Journal of Nonlinear Mathematical Physics Volume 11, Number 1 (2004), 47–74 Article

On Conditionally Invariant Solutions of Magnetohydrodynamic Equations. Multiple Waves.

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Received April 19, 2003; Accepted June 10, 2003

Abstract

We present a version of the conditional symmetry method in order to obtain multiple wave solutions expressed in terms of Riemann invariants. We construct an abelian distribution of vector fields which are symmetries of the original system of PDEs subjected to certain first order differential constraints. The usefulness of our approach is demonstrated on simple and double wave solutions of MHD equations. The paper also contains a comparison of the conditional symmetry method with the generalized method of characteristics.

1 Introduction

The objective of this paper is a development of the version of the conditional symmetry method (CSM) (proposed in [1],[2]) for the purpose of constructing simple Riemann wave solutions and their superpositions (*i.e.* multiples waves) admitted by the equations of magnetohydrodynamics (MHD) in $(3+1)$ dimensions. The flow under consideration is assumed to be ideal, nonstationary and isentropic for a compressible conductive fluid placed in magnetic field $\vec{H}$. The electrical conductivity of the fluid is assumed to be infinitely large (*i.e.* the fluid is an ideal conductor $\sigma \mapsto \infty$). We restrict our analysis to the case in which the dissipative effects, like viscosity and thermal conductivity, are negligible and no external forces are considered. Under the above assumptions the MHD model is governed by the system of equations

$$\frac{\partial \rho}{\partial t} + (\vec{\mathrm{v}} \cdot \nabla)\rho + (\nabla \cdot \vec{\mathrm{v}})\rho = 0\,, \tag{1.1}$$

$$\frac{\partial \vec{\mathrm{v}}}{\partial t} + (\vec{\mathrm{v}} \cdot \nabla)\vec{\mathrm{v}} + \frac{1}{\rho}\nabla p + \frac{1}{\rho}\vec{H} \times (\nabla \times \vec{H}) = 0\,, \tag{1.2}$$

$$\frac{\partial p}{\partial t} + (\vec{\mathrm{v}} \cdot \nabla)p + \kappa(\nabla \cdot \vec{\mathrm{v}})p = 0\,, \tag{1.3}$$

$$\frac{\partial \vec{H}}{\partial t} - \nabla \times (\vec{\mathrm{v}} \times \vec{H}) = 0\,, \tag{1.4}$$

$$\nabla \cdot \vec{H} = 0\,, \tag{1.5}$$

where we use the following notation: ρ and p represent the density and the pressure of the fluid, respectively; $\vec{\mathrm{v}} = (u, v, w)$ and $\vec{H} = \big(H_1, H_2, H_3\big)$ represent the velocity of the fluid and the magnetic field, respectively and κ is an adiabatic exponent. Without loss of generality in this model we can set the coefficient of the magnetic permeability to unity $\mu_e = 1$. The independent variables are denoted by $(x^\mu) = (t, x, y, z) \in E \subset \mathbb{R}^4$, $\mu = 0, 1, 2, 3$. The system of MHD equations (1.1)-(1.4) is of evolutionary form. It is well known [3] that, if the magnetic field $\vec{H}$ obeys equation (1.5) at time $t = 0$, then, by virtue of the MHD equations (1.1)-(1.4), it will retain this property for all $t > 0$. The system of MHD equations (1.1)-(1.5) is composed of nine equations involving eight dependent variables $\mathrm{u} = \big(\rho, p, \vec{\mathrm{v}}, \vec{H}\big) \in U \subset \mathbb{R}^8$. In $(3+1)$ dimensions the system of equations (1.1)-(1.4) can be written in the matrix equivalent evolutionary form

$$\mathrm{u}_t + \sum_{i=1}^{3} A^i(\mathrm{u})\, \mathrm{u}_{x^i} = 0\,, \tag{1.6}$$

where 8 by 8 matrix functions A^1, A^2 and A^3 take the form

$$A^1 = \begin{pmatrix} u & 0 & \rho & 0 & 0 & 0 & 0 & 0 \\ 0 & u & \kappa p & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/\rho & u & 0 & 0 & 0 & H_2/\rho & H_3/\rho \\ 0 & 0 & 0 & u & 0 & 0 & -H_1/\rho & 0 \\ 0 & 0 & 0 & 0 & u & 0 & 0 & H_1/\rho \\ 0 & 0 & 0 & 0 & 0 & u & 0 & 0 \\ 0 & 0 & H_2 & -H_1 & 0 & 0 & u & 0 \\ 0 & 0 & H_3 & 0 & -H_1 & 0 & 0 & u \end{pmatrix}, \tag{1.7}$$

$$A^2 = \begin{pmatrix} v & 0 & \rho & 0 & 0 & 0 & 0 & 0 \\ 0 & v & \kappa p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & v & 0 & 0 & -H_2/\rho & 0 & 0 \\ 0 & 1/\rho & 0 & v & 0 & H_1/\rho & 0 & H_3/\rho \\ 0 & 0 & 0 & 0 & v & 0 & 0 & -H_2/\rho \\ 0 & 0 & -H_2 & H_1 & 0 & v & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & v & 0 \\ 0 & 0 & 0 & H_3 & -H_2 & 0 & 0 & v \end{pmatrix}, \tag{1.8}$$

$$A^3 = \begin{pmatrix} w & 0 & 0 & 0 & \rho & 0 & 0 & 0 \\ 0 & w & 0 & 0 & \kappa p & 0 & 0 & 0 \\ 0 & 0 & w & 0 & 0 & -H_3/\rho & 0 & 0 \\ 0 & 0 & 0 & w & 0 & 0 & -H_3/\rho & 0 \\ 0 & 1/\rho & 0 & 0 & w & H_1/\rho & H_2/\rho & 0 \\ 0 & 0 & -H_3 & 0 & H_1 & w & 0 & 0 \\ 0 & 0 & 0 & -H_3 & H_2 & 0 & w & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & w \end{pmatrix} . \tag{1.9}$$

The MHD system (1.1)-(1.5) is an hyperbolic one and it is invariant under the Galilean-similitude Lie algebra [4].

In this paper we seek for solutions describing the propagation and nonlinear superpositions of waves which can be realized in the MHD system (1.1)-(1.5). A wave vector of system (1.6) is a nonzero vector function

$$\lambda\,(\mathrm{u}) = \big(\,\lambda_o\,(\mathrm{u}),\,\ldots\,,\lambda_p\,(\mathrm{u})\,\big)\,, \tag{1.10}$$

such that

$$\ker\big(\lambda_o\,(\mathrm{u})\mathbf{I} + \lambda_i\,(\mathrm{u})A^i\big) \neq 0 \tag{1.11}$$

holds. This means that there exists an eight-component vector $\gamma = \big(\gamma^1,\,\ldots\,,\gamma^8\big)$ for which the condition

$$\big(\lambda_o\,(\mathrm{u})\mathbf{I} + \lambda_i\,(\mathrm{u})A^i\big)\,\gamma = 0 \tag{1.12}$$

is satisfied. Note that throughout this paper we use the summation convention over the repeated lower and upper indices, except in cases when the index is taken in brackets. The necessary and sufficient condition for the existence of a nonzero solution γ of equations (1.12) is

$$\operatorname{rank}\,\big(\lambda_o\,(\mathrm{u})\mathbf{I} + \lambda_i\,(\mathrm{u})A^i\big) < 8\,. \tag{1.13}$$

The relations (1.12) and (1.13) are called the wave relation and the dispersion relation, respectively [5]. The wave vector λ can be written in the form $\lambda = \big(\lambda_o, \vec{\lambda}\big)$, where $\vec{\lambda} = \big(\lambda_1, \lambda_2, \lambda_3\big)$ denotes the direction of wave propagation and λ_o is the phase velocity of the considered wave. The dispersion relation (1.13) for MHD equations (1.1)-(1.5) takes the form

$$\delta^2|\vec{\lambda}|^2\left(\delta^2|\vec{\lambda}|^2 - \frac{(\vec{H}\cdot\vec{\lambda})^2}{\rho}\right)\left[\delta^4|\vec{\lambda}|^4 - \delta^2|\vec{\lambda}|^2\left(\frac{|\vec{H}|^2}{\rho} + a^2\right) + a^2\,\frac{(\vec{H}\cdot\vec{\lambda})^2}{\rho}\right] = 0\,, \tag{1.14}$$

where $\delta|\vec{\lambda}| = \lambda_o + \vec{\mathrm{v}}\cdot\vec{\lambda}$ denotes the wave velocity with respect to a moving fluid and $a = \big(\kappa p/\rho\big)^{\frac{1}{2}}$ is the velocity of sound. Solving the dispersion equation (1.14) with respect to $\delta|\vec{\lambda}|$ we obtain the following eigenfunctions

$$\delta_E\,|\vec{\lambda}| = 0\,, \tag{1.15}$$

$$\delta_A\,|\vec{\lambda}| = \varepsilon\frac{(\vec{H}\cdot\vec{\lambda})}{\sqrt{\rho}}\,, \tag{1.16}$$

$$\delta_S\,|\vec{\lambda}| = \frac{\varepsilon}{2}\left[\left[\left(a\vec{\lambda}+\frac{\vec{H}}{\sqrt{\rho}}\right)^2\right]^{\frac{1}{2}} - \left[\left(a\vec{\lambda}-\frac{\vec{H}}{\sqrt{\rho}}\right)^2\right]^{\frac{1}{2}}\right], \tag{1.17}$$

$$\delta_F\,|\vec{\lambda}| = \frac{\varepsilon}{2}\left[\left[\left(a\vec{\lambda}+\frac{\vec{H}}{\sqrt{\rho}}\right)^2\right]^{\frac{1}{2}} + \left[\left(a\vec{\lambda}-\frac{\vec{H}}{\sqrt{\rho}}\right)^2\right]^{\frac{1}{2}}\right], \tag{1.18}$$

where $\varepsilon = \pm 1$ implies that the wave propagates in the right or in the left direction with respect to the medium. The eigenvalues (1.15)-(1.18) correspond to entropic waves δ_E, Alfvén waves δ_A, and magnetoacoustic slow δ_S and fast δ_F waves, respectively. The eigenvectors γ and λ, corresponding to the four types of eigenvalues (1.15)-(1.18) admitted by the MHD system of equations (1.1)-(1.5), have the following form [5], [6]

