessary.

We next construct the difference formula.

Lemma 1 (Difference formula I).

$$
|\mathbf{0}, \mathbf{1}, \cdots, \mathbf{N-1}| = \phi_\nu^N(n), \tag{3.29}
$$

$$
|\mathbf{0}_{n+1}, \mathbf{0}, \cdots, \mathbf{N-2}| = \left(-a_0^2 a_1^2\right)^{-(N-2)} q^{-(2\nu+N-2)(N-1)} \phi_\nu^N(n+1), \tag{3.30}
$$

$$
|\mathbf{1}_{n+1}, \mathbf{0}, \cdots, \mathbf{N-2}| = \left(-a_0^2 a_1^2\right)^{-(N-2)} q^{-(2\nu+N-2)(N-1)} \phi_\nu^N(n+1). \tag{3.31}
$$

Proof. Equation (3.29) is nothing but the definition equation (3.27). Subtracting the $(k-1)$-th column from the k-th column of the left hand side of equation (3.29) and using the contiguity relation (3.14) for $k = N, N-1, \ldots, 2$, we have

$$
\phi_\nu^N(n) = \begin{vmatrix} \cdots & H_{\nu+N-2}(n) & H_{\nu+N-1}(n) - H_{\nu+N-2}(n) \\ \cdots & H_{\nu+N-2}(n+1) & H_{\nu+N-1}(n+1) - H_{\nu+N-2}(n+1) \\ \vdots & \vdots & \vdots \\ \cdots & H_{\nu+N-2}(n+N-1) & H_{\nu+N-1}(n+N-1) - H_{\nu+N-2}(n+N-1) \end{vmatrix}
$$

$$= \begin{vmatrix} \cdots & H_{\nu+N-2}(n) & -a_0^2 a_1^2 q^{2(\nu+N-2)} H_{\nu+N-2}(n-1) \\ \cdots & H_{\nu+N-2}(n+1) & -a_0^2 a_1^2 q^{2(\nu+N-2)} H_{\nu+N-2}(n) \\ \vdots & \vdots & \vdots \\ \cdots & H_{\nu+N-2}(n+N-1) & -a_0^2 a_1^2 q^{2(\nu+N-2)} H_{\nu+N-2}(n+N-2) \end{vmatrix}$$

$$= -a_0^2 a_1^2 q^{2(\nu+N-2)}$$

$$\times \begin{vmatrix} H_\nu(n) & \cdots & H_{\nu+N-2}(n) & H_{\nu+N-2}(n-1) \\ H_\nu(n+1) & \cdots & H_{\nu+N-2}(n+1) & H_{\nu+N-2}(n) \\ \vdots & \vdots & \vdots & \vdots \\ H_\nu(n+N-1) & \cdots & H_{\nu+N-2}(n+N-1) & H_{\nu+N-2}(n+N-2) \end{vmatrix}$$

$$= \cdots\cdots\cdots\cdots\cdots$$

$$= \left(-a_0^2 a_1^2\right)^{N-1} q^{(2\nu+N-2)(N-1)}$$

$$\times \begin{vmatrix} H_\nu(n) & H_\nu(n-1) & \cdots & H_{\nu+N-2}(n-1) \\ H_\nu(n+1) & H_\nu(n) & \cdots & H_{\nu+N-2}(n) \\ \vdots & \vdots & \vdots & \vdots \\ H_\nu(n+N-1) & H_\nu(n+N-2) & \cdots & H_{\nu+N-2}(n+N-2) \end{vmatrix}, \quad (3.32)$$

which is nothing but equation (3.30). Moreover, subtracting the second column multiplied by $a_0^2 a_1^2 q^{2\nu}$ from the first column of the right hand side of equation (3.32) and using the contiguity relation (3.14), we obtain

$$\phi_\nu^N(n) = \left(-a_0^2 a_1^2\right)^{N-1} q^{(2\nu+N-2)(N-1)}$$

$$\times \begin{vmatrix} H_{\nu+1}(n) & H_\nu(n-1) & \cdots & H_{\nu+N-2}(n-1) \\ H_{\nu+1}(n+1) & H_\nu(n) & \cdots & H_{\nu+N-2}(n) \\ \vdots & \vdots & \vdots & \vdots \\ H_{\nu+1}(n+N-1) & H_\nu(n+N-2) & \cdots & H_{\nu+N-2}(n+N-2) \end{vmatrix}, \quad (3.33)$$