(1) for the entropic waves, we have three types of eigenvectors

$$E_1\ :\quad \gamma_{E_1} = \left(\gamma_\rho, -(\vec{H}\cdot\vec{\lambda}), \vec{\gamma}, \vec{h}\right), \qquad \lambda^{E_1} = \left(-\vec{\mathrm{v}}\cdot\vec{\lambda}, \vec{\lambda}\right), \tag{1.19}$$

$$E_2\ :\quad \gamma_{E_2} = \left(\gamma_\rho, -(\vec{H}\cdot\vec{h}), \vec{\gamma}, \vec{h}\right), \quad \lambda^i = \left(-\vec{\mathrm{v}}\cdot(\vec{\alpha}^{\,i}\times\vec{H}),\ \vec{\alpha}^{\,i}\times\vec{H}\right), \quad i=1,2 \tag{1.20}$$

where $\vec{\alpha}^{\,i}$ are two linearly independent vectors,

$$E_3\ :\quad \gamma_{E_3} = \left(\gamma_\rho, 0, \vec{0}, \vec{0}\right), \qquad \lambda^i = \left(-\vec{\mathrm{v}}\cdot\vec{e}_i, \vec{e}_i\right), \quad i=1,2,3 \tag{1.21}$$

where $\vec{e}_i$ are three linearly independent unit vectors,

(2) for the Alfvén waves

$$A^\varepsilon\ :\quad \gamma_A = \left(0, 0, \frac{\varepsilon\vec{h}}{\sqrt{\rho}}, \vec{h}\right), \qquad \lambda^A = \left(\varepsilon\frac{(\vec{H}\cdot\vec{\lambda})}{\sqrt{\rho}} - \vec{\mathrm{v}}\cdot\vec{\lambda}, \vec{\lambda}\right), \quad \varepsilon = \pm 1\,, \tag{1.22}$$

(3) for the magnetoacoustic waves S and F

$$\gamma_S = \left(\rho\delta_S^2|\vec{\lambda}|^2 - (\vec{H}\cdot\vec{\lambda})^2, \kappa p\left[\delta_S^2|\vec{\lambda}|^2 - \frac{1}{\rho}(\vec{H}\cdot\vec{\lambda})^2\right], -\varepsilon\delta_S|\vec{\lambda}|\left[\delta_S^2\vec{\lambda} - (\vec{H}\cdot\vec{\lambda})\frac{\vec{H}}{\rho}\right], \delta_S^2\left[|\vec{\lambda}|^2\vec{H} - (\vec{H}\cdot\vec{\lambda})\vec{\lambda}\right]\right),$$

$$\gamma_F = \left(\rho\delta_F^2|\vec{\lambda}|^2 - (\vec{H}\cdot\vec{\lambda})^2, \kappa p\left[\delta_F^2|\vec{\lambda}|^2 - \frac{1}{\rho}(\vec{H}\cdot\vec{\lambda})^2\right], -\varepsilon\delta_F|\vec{\lambda}|\left[\delta_F^2\vec{\lambda} - (\vec{H}\cdot\vec{\lambda})\frac{\vec{H}}{\rho}\right], \delta_F^2\left[|\vec{\lambda}|^2\vec{H} - (\vec{H}\cdot\vec{\lambda})\vec{\lambda}\right]\right),$$

$$\lambda^S = \left(\delta_S|\vec{\lambda}| - \vec{\mathrm{v}}\cdot\vec{\lambda}\right), \qquad \lambda^F = \left(\delta_F|\vec{\lambda}| - \vec{\mathrm{v}}\cdot\vec{\lambda}\right), \qquad \varepsilon = \pm 1, \tag{1.23}$$

where we have used the following notation for eigenvectors $\boldsymbol{\gamma} = \left(\gamma_\rho, \gamma_p, \vec{\gamma}, \vec{h}\right)$. They are related to unknown functions $\mathrm{u} = \left(\rho, p, \vec{\mathrm{v}}, \vec{H}\right) \in U$. Similarly the wave vector λ is related to the independent variables $\mathrm{x} = (t, x, y, z) = (t, \vec{\mathrm{x}}) \in E$.

The functions

$$r^s(\mathrm{x},\mathrm{u}) = \lambda^s_o(\mathrm{u})t + \lambda^s_i(\mathrm{u})x^i\,, \qquad s = 1,\,\ldots\,,8 \tag{1.24}$$

are called the Riemann invariants of the wave vectors λ^s (see *e.g.* [3],[7]). So according to equation (1.24), the MHD model (1.1)-(1.5) admits the following Riemann invariants associated with wave vectors λ^s

$$\begin{aligned}
r_{E_1}(\mathrm{x},\mathrm{u}) &= \vec{\lambda}(\mathrm{u})\cdot\vec{\mathrm{x}} - (\vec{\lambda}(\mathrm{u})\cdot\vec{\mathrm{v}})t\,, && (1.25)\\
r_{E_2}(\mathrm{x},\mathrm{u}) &= x + y - (u+v)t\,, && (1.26)\\
r_{E_3}(\mathrm{x},\mathrm{u}) &= x + y + z - (u+v+w)t\,, && (1.27)\\
r_A(\mathrm{x},\mathrm{u}) &= \vec{\lambda}(\mathrm{u})\cdot\vec{\mathrm{x}} + \left(\varepsilon\frac{(\vec{H}\cdot\vec{\lambda}(\mathrm{u}))}{\sqrt{\rho}} - \vec{\mathrm{v}}\cdot\vec{\lambda}(\mathrm{u})\right)t\,, \qquad \varepsilon = \pm 1\,, && (1.28)\\
r_S(\mathrm{x},\mathrm{u}) &= \vec{\lambda}(\mathrm{u})\cdot\vec{\mathrm{x}} + \Big(\delta_S|\vec{\lambda}(\mathrm{u})| - \vec{\mathrm{v}}\cdot\vec{\lambda}(\mathrm{u})\Big)t\,, && (1.29)\\
r_F(\mathrm{x},\mathrm{u}) &= \vec{\lambda}(\mathrm{u})\cdot\vec{\mathrm{x}} + \Big(\delta_F|\vec{\lambda}(\mathrm{u})| - \vec{\mathrm{v}}\cdot\vec{\lambda}(\mathrm{u})\Big)t\,. && (1.30)
\end{aligned}$$

For any function, $f:\mathbb{R}\to\mathbb{R}^8$, the equation

$$\mathrm{u} = f\big(r(\mathrm{x},\mathrm{u})\big) \tag{1.31}$$

defines a unique function $\mathrm{u}(\mathrm{x})$ on a neighborhood of $\mathrm{x} = 0$ and the Jacobian matrix is

$$\frac{\partial u^\alpha}{\partial x^\mu} = \phi(\mathrm{x})^{-1}\,\lambda^s_\mu\big(\mathrm{u}(\mathrm{x})\big)\,\gamma^\alpha_{(s)}\,, \qquad \mu = 0,1,2,3\,;\ \alpha, s = 1,\,\ldots\,,8 \tag{1.32}$$

where we have introduced the notation

$$\phi(\mathrm{x}) = 1 - \frac{\partial r^s}{\partial u^\alpha}\big(\mathrm{x},\mathrm{u}(\mathrm{x})\big)\,\gamma^\alpha_{(s)}\Big(r\,\big(\mathrm{x},\mathrm{u}(\mathrm{x})\big)\Big)\,, \tag{1.33}$$

and

$$\gamma^\alpha_s = \frac{df^{\,\alpha}}{dr^s}\,. \tag{1.34}$$

Note that the Jacobian matrix (1.32) has rank at most equal to one. The rank-one solutions of the hyperbolic system (1.6) are called simple waves and always exist [3]. This type of solution was introduced by S.D. Poisson [8] at the beginning of the 19th century in connection with the equations describing a compressible isothermal gas flow.

In this paper we construct several classes of exact solutions in the form (1.31) for the MHD equations (1.1)-(1.5). In particular we focus on constructing simple wave solutions, scattering and nonscattering double wave solutions (rank-2 solutions) obtained by the conditional symmetry method described below.

This paper is organized as follows. **Section 2** presents an adapted version of the CSM. We use it to obtain simple waves of MHD system of equations. **Section 3** contains a detailed description of how to construct certain classes of double wave solutions admitted by the MHD equations. **Section 4** summarizes the obtained results and contains a comparison of these results with the generalized method of characteristics (GMC).

2 Conditional symmetries and simple wave solutions of MHD equations.

The methodological approach adopted here is a new variant of the CSM presented in [1] and [2]. The notion of conditional symmetries evolved in the process of extending the classical Lie theory of symmetries to partial differential equations (PDEs). This approach consists basically in modifying the original system by adding to it certain differential constraints of the first order for which a symmetry criterion is identically satisfied [9]. The overdetermined system of equations obtained in this way admits, in some cases, a larger class of Lie point symmetries and, consequently, it can provide new classes of solutions of the original system. This basic idea was developed and implemented recently by many authors, among others G. Bluman and J.D. Cole [10], P. Olver and Ph. Rosenau [11], W. Fushchych [12], P. Clarkson and P. Winternitz [13]. For a comprehensive review of this subject see the chapter by P. Olver and E.M. Vorobev in [14], (Vol. 3, Chapter 11).

The method proposed in this work is distinguished by the specific choice of multiple differential constraints (DCs) of the first order, compatible with the initial system (1.6) for which the invariance criterion is identically satisfied. Consider a set of vectors

$$\xi_a\,(\mathrm{u}) = \Big(\xi_a^1\,(\mathrm{u}), \dots, \xi_a^p\,(\mathrm{u})\Big)^T, \qquad a = 1, \dots, p-1 \tag{2.1}$$

which are orthogonal to a given wave vector λ

$$\lambda_\mu \cdot \xi_a^\mu = 0\,, \qquad \mu = 1, \dots, p\,, \quad a = 1, \dots, p-1\,. \tag{2.2}$$

Note that ξ_a is not uniquely defined and the set $\{\lambda, \xi_1, \dots, \xi_{p-1}\}$ forms a basis in the space of independent variables $E \subset \mathbb{R}^p$. Multiplying equation (1.32) by the vectors ξ_a and using the orthogonality property (2.2) we obtain for any $\alpha = 1, \dots, q$ and $a = 1, \dots, p-1$

$$\xi_a^\mu\big(\mathrm{u}(\mathrm{x})\big)\frac{\partial u^\alpha}{\partial x^\mu} = 0\,. \tag{2.3}$$

This means that the functions $\mathrm{u} = \big(u^1(\mathrm{x}), \dots, u^q(\mathrm{x})\big) \in U \subset \mathbb{R}^q$ are invariants of the vector fields $\xi_a^\mu\,\big(\mathrm{u}(\mathrm{x})\big)\partial/\partial x^\mu$ acting in E space. Hence the graph of the solution $\Gamma = \big\{\big(\mathrm{x}, \mathrm{u}(\mathrm{x})\big)\big\}$ is invariant under the vector fields

$$X_a = \xi_a^\mu\,(\mathrm{u})\frac{\partial}{\partial x^\mu}\,, \qquad a = 1, \dots, p-1 \tag{2.4}$$

in the space of independent and dependent variables $E \times U \subset \mathbb{R}^p \times \mathbb{R}^q$. Note that the vector fields X_a of the form (2.4) commute, *i.e.* they form an abelian distribution.

Now, if the function $\mathrm{u}\,(\mathrm{x})$ is a solution of equations (1.31), then $\mathrm{u}\,(\mathrm{x})$ is a solution of (1.6) if and only if the equation

$$\big[\lambda_o\,(f)\mathbf{I} + \lambda_i\,(f)\,A^i\,(f)\big]\frac{df^\alpha}{dr} = 0 \tag{2.5}$$

holds. This means that df/dr has to be an element of $\ker\,(\lambda_o\,(f)\mathbf{I}+\lambda_i\,(f)A^i\,(f))$. Equations (2.5) form an underdetermined system of first order ordinary differential equations (ODEs) for f. Note that the differential constraints depend on the dimension of $\ker(\lambda_o(f)\mathbf{I} + \lambda_i(f)A^i(f))$. For example, if

$$\lambda_o(f)\mathbf{I} + \lambda_i(f)A^i(f) = 0\,, \tag{2.6}$$

then there is no differential constraint on the function f at all.

Thus, putting it all together, we can conclude that a function $\mathrm{u}(\mathrm{x})$ defined on a neighborhood of $\mathrm{x} = 0$ satisfies an equation of the form (1.31)

$$\mathrm{u} = f\left(\lambda_o(\mathrm{u})t + \vec{\lambda}\cdot\vec{\mathrm{x}}\right) \tag{2.7}$$

for some $f : \mathbb{R} \to \mathbb{R}^8$ if and only if the graph $\Gamma = \left\{\left(\mathrm{x}, \mathrm{u}(\mathrm{x})\right)\right\}$ is invariant under the vector fields X_a of the form (2.4). Such a function $\mathrm{u}(\mathrm{x})$ is a solution of the system (1.6) if and only if the ODEs (2.5) are satisfied. Note also that, due to the homogeneity of equations (2.5), the rescaling of the wave vector λ produces the same solution of the form (1.31).

Now we choose an appropriate system of coordinates on $E \times U$ space in order to rectify the vector fields (2.4) and find the invariance conditions which guarantee the existence of a rank-one solution (1.31) of equation (1.6). We assume that one of the components of the wave vector λ is different from zero, say $\lambda_1 \neq 0$. Then the independent vector fields

$$X_2 = \frac{\partial}{\partial x^2} - \frac{\lambda_2}{\lambda_1}\frac{\partial}{\partial x^1}\,, \ldots, X_p = \frac{\partial}{\partial x^p} - \frac{\lambda_p}{\lambda_1}\frac{\partial}{\partial x^1} \tag{2.8}$$

have the form (2.4) with the orthogonality property (2.2). If we change the independent and dependent variables as follows

$$\overline{x}^1 = r(\mathrm{x}, \mathrm{u})\,,\ \ \overline{x}^2 = x^2, \ldots, \overline{x}^p = x^p\,, \quad \overline{u}^1 = u^1,\ \overline{u}^2 = u^2, \ldots, \overline{u}^q = u^q\,, \tag{2.9}$$

then the vector fields (2.8) take the rectified form

$$X_2 = \frac{\partial}{\partial \overline{x}^2}\,, \ldots, X_p = \frac{\partial}{\partial \overline{x}^p}\,, \tag{2.10}$$

and the corresponding invariant conditions are

$$\overline{\mathrm{u}}_{\overline{x}^2} = 0\,, \ldots, \overline{\mathrm{u}}_{\overline{x}^p} = 0\,. \tag{2.11}$$

So, in this new coordinate system on $\mathbb{R}^p \times \mathbb{R}^q$, we subject the original system (1.6) to the invariance conditions (2.11) and produce an overdetermined quasilinear system of the form

$$\begin{cases} \left[\lambda_o(\overline{\mathrm{u}})\mathbf{I} + \lambda_i(\overline{\mathrm{u}})\,A^i(\overline{\mathrm{u}})\right]\overline{\mathrm{u}}_{\overline{x}^1} = 0\,, \\[2ex] \qquad \overline{\mathrm{u}}_{\overline{x}^2} = 0\,, \ldots, \overline{\mathrm{u}}_{\overline{x}^p} = 0\,. \end{cases} \tag{2.12}$$

with the general solution

$$\overline{\mathrm{u}}(\overline{\mathrm{x}}) = f(\overline{x}^1)\,, \tag{2.13}$$

where the function $f : \mathbb{R} \to \mathbb{R}^q$ has to satisfy the ODEs (2.5).

Suppose now that the integral curve Γ of the vector field $\gamma^\alpha(\mathrm{u})\partial/\partial u^\alpha$ on U satisfies the ODEs (1.34). Suppose also that the wave vector $\lambda(\mathrm{u})$ is pulled back to the curve Γ, *i.e.* $f^*(\lambda)$. Then the functions $\lambda_\mu(\mathrm{u})$ become functions of the parameter r defined on the curve Γ. We denote $f^*(\lambda_\mu)$ by $\lambda_\mu(r)$. The set (1.24) and (1.31) of implicitly defined relations between the variables u^α, x^μ and r can be written as

$$u^\alpha = f^\alpha(r), \qquad r = \lambda_\mu(r)x^\mu. \tag{2.14}$$

Expression (2.14) constitutes a rank-one solution (a Riemann wave) of system (1.6) and the scalar function $r(\mathrm{x})$ is the Riemann invariant associated with the wave vector $\lambda(r)$.

Now we present rank-one solutions of the MHD equations (1.1)-(1.5) which illustrate these theoretical considerations. Equations (1.12), (1.13) and (2.2) determine the characteristic directions λ^s, γ_s and ξ_a, each of which are of four types, and lead to different physical properties. We now discuss the rank-one solutions of MHD equations obtained from the reduced equations (2.5). The different symmetries associated with the vector fields X_a of the form (2.4) lead us to four different types of solutions. The results can be summarized as follows.

2.1 Simple entropic waves.

We consider three types of entropic waves generated by the wave vector $\vec{\lambda}$. We denote by E_1, E_2 and E_3 the entropic waves related to wave vectors which are of one, two or three dimensions, respectively.

1). For the entropic wave of type E_1 the vector fields X_a satisfying condition (2.2) are given by

$$X_1 = \frac{\lambda_1}{(\vec{\mathrm{v}}\cdot\vec{\lambda})}\frac{\partial}{\partial t} + \frac{\partial}{\partial x}, \quad X_2 = \frac{\lambda_2}{(\vec{\mathrm{v}}\cdot\vec{\lambda})}\frac{\partial}{\partial t} + \frac{\partial}{\partial y}, \quad X_3 = \frac{\lambda_3}{(\vec{\mathrm{v}}\cdot\vec{\lambda})}\frac{\partial}{\partial t} + \frac{\partial}{\partial z}. \tag{2.15}$$

The solutions invariant under $\{X_1, X_2, X_3\}$ have the form

$$\rho = \rho(r), \quad p(r) + \frac{|\vec{H}|^2}{2} = p_o, \quad \vec{\mathrm{v}} = \vec{\mathrm{v}}(r), \quad \vec{H} = \vec{H}(r), \tag{2.16}$$

where the Riemann invariant is

$$r = \vec{\lambda}(r)\cdot\vec{\mathrm{x}} - \big(\vec{\lambda}(r)\cdot\vec{\mathrm{v}}\big)t, \tag{2.17}$$

p_o is an arbitrary constant, the vectors $\vec{\mathrm{v}}$ and $\vec{H}$ have to satisfy

$$\left(\frac{d\vec{H}}{dr}\times\frac{d\vec{\mathrm{v}}}{dr}\right)\cdot\vec{H} = 0, \tag{2.18}$$

and the wave vector $\vec{\lambda}$ is given by

$$\vec{\lambda} = \frac{\vec{H}}{|\vec{H}|^2}\times\frac{d\vec{H}}{dr}. \tag{2.19}$$

The relations (2.18) and (2.19) imply that the entropic wave solution (2.16) is constant on the planes perpendicular to $\vec{\lambda}$. Thus the entropic wave E_1 is a plane wave which propagates in an incompressible fluid. The quantities ρ, p, $\vec{v}$ and $\vec{H}$ are conserved along the flow. The Lorentz force $\vec{F}_m$ associated with the entropic wave E_1 is given by

$$\vec{F}_m = -\frac{1}{2}\nabla\left[|\vec{H}|^2\right], \tag{2.20}$$

and is cancelled out by the hydrodynamic pressure, ∇p, so that the fluid is force-free. Consequently, by virtue of Kelvin's theorem, the circulation

$$\Gamma_c = \oint_C \vec{v}\cdot d\vec{\ell} \tag{2.21}$$

around a fluid element is preserved [15]. Note that $\vec{F}_m \cdot \vec{v} \neq 0$, which indicates the existence of coupling between hydrodynamic and magnetic effects.