which is nothing but equation (3.31). ∎

Now consider the Plücker relation,

$$\begin{aligned} 0 = & |\mathbf{1}_{n+1}, \mathbf{1}, \cdots, \mathbf{N-1}, \mathbf{N}| \times |\mathbf{0}, \mathbf{1}, \cdots, \mathbf{N-1}, \varphi_2| \\ & - |\mathbf{0}, \mathbf{1}, \cdots, \mathbf{N-1}, \mathbf{N}| \times |\mathbf{1}_{n+1}, \mathbf{1}, \mathbf{1}, \cdots, \mathbf{N-1}, \varphi_2| \\ & - |\mathbf{1}, \cdots, \mathbf{N-1}, \mathbf{N}, \varphi_2| \times |\mathbf{1}_{n+1}, \mathbf{0}, \mathbf{1}, \cdots, \mathbf{N-1}|, \end{aligned} \quad (3.34)$$

where φ_2 is the column vector,

$$\varphi_2 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (3.35)$$

We obtain equation (3.24) by applying Lemma 1 to equation (3.34) after expanding with respect to the column φ_2. This completes the proof of bilinear equation (3.24).

A   Difference formulas and Plücker relations

In this appendix, we provide with data which are necessary for the proof of Proposition 7.
We first give the difference formulas.

Lemma 2 (Difference formula II).

$$|\mathbf{0}', \mathbf{1}', \cdots, \mathbf{N-1}'| = \phi_\nu^N(n), \tag{A.1}$$

$$|\mathbf{0}'_{\nu+1}, \mathbf{0}', \cdots, \mathbf{N-2}'| = \left(-a_0^2\right)^{-(N-2)} q^{-(2n+N-2)(N-1)} \phi_{\nu+1}^N(n), \tag{A.2}$$

$$|\mathbf{1}'_{\nu+1}, \mathbf{0}', \cdots, \mathbf{N-2}'| = \left(-a_0^2\right)^{-(N-2)} q^{-(2n+N-2)(N-1)} \phi_{\nu+1}^N(n), \tag{A.3}$$

where

$$\mathbf{k}'_\nu = \begin{pmatrix} H_\nu(n+k) \\ H_{\nu+1}(n+k) \\ \vdots \end{pmatrix}. \tag{A.4}$$

We next rewrite $\phi_\nu^N(n)$ in terms of Toeplitz determinant with respect to n by using the contiguity relations (3.13) and (3.14) as

$$\phi_\nu^N(n) = \left(-a_0^2 a_1^2\right)^{N(N-1)/2} q^{N(N-1)(3\nu+N-2)/3}$$

$$\times \begin{vmatrix} H_\nu(n) & H_\nu(n+1) & \cdots & H_\nu(n+N-1) \\ H_\nu(n-1) & H_\nu(n) & \cdots & H_\nu(n+N-2) \\ \vdots & \vdots & \ddots & \vdots \\ H_\nu(n-N+1) & H_\nu(n-N+2) & \cdots & H_\nu(n) \end{vmatrix}$$

$$\equiv \left(-a_0^2 a_1^2\right)^{N(N-1)/2} q^{N(N-1)(3\nu+N-2)/3} \psi_\nu^N(n). \tag{A.5}$$

Then we obtain the third difference formula:

Lemma 3 (Difference formula III).

$$|\overline{\mathbf{0}}, \overline{\mathbf{1}}, \cdots, \overline{\mathbf{N-1}}| = \psi_\nu^N(n), \tag{A.6}$$

$$|\widetilde{\overline{\mathbf{0}}}_{\nu+1}, \overline{\mathbf{0}}, \cdots, \overline{\mathbf{N-2}}| = \left(-a_0^2\right)^{-(N-2)} q^{-2(n-1)(N-1)} \psi_{\nu+1}^N(n), \tag{A.7}$$

$$|\widetilde{\overline{\mathbf{1}}}_{\nu+1}, \overline{\mathbf{0}}, \cdots, \overline{\mathbf{N-2}}| = \left(-a_0^2\right)^{-(N-2)} q^{-2(n-1)(N-1)} \psi_{\nu+1}^N(n), \tag{A.8}$$

where

$$\overline{\mathbf{k}}_\nu = \begin{pmatrix} H_\nu(n+k) \\ H_\nu(n+k-1) \\ H_\nu(n+k-2) \\ \vdots \end{pmatrix}, \qquad \widetilde{\overline{\mathbf{k}}}_\nu = \begin{pmatrix} H_\nu(n+k) \\ q^{-2} H_\nu(n+k-1) \\ q^{-4} H_\nu(n+k-2) \\ \vdots \end{pmatrix}. \tag{A.9}$$