2). The entropic wave E_2 is characterized by (1.20) and the corresponding vector fields X_a satisfying condition (2.2) are

$$X_1 = \frac{1}{v}\frac{\partial}{\partial t} + \frac{u}{v}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \qquad X_2 = \frac{\partial}{\partial z}. \tag{2.22}$$

The solutions invariant under $\{X_1, X_2\}$ have the form

$$\rho = \rho(r), \quad p(r) + \frac{1}{2}|\vec{H}|^2 = p_o, \quad \vec{v} = u(r)\vec{e}_1 + v(r)\vec{e}_2 + w(r)\vec{e}_3, \quad \vec{H} = H(r)\vec{e}_3, \tag{2.23}$$

with the constraint

$$u(r) + v(r) = U_o, \tag{2.24}$$

where p_o and U_o are arbitrary constants, and ρ, p u, v, w and $\vec{H}$ are arbitrary functions of the Riemann invariant

$$r = x + y - U_o t. \tag{2.25}$$

The Lorentz force related to the entropic wave E_2 is given by

$$\vec{F}_m = -\frac{1}{2}\nabla\left[|\vec{H}|^2\right] = \nabla p, \qquad \vec{F}_m \cdot \vec{v} = 0, \tag{2.26}$$

and therefore there is no coupling between hydrodynamic and magnetic effects. The entropic wave E_2 propagates in an incompressible fluid without vorticity.

3). An entropic wave E_3 is invariant under the vector field

$$X = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}. \tag{2.27}$$

Solving the MHD system (1.1)-(1.5) we obtain

$$\rho = \rho(r), \quad p(r) + \frac{|\vec{H}|^2}{2} = p_o, \quad \vec{v} = \vec{v}(r), \quad \vec{H} = \vec{H}(r), \tag{2.28}$$

where the following constraints

$$\begin{aligned} u(r) + v(r) + w(r) &= C_o, \quad H_1(r) + H_2(r) + H_3(r) = \mathcal{H}_o, \\ \mathcal{H}_o\frac{d\vec{v}}{dr} &= 0, \quad \nabla\left[p + \frac{1}{2}|\vec{H}|^2\right] = (\vec{H}\cdot\nabla)\vec{H} \end{aligned} \tag{2.29}$$

hold. Here C_o and $\mathcal{H}_o$ are arbitrary constants. By taking into consideration the eigenvector $\gamma_{E_3} = \left(\gamma_\rho, 0, \vec{0}, \vec{0}\right)$, we obtain the solutions

$$\rho = \rho\,(r)\,, \quad p = p_o\,, \quad \vec{v} = \vec{v}_o\,, \quad \vec{H} = \vec{H}_o\,, \tag{2.30}$$

where p_o is an arbitrary constant, $\vec{v}_o$ and $\vec{H}_o$ are constant vectors and ρ is an arbitrary function of the Riemann invariant

$$r = x + y + z - C_o t\,. \tag{2.31}$$

It is well known [15] that, after an appropriate Galilean transformation, the solution given by (2.30) and (2.31) becomes stationary.

2.2 Simple Alfvén waves.

The vector fields X_a associated with the Alfvén wave A^ε are

$$X_1 = -\frac{\lambda_1}{\lambda_o}\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\,, \quad X_2 = -\frac{\lambda_2}{\lambda_o}\frac{\partial}{\partial t} + \frac{\partial}{\partial y}\,, \quad X_3 = -\frac{\lambda_3}{\lambda_o}\frac{\partial}{\partial t} + \frac{\partial}{\partial z}\,, \tag{2.32}$$

where

$$\lambda_o = \varepsilon\frac{(\vec{H}\cdot\vec{\lambda})}{\sqrt{\rho}} - \vec{v}\cdot\vec{\lambda}\,, \qquad \varepsilon = \pm 1\,. \tag{2.33}$$

The solutions invariant under $\{X_1, X_2, X_3\}$ are given by

$$\rho = \rho_o\,, \qquad p = p_o\,, \qquad \vec{v} = \frac{\varepsilon\vec{H}}{\sqrt{\rho_o}} + \vec{v}_o\,, \qquad |\vec{H}|^2 = \mathcal{H}_o^2\,, \qquad \varepsilon = \pm 1\,, \tag{2.34}$$

where ρ_o, p_o, $\vec{v}_o$ and $\mathcal{H}_o$ are arbitrary constants and $\vec{H}$ is an arbitrary vector function of the Riemann invariant

$$r = \vec{\lambda}\,(r)\cdot\vec{x} + \left(\vec{\lambda}\,(r)\cdot\vec{v}_o\right)t\,. \tag{2.35}$$

The vectors $\vec{v}$ and $\vec{H}$ satisfy the constraints

$$\frac{d\vec{v}}{dr}\cdot\vec{\lambda} = 0\,, \qquad \frac{d\vec{H}}{dr}\cdot\vec{\lambda} = 0\,. \tag{2.36}$$

The Lorentz force $\vec{F}_m$ associated with the Alfvén A^ε wave is

$$\vec{F}_m = (\vec{H}\cdot\vec{\lambda})\frac{d\vec{H}}{dr}\,. \tag{2.37}$$

The solution (2.34), along with (2.37), describes the well known properties of the Alfvén waves [16], namely, they have no group velocity (the solution (2.34) is stationary, after an appropriate Galilean transformation) and propagate in an incompressible fluid. The Lorentz force $\vec{F}_m$ acts on the fluid transversally to the direction of the wave vector, $\vec{\lambda}$, so that the magnetic field lines twist relatively to one another, but do not compress. Finally the fact that $\vec{F}_m\cdot\vec{v} = 0$ indicates that there is no coupling between hydrodynamic and magnetic effects.

2.3 Slow and fast simple magnetoacoustic waves.

We consider now the slow S and fast F magnetoacoustic waves which are invariant under the vectors fields

$$X_1 = -\frac{\lambda_1}{\lambda_o}\frac{\partial}{\partial t} + \frac{\partial}{\partial x}, \quad X_2 = -\frac{\lambda_2}{\lambda_o}\frac{\partial}{\partial t} + \frac{\partial}{\partial y}, \quad X_3 = -\frac{\lambda_3}{\lambda_o}\frac{\partial}{\partial t} + \frac{\partial}{\partial z}, \tag{2.38}$$

where

$$\lambda_o = \delta_{S/F}\,|\vec{\lambda}| - \vec{\mathrm{v}}\cdot\vec{\lambda}^{S/F}. \tag{2.39}$$

The functions $\delta_S\,|\vec{\lambda}|$ and $\delta_F\,|\vec{\lambda}|$ are given by equations (1.17) and (1.18), respectively. We denote them by $\delta_{S/F}\,|\vec{\lambda}|$ and the wave vectors $\vec{\lambda}^S$ and $\vec{\lambda}^F$ by $\vec{\lambda}^{S/F}$. Under the conditions (2.38) the reduced system (2.5) takes the form of the following nonlinear system of ODEs

$$\begin{aligned}
\frac{d\rho}{dr} &= \frac{\eta(r)\rho}{\delta^2_{S/F}}\left[\delta^2_{S/F} - \frac{(\vec{H}\cdot\vec{\lambda}^{S/F})^2}{|\vec{\lambda}^{S/F}|^2\,\rho}\right], \\
\frac{d\vec{\mathrm{v}}}{dr} &= -\frac{\eta(r)}{\delta_{S/F}}\left[\delta^2_{S/F}\frac{\vec{\lambda}^{S/F}}{|\vec{\lambda}^{S/F}|} - \frac{(\vec{H}\cdot\vec{\lambda}^{S/F})}{|\vec{\lambda}^{S/F}|\,\rho}\,\vec{H}\right], \\
\frac{d\vec{H}}{dr} &= \eta\,(r)\left[\vec{H} - (\vec{H}\cdot\vec{\lambda}^{S/F})\,\frac{\vec{\lambda}^{S/F}}{|\vec{\lambda}^{S/F}|^2}\right],
\end{aligned} \tag{2.40}$$

where

$$\eta(r) = a^2\left[\delta^2_{S/F} - \frac{|\vec{H}|^2}{\rho}\right]^{-1}, \quad \vec{H}\cdot\left(\vec{\lambda}^{S/F}\times\frac{d\vec{H}}{dr}\right) = 0, \quad \vec{H}\cdot\left(\vec{\lambda}^{S/F}\times\frac{d\vec{\mathrm{v}}}{dr}\right) = 0, \quad p = A_o\rho^{\kappa}, \tag{2.41}$$

and A_o is an arbitrary constant. The determination of the general solution of system (2.40) is a difficult task. To simplify it we decouple the system (2.40) by considering two separate cases of the fixed orientation of the magnetic field $\vec{H}$ relatively to wave vectors $\vec{\lambda}^S$ and $\vec{\lambda}^F$.

1). The magnetic field $\vec{H}$ is orthogonal to $\vec{\lambda}^{S/F}$.

Here the slow magnetoacoustic wave S becomes an entropic wave E_1. The corresponding solution for the fast magnetoacoustic wave F is

$$\rho = \rho\,(r), \qquad p = A_o\,\rho^{\kappa}, \qquad \vec{H} = \rho\vec{H}_o, \tag{2.42}$$

where ρ is an arbitrary function of the Riemann invariant

$$r = \vec{\lambda}^F\cdot\vec{\mathrm{x}} + \left(\sqrt{\kappa\,A_o\,\rho^{\kappa-1} + \rho|\vec{H}_o|^2} - \vec{\mathrm{v}}\cdot\vec{\lambda}^F\right)t. \tag{2.43}$$

Here $\vec{H}_o$ is a constant vector perpendicular to the direction of wave propagation $\vec{\lambda}^F$ which is also constant. The velocity of the flow is parallel to $\vec{\lambda}^F$ and is expressed in terms of the density

$$\vec{\mathrm{v}} = \varepsilon\mathrm{v}\,(\rho)\,\vec{\lambda}^F, \qquad \varepsilon = \pm 1, \tag{2.44}$$

where the norm of $\vec{v}$ is given by

$$\mathrm{v}(\rho)=2\begin{cases}\sqrt{A_o}\left[\sqrt{\beta_o\rho+1}-\operatorname{arc\,tanh}\sqrt{\beta_o\rho+1}\right] & \text{for} \quad \kappa=1,\\ \sqrt{2A_o(\beta_o+1)}\sqrt{\rho} \quad \text{for } \kappa=2,\\ \dfrac{\sqrt{\kappa A_o}}{(\kappa-1)}\sqrt{\rho^{\kappa-1}+\beta_o\rho}\left[1+\dfrac{(\kappa-2)\sqrt{\beta_o}}{\sqrt{\rho^{\kappa-2}+\beta_o}}\right]{}_2F_1\left(a_1,b_1,c_1;\dfrac{-\rho^{\kappa-2}}{\beta_o}\right) & \text{for } \kappa\neq 1,2.\end{cases} \tag{2.45}$$

The function ${}_2F_1\left(a_1,b_1,c_1;z\right)$ is a hypergeometric function of the second kind of one variable $z=-\rho^{\kappa-2}/\beta_o$, where

$$\beta_o=\frac{|\vec{H}_o^2|}{\kappa A_o}\,, \tag{2.46}$$

with parameters $a_1=1/2(\kappa-2),\ b_1=\frac{1}{2}, c_1=\ 1+1/2(\kappa-2)$. Equation (2.46) gives the ratio of the magnetic pressure $\frac{1}{2}|\vec{H}_o|^2$ to the hydrodynamic pressure $p\sim A_o$. When β_o is of the order of one or larger, the flow will be affected noticeably by the magnetic fluid $\vec{H}$. If $\beta_o\ll 1$, the opposite is true.

The Lorentz force $\vec{F}_m$ related to the fast magnetoacoustic wave F is

$$\vec{F}_m=-\frac{1}{2}\nabla\left[|\vec{H}|^2\right]=-|\vec{H}_o|^2\rho\frac{d\rho}{dr}\,\vec{\lambda}^F, \tag{2.47}$$

and therefore the fast magnetoacoustic wave moving perpendicularly to $\vec{H}$ causes compressions and expansions of the distance between the lines of force without changing their direction. Since $\nabla\times\vec{v}=0$, the fluid has no vorticity and the fact that $\vec{F}_m$ is a conservative force implies, by virtue of Kelvin's Theorem, that the circulation (2.21) is constant.

2). The magnetic field $\vec{H}$ is parallel to $\vec{\lambda}^{S/F}$.

For $\delta_A\equiv|\,\vec{H}\,|^2/\rho<a$, the fast magnetoacoustic wave F becomes simply an acoustic wave that propagates along a constant magnetic field $\vec{H}_o$ and there are no magnetic effects [17]. The slow magnetoacoustic wave S (or fast F if $\delta_A>a$) is more interesting:

$$\rho=\left[\left(\frac{2\kappa+1}{2\beta_o}\right)^{(1+2/\kappa)}-\frac{(\kappa+2)}{\beta_o}r\right]^{-1/(\kappa+2)}, \quad p=A_o\,\rho^\kappa\,, \quad \vec{H}=H_o\,\vec{\lambda}(r)\,. \tag{2.48}$$

Here A_o and H_o are arbitrary constants and ρ is a function of the Riemann invariant

$$r=\vec{\lambda}\,(r)\cdot\vec{x}+|\vec{\lambda}\,(r)|\big(\delta_A-\mathrm{v}\sin\theta\big)t\,. \tag{2.49}$$

The flow velocity takes the form

$$\vec{v}=\varepsilon\mathrm{v}\big[\sin\theta\,\vec{\lambda}\,(r)-\cos\theta\,\vec{\lambda}_\perp(r)\big]\,, \qquad \varepsilon=\pm1\,, \tag{2.50}$$

where the index $\perp$ specifies the direction perpendicular to $\vec{\lambda}$ and the angle $\theta=\measuredangle(\vec{\lambda},\vec{\gamma})$ is defined by

$$\tan\theta=-\frac{\delta_A^2}{a^2}\sim-\beta_o\,\rho^{-\kappa}. \tag{2.51}$$

The norm of $\vec{\mathrm{v}}$ is given in terms of ρ by

$$\mathrm{v}\,(\rho) = 2H_o\,\rho^{-(\kappa+\frac{1}{2})}\left[\sqrt{\rho^{2\kappa}+{\beta_o}^2} - \frac{2\kappa\beta_o}{(1+2\kappa)}\;{}_2F_1\left(a_1, b_1, c_1\,; \frac{-\rho^{2\kappa}}{{\beta_o}^2}\right)\right], \tag{2.52}$$

where the function ${}_2F_1\,(a_1, b_1, c_1; z)$ is a hypergeometric function of the second kind of one variable $z = -\rho^{2\kappa}/{\beta_o}^2$, where $\beta_o = H_o^2/\kappa A_o$, with parameters $a_1 = -(1+2\kappa)/4\kappa$, $b_1 = \frac{1}{2}, c_1 = 1-(1+2\kappa)/4\kappa$.

For $\beta_o \sim \delta_A^2/a^2 \gtrsim 1$, when the magnetic pressure and hydrodynamic pressure are comparable, the magnetic field lines undergo compressions and expansions resulting from the behaviour of the longitudinal component of $\vec{\gamma}$ along the wave vector $\vec{\lambda}$ (similar to acoustic waves in hydrodynamics). In addition the tensions produced by the Lorentz force $\vec{F}_m$ are of the form

$$\vec{F}_m = \frac{H_o^2}{\rho}\,\vec{\lambda}_\perp\,(r)\,. \tag{2.53}$$

The slow magnetoacoustic wave S is often called the “compressional Alfvén wave” [18]. If $\beta_o \gg 1$, the magnetic field is so strong that

$$(\vec{\lambda}\cdot\vec{\gamma}) = \frac{a^2}{\delta_A^2} \longrightarrow 0 \quad\Leftrightarrow\quad \theta \rightarrow \frac{\pi}{2}\,, \tag{2.54}$$

and we obtain the simple Alfvén wave A^ε described in **Section 2.2** which propagates in an incompressible fluid. This wave is called the “torsional Alfvén wave” [18].

The simple wave solutions of the MHD equations described above are not new. They have been obtained in the past, mostly by the generalized method of characteristics (see *e.g.*[6]). It is worth mentioning that the treatment of MHD equations, or any hyperbolic equations for that matter, by the classical symmetry reduction method has never yielded simple wave type solutions. It has only become possible through the use of the conditional symmetry method, the variant of which we present here. Interestingly it delivers the solutions in a closed form, lending themselves easily to physical interpretation. The GMC, on other hand, usually provides solutions in much more complex form and a good deal of guesswork is often involved in bringing them to more serviceable form. Thus, it seems that there exists a certain affinity between conditional symmetries and the structure of simple wave solutions of hyperbolic equations. This fact is of more practical consequence when we proceed to multiple wave solutions.

3 Conditional symmetries and double waves in MHD equations.

Now we generalize the above construction to the case of superposition of many simple waves described in terms of Riemann invariants. For this purpose we fix k linearly independent wave vectors $\lambda^1, \ldots, \lambda^k$, $1 \leq k \leq p$, which correspond to the following Riemann invariants

$$r^s\,(\mathrm{x}, \mathrm{u}) = \lambda^s_\mu\,(\mathrm{u})\,x^\mu\,, \qquad s = 1, \ldots, k\,. \tag{3.1}$$

The equation

$$u = f\Big(r^1(\mathrm{x},\mathrm{u}), \ldots, r^k(\mathrm{x},\mathrm{u})\Big) \tag{3.2}$$

defines a unique function $\mathrm{u}(\mathrm{x})$ on a neighborhood of $\mathrm{x}=0$ and the Jacobian matrix is given by

$$\frac{\partial u^\alpha}{\partial x^\mu}(\mathrm{x}) = \Big(\phi^{-1}(\mathrm{x})\Big)^l_j \lambda^j_\mu(\mathrm{u}(\mathrm{x}))\, \frac{\partial f^\alpha}{\partial r^l}\big(r(\mathrm{x},\mathrm{u})\big), \tag{3.3}$$

with $l, j = 1, \ldots, k$; $\mu = 1, \ldots, p$; $\alpha = 1, \ldots, q$ where the matrix

$$\Big(\phi(\mathrm{x})\Big)^l_j = \delta^l_j - \frac{\partial r^l}{\partial u^\alpha}\frac{\partial f^\alpha}{\partial r^j}\big(r(\mathrm{x},\mathrm{u})\big) \tag{3.4}$$

is assumed to be invertible. This assumption excludes the gradient catastrophe phenomenon for the function u. Note that the rank of the Jacobian matrix (3.3) is at most equal to k. If the set of vectors

$$\xi_a(\mathrm{u}) = \big(\xi^1_a(\mathrm{u}), \ldots, \xi^p_a(\mathrm{u})\big)^T, \qquad a = 1, \ldots, p-k \tag{3.5}$$

satisfies the orthogonality condition

$$\lambda^s_\mu \cdot \xi^\mu_a = 0\,, \qquad s = 1, \ldots, k\,, \quad a = 1, \ldots, p-k \tag{3.6}$$

then we get

$$\xi^\mu_a(\mathrm{u}(\mathrm{x}))\frac{\partial u^\alpha}{\partial x^\mu} = 0\,. \tag{3.7}$$

This means that $u^1(\mathrm{x}), \ldots, u^q(\mathrm{x})$ are invariants of the vector fields

$$\xi^\mu_a(\mathrm{u}(\mathrm{x}))\frac{\partial}{\partial x^\mu}\,, \qquad a = 1, \ldots, p-k \tag{3.8}$$

in the space of independent variables $E \subset \mathbb{R}^p$. Hence the graph of the solution (3.2) $\Gamma = \big\{(\mathrm{x}, \mathrm{u}(\mathrm{x}))\big\}$ is invariant under the vector fields

$$X_a = \xi^\mu_a(\mathrm{u})\frac{\partial}{\partial x^\mu}\,, \qquad a = 1, \ldots, p-k \tag{3.9}$$

acting on the space of independent and dependent variables $E \times U \subset \mathbb{R}^p \times \mathbb{R}^q$.

If $\mathrm{u}(\mathrm{x})$ is a q-component function defined on a neighborhood of $\mathrm{x}=0$ such that the graph $\Gamma = \big\{(\mathrm{x}, \mathrm{u}(\mathrm{x}))\big\}$ is invariant under all vector fields (3.9) with the property (3.6), then $\mathrm{u}(\mathrm{x})$ is the solution of (3.2) for some function f. The functions $r^1, \ldots, r^k,\ u^1 \ \ldots, u^q$ constitute a complete set of invariants of the abelian algebra generated by the vector fields (3.9). Substituting expression (3.2) into system (1.6) we obtain the reduced system

$$\Big(\phi^{-1}(\mathrm{x})\Big)^l_j \lambda^j_\mu(\mathrm{u}(\mathrm{x}))\, A^\mu(\mathrm{u}(\mathrm{x}))\frac{\partial f^\alpha}{\partial r^l}\Big(r\big(\mathrm{x}, \mathrm{u}(\mathrm{x})\big)\Big) = 0\,. \tag{3.10}$$

Under these circumstances we can state the following.