Lemmas 2 and 3 are proved by applying the same procedure as that for Lemma 1 on the determinants,

$$\phi_\nu^N(n) = \begin{vmatrix} H_\nu(n) & H_\nu(n+1) & \cdots & H_\nu(n+N-1) \\ H_{\nu+1}(n) & H_{\nu+1}(n+1) & \cdots & H_{\nu+1}(n+N-1) \\ \vdots & \vdots & \ddots & \vdots \\ H_{\nu+N-1}(n) & H_{\nu+N-1}(n+1) & \cdots & H_{\nu+N-1}(n+N-1) \end{vmatrix}, \tag{A.10}$$

and

$$\psi_\nu^N(n) = \begin{vmatrix} H_\nu(n) & H_\nu(n+1) & \cdots & H_\nu(n+N-1) \\ H_\nu(n-1) & H_\nu(n) & \cdots & H_\nu(n+N-2) \\ \vdots & \vdots & \ddots & \vdots \\ H_\nu(n-N+1) & H_\nu(n-N+2) & \cdots & H_\nu(n) \end{vmatrix}, \tag{A.11}$$

respectively, by using the contiguity relation (3.13).

The bilinear equations (3.23)–(3.26) are derived by considering proper Plücker relations. We finally give the list the Plücker relations and difference formulas which are necessary for the derivations. This completes the proof of Proposition 7 and thus Theorem 4.

Equation (3.23):
Plücker relation

$$0 = |\widetilde{\overline{\mathbf{1}}}_{\nu+1}, \overline{\mathbf{1}}, \cdots, \overline{\mathbf{N-1}}, \overline{\mathbf{N}}| \times |\overline{\mathbf{0}}, \overline{\mathbf{1}}, \cdots, \overline{\mathbf{N-1}}, \varphi_2|$$
$$- |\overline{\mathbf{0}}, \overline{\mathbf{1}}, \cdots, \overline{\mathbf{N-1}}, \overline{\mathbf{N}}| \times |\widetilde{\overline{\mathbf{1}}}_{\nu+1}, \overline{\mathbf{1}}, \cdots, \overline{\mathbf{N-1}}, \varphi_2|$$
$$- |\overline{\mathbf{1}}, \cdots, \overline{\mathbf{N-1}}, \overline{\mathbf{N}}, \varphi_2| \times |\widetilde{\overline{\mathbf{1}}}_{\nu+1}, \overline{\mathbf{0}}, \overline{\mathbf{1}}, \cdots, \overline{\mathbf{N-1}}|.$$

Difference formula Lemma 3.

Equation (3.25):
Plücker relation

$$0 = |\mathbf{1}'_{\nu+1}, \mathbf{1}', \cdots, \mathbf{N-1}', \mathbf{N}'| \times |\mathbf{0}', \mathbf{1}', \cdots, \mathbf{N-1}', \varphi_1|$$
$$- |\mathbf{0}', \mathbf{1}', \cdots, \mathbf{N-1}', \mathbf{N}'| \times |\mathbf{1}'_{\nu+1}, \mathbf{1}', \mathbf{1}', \cdots, \mathbf{N-1}', \varphi_1|$$
$$- |\mathbf{1}', \cdots, \mathbf{N-1}', \mathbf{N}', \varphi_1| \times |\mathbf{1}'_{\nu+1}, \mathbf{0}', \mathbf{1}', \cdots, \mathbf{N-1}'|.$$

Difference formula Lemma 2.

Equation (3.26):
Plücker relation

$$0 = |\mathbf{1}_{n+1}, \mathbf{1}, \cdots, \mathbf{N-1}, \mathbf{N}| \times |\mathbf{0}, \mathbf{1}, \cdots, \mathbf{N-1}, \varphi_1|$$
$$- |\mathbf{0}, \mathbf{1}, \cdots, \mathbf{N-1}, \mathbf{N}| \times |\mathbf{1}_{n+1}, \mathbf{1}, \mathbf{1}, \cdots, \mathbf{N-1}, \varphi_1|$$
$$- |\mathbf{1}, \cdots, \mathbf{N-1}, \mathbf{N}, \varphi_1| \times |\mathbf{1}_{n+1}, \mathbf{0}, \mathbf{1}, \cdots, \mathbf{N-1}|.$$

Difference formula Lemma 1.

In the above data, φ_1 and φ_2 are column vectors given by

$$\varphi_1 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \qquad \varphi_2 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{A.12}$$

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