Proposition. A function u(x) defined on a neighborhood of x = 0 satisfies an equation (3.2) for some function $f : \mathbb{R}^k \to \mathbb{R}^q$ if and only if the graph $\Gamma = \left\{\left(\mathrm{x}, \mathrm{u}(\mathrm{x})\right)\right\}$ is invariant under all vector fields (3.9). Such a function u(x) is a solution of the system (1.6) if and only if the partial differential equations (3.10) are satisfied.

Now we show that a proper change of variables allows us to rectify the vector fields (3.9) and, consequently, to derive a reduced system which admits multiple wave solutions. If the k by k matrix

$$\Lambda = \left(\lambda^i_j\right), \qquad 1 \leq i,\, j \leq k \tag{3.11}$$

is invertible, then the independent vector fields

$$X_{k+1} = \frac{\partial}{\partial x^{k+1}} - \sum_{i,j=1}^{k} \left(\Lambda^{-1}\right)^j_i \lambda^i_{k+1} \frac{\partial}{\partial x^j}\,, \,\ldots\,,\, X_p = \frac{\partial}{\partial x^p} - \sum_{i,j=1}^{k} \left(\Lambda^{-1}\right)^j_i \lambda^i_p \frac{\partial}{\partial x^j} \tag{3.12}$$

have the required form (3.9) for which the orthogonality conditions (3.2) are satisfied. The change of independent and dependent variables

$$\overline{x}^1 = r^1(\mathrm{x}, \mathrm{u}),\, \ldots\,, \overline{x}^k = r^k(\mathrm{x}, \mathrm{u}),\ \overline{x}^{k+1} = x^{k+1},\, \ldots\,, \overline{x}^p = x^p,\ \overline{u}^1 = u^1,\, \ldots\,, \overline{u}^q = u^q \tag{3.13}$$

allows us to rectify the vector fields X_a and we get

$$X_{k+1} = \frac{\partial}{\partial \overline{x}^{\,k+1}}\,, \,\ldots\,,\, X_p = \frac{\partial}{\partial \overline{x}^{\,p}}\,. \tag{3.14}$$

The corresponding invariance conditions are

$$\overline{\mathrm{u}}_{\overline{x}^{k+1}} = 0\,, \,\ldots\,,\, \overline{\mathrm{u}}_{\overline{x}^p} = 0\,. \tag{3.15}$$

The general solution of the invariance conditions (3.15) is given by

$$\overline{\mathrm{u}}\left(\overline{\mathrm{x}}\right) = f\left(\overline{x}^1, \,\ldots\,, \overline{x}^k\right), \tag{3.16}$$

where $f : \mathbb{R}^k \to \mathbb{R}^q$ is arbitrary. The system (1.6) is subjected to the invariance conditions (3.15) and, when written in terms of new coordinates $(\overline{\mathrm{x}}, \overline{\mathrm{u}})$, takes the form

$$\Delta\ :\ \begin{cases} \displaystyle\sum_{i,j=1}^{k} \sum_{\mu=1}^{p} \left(\Phi^{-1}\right)^i_j \lambda^j_\mu A^\mu \overline{\mathrm{u}}_{\overline{x}^i} = 0\,, \\[2ex] \overline{\mathrm{u}}_{\overline{x}^{k+1}} = 0\,, \,\ldots\,,\, \overline{\mathrm{u}}_{\overline{x}^p} = 0\,. \end{cases} \tag{3.17}$$

Note that for $k \geq 2$ the system (3.17) is more sophisticated than its analogue (2.12). This is due to the presence of the k by k matrix Φ.

As it was shown in [1] (page 885, Theorem 2), that the symmetry criterion for the existence of group invariant solutions of the form (3.16) (*i.e.* rank-k solutions) for the overdetermined system (3.17) states that

$$\mathrm{pr}^{(1)}\, X_a\, \Delta = 0\,, \qquad a = 1,\, \ldots\,,\, p - k \tag{3.18}$$

whenever equations $\Delta = 0$ are satisfied.

We complete our construction of rank-k solutions of system (1.6) by pulling back the wave vectors $\lambda^1, \dots, \lambda^k$ to the k-dimensional submanifold $\mathcal{S} \subset U$ obtained by solving the system (3.17). Then the wave vectors $\lambda^1(\mathrm{u}), \dots, \lambda^k(\mathrm{u})$ become functions of the parameters $r^1, \dots, r^k$. The set (3.1) and (3.2) of implicitly defined relations between the variables u^α, x^μ and $r^1, \dots, r^k$ can be written as follows

$$\mathrm{u} = f(r), \qquad r^s = \lambda^s_\mu(r)\, x^\mu, \qquad s = 1, \dots, k \tag{3.19}$$

where $r = \left(r^1, \dots, r^k\right)$. Note that the rank of function $\mathrm{u}(\mathrm{x})$ is at most equal to k. In particular solutions (3.19) include Riemann k-waves (as presented in **Section 4**).

The construction procedure outlined above has been applied by us to the MHD equations (1.1)-(1.5) in order to obtain rank-2 solutions representing nonlinear superpositions of two simple waves. The results of our analysis are summarized in Table 1, which shows the possibility of existence of these solutions obtained from different combinations of the vector fields X_a.

The remaining part of this section is devoted to the discussion of some more interesting solutions.

We denote by E_iE_j, $A^\varepsilon A^\varepsilon$, $A^\varepsilon E_i$, FF, FE_i, $\dots$, $i, j = 1, 2, 3$, the solutions which are the result of nonlinear superpositions of given waves. The indices i, j are related to the dimension generated by the wave vectors λ^s. Moreover we denote by r and s the Riemann invariants (which coincide with group invariants of X_a) of the waves under consideration.

3.1 Double entropic waves.

Analyzing the double entropic waves E_iE_j, we consider separately two cases.

1). In the first case two wave vectors $\vec{\lambda}^1$ and $\vec{\lambda}^2$ are located on the plane spanned by the unit vectors $\vec{e}_1$ and $\vec{e}_2$. Hence we have

$$\lambda^1 = \big(-(\vec{\lambda}^1 \cdot \vec{\mathrm{v}}),\, \vec{\lambda}^1\big), \qquad \lambda^2 = \big(-(\vec{\lambda}^2 \cdot \vec{\mathrm{v}}),\, \vec{\lambda}^2\big), \tag{3.20}$$

$$\vec{\lambda}^1 = (\cos\varphi, \sin\varphi, 0), \qquad \vec{\lambda}^2 = (\cos\theta, \sin\theta, 0). \tag{3.21}$$

Here φ and θ are functions of the Riemann invariants s and r which are to be determined. The general solution of the type E_1E_1 invariant under the vector fields (3.9)

$$X_1 = \frac{1}{v}\frac{\partial}{\partial t} + \frac{u}{v}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \qquad X_2 = \frac{\partial}{\partial z}, \tag{3.22}$$

is determined by the following system of equations

$$\frac{d}{dt}\{\rho, p, \vec{\mathrm{v}}, \vec{H}\} = 0, \;\; \nabla\cdot\vec{\mathrm{v}} = 0, \;\; \nabla p = -\frac{1}{2}\nabla|\vec{H}|^2 + (\vec{H}\cdot\nabla)\vec{H}, \;\; (\vec{H}\cdot\nabla)\vec{\mathrm{v}} = 0, \;\; \nabla\cdot\vec{H} = 0, \tag{3.23}$$

where $d/dt = \partial/\partial t + (\vec{\mathrm{v}}\cdot\nabla)$; ρ, p, $\vec{\mathrm{v}}$ and $\vec{H}$ are arbitrary functions of the Riemann invariants

$$s = \vec{\lambda}^1(s, r)\cdot(\vec{\mathrm{x}} - \vec{\mathrm{v}}t), \qquad r = \vec{\lambda}^2(s, r)\cdot(\vec{\mathrm{x}} - \vec{\mathrm{v}}t). \tag{3.24}$$

The reduced system (3.23) describes the propagation of a double entropic wave E_1E_1 in an incompressible fluid for which the quantities ρ, p, $\vec{\mathrm{v}}$ and $\vec{H}$ are conserved along the flow.

The fluid is force free since we have an equilibrium between the gradient of hydrodynamic pressure and the Lorentz force $\vec{F}_m$ which is the sum of the gradient of magnetic pressure $|\vec{H}|^2/2$ and a tension force $(\vec{H}\cdot\nabla)\vec{H}$. The constraint $(\vec{H}\cdot\nabla)\vec{\mathrm{v}}=0$ implies that the velocity of the fluid $\vec{\mathrm{v}}$ remains constant along $\vec{H}$. The double entropic wave produces vorticity and the circulation (2.21) of the fluid is preserved since $d\vec{\mathrm{v}}/dt=0$. If we impose the condition that the Lorentz force depends only on the magnetic pressure, then we obtain the solution

$$\rho=\rho(s,r)\,,\quad p(s,r)+\frac{1}{2}|\vec{H}(s,r)|^2=p_o\,,\quad \vec{\mathrm{v}}=w(s,r)\vec{e}_3\,,\quad \vec{H}=H(s,r)\vec{e}_3\,,\qquad(3.25)$$

where ρ, w and H are arbitrary functions of the Riemann invariants $s=\vec{\lambda}^1(s,r)\cdot\vec{\mathrm{x}}$ and $r=\vec{\lambda}^2(s,r)\cdot\vec{\mathrm{x}}$; p_o is an arbitrary constant. The double entropic wave propagates in an incompressible fluid in which the flow is one-dimensional and stationary without vorticity.

2). In the second case we fix the direction of the wave vector $\vec{\lambda}^1$ along $\vec{e}_3$ and we let the wave vector $\vec{\lambda}^2$ circulate on the plane perpendicular to $\vec{\lambda}^1$. The entropic wave vectors have the form

$$\lambda^1=\big(-w,0,0,1\big)\,,\qquad \lambda^2=\big(-(\vec{\lambda}^2\cdot\vec{\mathrm{v}}),\,\vec{\lambda}^2\big)\,,\qquad(3.26)$$

where $\vec{\lambda}^2=(\cos\theta,\sin\theta,0)$ and θ is a function of the Riemann invariants s and r which are to be determined. The corresponding vector fields, (3.9), are given by

$$X_1=-\tan\theta\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\,,\qquad X_2=\frac{\partial}{\partial t}+(u+v\tan\theta)\frac{\partial}{\partial x}+w\frac{\partial}{\partial z}\,.\qquad(3.27)$$

We can now consider two types of MHD solutions, invariant under $\{X_1,X_2\}$, which depend on the direction of the magnetic field $\vec{H}$ with respect to the wave vectors $\vec{\lambda}^1$ and $\vec{\lambda}^2$.

2.a). The magnetic field $\vec{H}$ is perpendicular to the wave vectors $\vec{\lambda}^1$ and $\vec{\lambda}^2$. The solution is given by

$$\begin{aligned}\rho&=\rho(s,r)\,,\qquad p(s,r)+\frac{1}{2}|\vec{H}(s,r)|^2=p_o\,,\qquad \vec{\mathrm{v}}=\mathrm{v}(s,r)\,\vec{\lambda}^2_\perp(r)+w(s)\,\vec{e}_3\,,\\ \vec{H}&=H(s,r)\,\vec{\lambda}^2_\perp(r)\,,\end{aligned}\qquad(3.28)$$

where p_o is an arbitrary constant; ρ, v, w and H are arbitrary functions of their arguments, the unit vector $\vec{\lambda}^2_\perp=\big(-\sin\theta\,(r),\cos\theta\,(r),0\big)$ is perpendicular to the wave vector $\vec{\lambda}^2$ and θ is an arbitrary function of r. The Riemann invariants s and r are given by

$$s=\vec{\lambda}^2\,(r)\cdot\vec{\mathrm{x}}\,,\qquad r=z-w(s)t\,.\qquad(3.29)$$

2.b). The magnetic field $\vec{H}$ is perpendicular to the constant wave vector $\vec{\lambda}^2_o$. Here the solution takes the form

$$\rho=\rho(s,r),\;\; p(s)+\frac{1}{2}|\vec{H}(s)|^2=p_o\,,\;\; \vec{\mathrm{v}}=\mathrm{v}(s)\vec{e}_o+w(s)\vec{e}_3\,,\;\; \vec{H}=H_\perp(s)\vec{e}_o+H_3(s)\,\vec{e}_3\,,\;(3.30)$$

where p_o is an arbitrary constant, the constant vector $\vec{e}_o=\big(-\sin\theta_o,\cos\theta_o,0\big)$ is perpendicular to the wave vector $\vec{\lambda}^2_o$, ρ is an arbitrary function of two Riemann invariants s and

r; and v, w, $H_\perp$ and H_3 are arbitrary functions of the Riemann invariant s. The invariants s and r have the form

$$s = \vec{\lambda}_o^2 \cdot \vec{x}\,, \qquad r = z - w(s)t\,. \tag{3.31}$$

For both cases **2.a)** and **2.b)** the Lorentz force is cancelled by the gradient of the hydrodynamic pressure. Thus the fluid is force free. By virtue of Kelvin's Theorem the circulation (2.21) of the fluid is preserved.

3.2 Double Alfvén wave AA.

We discuss the superposition of two Alfvén waves A^ε for which the wave vectors have the form (1.22)

$$\lambda^1 = \left(\lambda_o^1\,,\ \lambda_1^1, \lambda_2^1, \lambda_3^1\right), \qquad \lambda^2 = \left(\lambda_o^2\,,\ \lambda_1^2, \lambda_2^2, \lambda_3^2\right), \tag{3.32}$$

where

$$\lambda_o^i = \varepsilon \frac{(\vec{H}\cdot\vec{\lambda}^i)}{\sqrt{\rho}} - \vec{v}\cdot\vec{\lambda}^i\,, \qquad i=1,2, \quad \varepsilon = \pm 1\,. \tag{3.33}$$

We assume that the wave vectors $\vec{\lambda}^1$ and $\vec{\lambda}^2$ are linearly independent. The corresponding vector fields (3.9) are given by

$$\begin{aligned} X_1 &= \frac{(\lambda_1^1\lambda_2^2 - \lambda_2^1\lambda_1^2)}{(\lambda_o^1\lambda_1^2 - \lambda_1^1\lambda_o^2)}\frac{\partial}{\partial t} + \frac{(\lambda_2^1\lambda_o^2 - \lambda_o^1\lambda_2^2)}{(\lambda_o^1\lambda_1^2 - \lambda_1^1\lambda_o^2)}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\,, \\ X_2 &= \frac{(\lambda_1^1\lambda_3^2 - \lambda_3^1\lambda_1^2)}{(\lambda_o^1\lambda_1^2 - \lambda_1^1\lambda_o^2)}\frac{\partial}{\partial t} + \frac{(\lambda_3^1\lambda_o^2 - \lambda_o^1\lambda_3^2)}{(\lambda_o^1\lambda_1^2 - \lambda_1^1\lambda_o^2)}\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\,. \end{aligned} \tag{3.34}$$

The solutions invariant under $\{X_1, X_2\}$ are

$$\rho = \rho_o\,, \qquad p = p_o\,, \qquad \vec{v}\,(s,r) = \frac{\varepsilon}{\sqrt{\rho_o}}\vec{H}\,(s,r)\,, \qquad |\vec{H}|^2 = \mathcal{H}_o^2\,, \qquad \varepsilon = \pm 1\,, \tag{3.35}$$

where ρ_o, p_o and $\mathcal{H}_o$ are arbitrary constants, $\vec{v}$ and $\vec{H}$ are vector functions of the Riemann invariants

$$s = \vec{\lambda}^1\,(s,r)\cdot\vec{x}\,, \qquad r = \vec{\lambda}^2\,(s,r)\cdot\vec{x}\,. \tag{3.36}$$

The magnetic field $\vec{H}$ must satisfy the constraints

$$\frac{d\vec{H}}{ds}\cdot\vec{\lambda}^1 = 0\,, \qquad \frac{d\vec{H}}{dr}\cdot\vec{\lambda}^2 = 0\,. \tag{3.37}$$

The double Alfvén wave AA propagates in an incompressible and stationary fluid which flows along the lines of force of the magnetic field $\vec{H}$.

3.3 Double Alfvén entropic waves AE_1.

We now consider the superposition of the Alfvén wave A^{ε} with the entropic wave E_1. The wave vectors (1.19) and (1.22) are given by

$$\lambda^1 = \left(\lambda^1_o,\ \lambda^1_1, \lambda^1_2, \lambda^1_3\right), \qquad \lambda^2 = \left(\lambda^2_o,\ \lambda^2_1, \lambda^2_2, \lambda^2_3\right), \tag{3.38}$$

where the vectors $\vec{\lambda}^1$ and $\vec{\lambda}^2$ correspond to the Alfvén $\vec{\lambda}^A$ and the entropic $\vec{\lambda}^E$ wave vectors, respectively, and

$$\lambda^1_o = \varepsilon\frac{(\vec{H}\cdot\vec{\lambda}^1)}{\sqrt{\rho}} - \vec{\mathrm{v}}\cdot\vec{\lambda}^1, \quad \varepsilon = \pm 1, \qquad \lambda^2_o = -\vec{\lambda}^2\cdot\vec{\mathrm{v}}. \tag{3.39}$$

The invariant solutions under the vector fields (3.9),

$$\begin{aligned} X_1 &= \frac{(\lambda^1_1\lambda^2_2 - \lambda^1_2\lambda^2_1)}{(\lambda^1_o\lambda^2_1 - \lambda^1_1\lambda^2_o)}\frac{\partial}{\partial t} + \frac{(\lambda^1_2\lambda^2_o - \lambda^1_o\lambda^2_2)}{(\lambda^1_o\lambda^2_1 - \lambda^1_1\lambda^2_o)}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \\ X_2 &= \frac{(\lambda^1_1\lambda^2_3 - \lambda^1_3\lambda^2_1)}{(\lambda^1_o\lambda^2_1 - \lambda^1_1\lambda^2_o)}\frac{\partial}{\partial t} + \frac{(\lambda^1_3\lambda^2_o - \lambda^1_o\lambda^2_3)}{(\lambda^1_o\lambda^2_1 - \lambda^1_1\lambda^2_o)}\frac{\partial}{\partial x} + \frac{\partial}{\partial z}, \end{aligned} \tag{3.40}$$

take the form

$$\begin{aligned} \rho &= \rho(r), \quad p(r) + \frac{1}{2}\mathcal{H}^2(r) = p_o, \quad |\vec{H}|^2 = \mathcal{H}^2(r), \\ \vec{\mathrm{v}} &= \frac{\varepsilon\vec{H}}{\sqrt{\rho(r)}} + \vec{\varphi}(r), \quad \vec{H} = \alpha(s,r)\dot{\vec{\varphi}} + \beta(s,r)\ddot{\vec{\varphi}} + \vec{\psi}(r), \qquad \varepsilon = \pm 1. \end{aligned} \tag{3.41}$$

They are expressed in terms of the Riemann invariants

$$s = \vec{\lambda}^1(s,r)\cdot\vec{\mathrm{x}}, \qquad r = \vec{\lambda}^2(s,r)\cdot\vec{\mathrm{x}} - \left(\vec{\lambda}^2(s,r)\cdot\vec{\mathrm{v}}\right)t, \tag{3.42}$$

where p_o is an arbitrary constant, ρ, p and $\mathcal{H}$ are arbitrary functions of their arguments. Furthermore the vector functions $\vec{\varphi}$ and $\vec{\psi}$ must satisfy the algebraic relations

$$\vec{\varphi}\cdot\vec{\lambda}^1 = 0, \quad \dot{\vec{\varphi}}\cdot\vec{\lambda}^k = 0, \quad \ddot{\vec{\varphi}}\cdot\vec{\lambda}^k = 0, \quad \dddot{\vec{\varphi}}\cdot\vec{\lambda}^2 = 0, \quad \vec{\psi}\cdot\vec{\lambda}^2 = 0, \quad \dot{\vec{\psi}}\cdot\vec{\lambda}^2 = 0, \tag{3.43}$$

for $k = 1, 2$. In expressions (3.41) and (3.43) we have used the following notation

$$\dot{\vec{\varphi}} = \frac{d\vec{\varphi}}{dr}, \quad \ddot{\vec{\varphi}} = \frac{d^2\vec{\varphi}}{dr^2}, \quad \dddot{\vec{\varphi}} = \frac{d^3\vec{\varphi}}{dr^3}, \quad \dot{\vec{\psi}} = \frac{d\vec{\psi}}{dr}. \tag{3.44}$$

The function α is expressed in terms of the function β, the vector functions $\vec{\varphi}$ and $\vec{\psi}$ and their derivatives

$$\alpha = \frac{-[(\dot{\vec{\varphi}}\cdot\ddot{\vec{\varphi}})\beta + (\dot{\vec{\varphi}}\cdot\vec{\psi})] \pm \sqrt{\Delta}}{|\dot{\vec{\varphi}}|^2}, \tag{3.45}$$

where the solution (3.45) is real if the discriminant Δ is positive, *i.e.*

$$\Delta = \left[(\dot{\vec{\varphi}}\cdot\ddot{\vec{\varphi}})\beta + (\dot{\vec{\varphi}}\cdot\vec{\psi})\right]^2 - |\dot{\vec{\varphi}}|^2\left[|\ddot{\vec{\varphi}}|^2\beta^2 + 2(\ddot{\vec{\varphi}}\cdot\vec{\psi})\beta + |\vec{\psi}|^2 - \mathcal{H}^2(r)\right] \geq 0. \tag{3.46}$$

To calculate the function β we must solve the following nonlinear ODE

$$\Big[(\dot{\vec{\varphi}}\times\vec{\psi})\cdot\ddot{\vec{\varphi}}\Big]\frac{\partial\beta}{\partial r}+\Big[(\dot{\vec{\varphi}}\times\ddot{\vec{\varphi}})\cdot\dddot{\vec{\varphi}}\Big]\beta^2+\Big[(\dot{\vec{\varphi}}\times\vec{\psi})\cdot\dddot{\vec{\varphi}}+(\dot{\vec{\varphi}}\times\ddot{\vec{\varphi}})\cdot\dot{\vec{\psi}}-|\dot{\vec{\varphi}}|^{-2}(\dot{\vec{\varphi}}\cdot\ddot{\vec{\varphi}})[(\dot{\vec{\varphi}}\times\vec{\psi})\cdot\ddot{\vec{\varphi}}]\Big]\beta$$
$$-|\dot{\vec{\varphi}}|^{-2}\big[(\dot{\vec{\varphi}}\times\vec{\psi})\cdot\ddot{\vec{\varphi}}\big]\Big[(\dot{\vec{\varphi}}\cdot\vec{\psi})\mp\sqrt{\Delta}\Big]+\big[(\dot{\vec{\varphi}}\times\vec{\psi})\cdot\dot{\vec{\psi}}\big]=0\,, \tag{3.47}$$

where the coefficients are expressed in terms of $\vec{\varphi}$, $\vec{\psi}$ and their derivatives. Analyzing the solutions (3.41), we notice the synergetic effects which result from the superposition of the two simple waves. This superposition constitutes the double wave AE_1. We have an equilibrium between hydrodynamic and magnetic pressures resulting from the entropic wave E_1. In addition we observe that the density ρ and the pressure p are unchanged by the Alfvén wave A^ε. Both waves modify the flow direction that is not parallel to the magnetic field $\vec{H}$, in which case the flow is incompressible and rotational (*i.e.* $\nabla\times\vec{\mathrm{v}}\neq 0$). However, the circulation of the fluid is not conserved because of the existence of tension forces originating from the Lorentz force (due to the contribution of the Alfvén wave A^ε).

3.4 Double magnetoacoustic waves FF.

In the study of the double magnetoacoustic waves FF we limit our analysis to the situation in which the magnetoacoustic wave vectors $\vec{\lambda}^1$ and $\vec{\lambda}^2$ are orthogonal to the magnetic field $\vec{H}$. We consider two cases.

1). We assume that $\vec{H}=\big(0,0,H\big)$ and the wave vectors are of the form

$$\lambda^1=\big(\delta_F-u,1,0,0\big)\,,\qquad \lambda^2=\big(\delta_F-v,0,1,0\big)\,, \tag{3.48}$$

where

$$\delta_F=\varepsilon\left[\frac{\kappa p}{\rho}+\frac{|\vec{H}|^2}{\rho}\right]^{\frac12}\,,\qquad \varepsilon=\pm1\,. \tag{3.49}$$

Then the corresponding vector fields X_a are given by

$$X_1=\frac{\partial}{\partial t}+(u-\delta_F)\frac{\partial}{\partial x}+(v-\delta_F)\frac{\partial}{\partial y}\,,\qquad X_2=\frac{\partial}{\partial z}\,.$$

The solutions invariant under $\{X_1,X_2\}$ are

$$\begin{aligned}\rho &= \rho(s,r)\,,\quad p=A_o\,\rho^\kappa\,,\quad \vec{\mathrm{v}}=\Big[v(s,r)+\frac12\big[f(r)+g(s)\big]\Big]\vec{e}_1+v(s,r)\vec{e}_2+w(s-r)\vec{e}_3\,,\\ \vec{H} &= \rho H_o\,\vec{e}_3\,,\end{aligned} \tag{3.50}$$

where A_o and H_o are arbitrary constants; ρ, v and w are arbitrary functions of their arguments. The Riemann invariants are of the form

$$s=x+\left(\delta_F-\frac12\big[f(r)+g(s)\big]-v\right)t\,,\qquad r=y+(\delta_F-v)t\,, \tag{3.51}$$

and the functions f and g are given by

$$f(r)-g(s)=\varepsilon\begin{cases}4\sqrt{A_o}\Big[\sqrt{\beta_o\rho+1}-\operatorname{arc\,tanh}[\sqrt{\beta_o\rho+1}]\Big] & \text{for } \kappa=1,\\ 4\sqrt{2A_o(\beta_o+1)\rho} & \text{for } \kappa=2,\\ \dfrac{4\sqrt{\kappa A_o}}{(\kappa-1)}\sqrt{\rho^{\kappa-1}+\beta_o\rho}\left[1+\dfrac{(\kappa-2)\sqrt{\beta_o}}{\sqrt{\rho^{\kappa-2}+\beta_o}}\right]{}_2F_1\left(a_1,b_1,c_1;\dfrac{-\rho^{\kappa-2}}{\beta_o}\right) & \text{for } \kappa\neq1,2\end{cases} \tag{3.52}$$

where $\varepsilon = \pm 1$, the function ${}_2F_1\,(a_1, b_1, c_1; z)$ is a hypergeometric function of the second kind of one variable $z = -\rho^{\kappa-2}/\beta_o$ and $\beta_o = H_o^2/\kappa A_o$, with parameters $a_1 = 1/2(\kappa-2)$, $b_1 = \frac{1}{2}, c_1 = 1 + 1/2(\kappa-2)$.

The Lorentz force associated with the double wave FF is

$$\vec{F}_m = -\frac{1}{2}\nabla\big[|\vec{H}|^2\big] = -H_o^2\,\rho\left[\frac{\partial\rho}{\partial s}\,\vec{e}_1 + \frac{\partial\rho}{\partial r}\,\vec{e}_2\right]. \tag{3.53}$$

Consequently the double wave FF causes compressions and expansions of magnetic field lines in two directions $\vec{e}_1$ and $\vec{e}_2$, perpendicular to the magnetic field $\vec{H}$. Since $\vec{F}_m$ is a conservative force (*i.e.* it can be derived from the gradient of the magnetic pressure $\frac{1}{2}|\vec{H}|^2$), the circulation of the flow (2.21) is preserved.

2). We consider the one-dimensional case of a superposition of two magnetoacoustic fast waves which propagate with local velocities $\lambda_o^\varepsilon = \delta_F + \varepsilon u$, $\varepsilon = \pm 1$. So we have

$$\lambda^1 = \big(\delta_F - u, 1, 0, 0\big), \qquad \lambda^2 = \big(-(\delta_F + u), 1, 0, 0\big), \tag{3.54}$$

where

$$\delta_F = \left[\frac{\kappa p}{\rho} + \frac{|\vec{H}|^2}{\rho}\right]^{\frac{1}{2}}. \tag{3.55}$$

The vector fields, (3.9), take the forms

$$X_1 = \frac{\partial}{\partial y}, \qquad X_2 = \frac{\partial}{\partial z}. \tag{3.56}$$

Here we assume that the unknown functions $\mathrm{u} = \big(\rho, p, \vec{\mathrm{v}}, \vec{H}\big)$ are some functions of the Riemann invariants $s = x + (\delta_F - u)t$ and $r = x - (\delta_F + u)t$. Substituting these functions into the MHD equations (1.1)-(1.5) we obtain the solutions

$$\begin{aligned} \rho &= \rho\,(s,r), \quad p = A_o\,\rho^\kappa, \quad \vec{\mathrm{v}} = \frac{1}{2}\big[f(r) + g(s)\big]\,\vec{e}_1 + v(s+r)\,\vec{e}_2 + w(s+r)\,\vec{e}_3, \\ \vec{H} &= \rho H_o\big[\cos\varphi(s+r)\,\vec{e}_2 + \sin\varphi(s+r)\,\vec{e}_3\big], \end{aligned} \tag{3.57}$$

where A_o and H_o are arbitrary constants; ρ, v, w and φ are arbitrary functions of their arguments. The Riemann invariants are of the form

$$s = x + \left(\delta_F - \frac{1}{2}\big[f(r) + g(s)\big]\right)t, \qquad r = x - \left(\delta_F + \frac{1}{2}\big[f(r) + g(s)\big]\right)t, \tag{3.58}$$

where the functions f and g retain the form (3.52).

For the particular case $\kappa = 2$ the solutions (3.57) can be expressed as

$$\rho = \frac{\big[f(r) - g(s)\big]^2}{16(2A_o + H_o^2)}, \quad p = A_o\,\rho^2, \quad \vec{\mathrm{v}} = \frac{1}{2}\big[f(r) + g(s)\big]\,\vec{e}_1, \quad \vec{H} = \rho H_o\,\vec{e}_3,$$

where A_o and H_o are arbitrary constants, f and g are arbitrary functions of the Riemann invariants r and s, respectively. These invariants are given by

$$s = x - \frac{1}{4}(3g + f)t, \qquad r = x - \frac{1}{4}(3f + g)t. \tag{3.59}$$

The Lorentz force is

$$\vec{F}_m = -\frac{1}{2}\nabla\big[|\vec{H}|^2\big] = -H_o^2\,\rho\left[\frac{\partial\rho}{\partial s} + \frac{\partial\rho}{\partial r}\right]\vec{e}_1\,. \tag{3.60}$$

Thus the double wave FF propagates in the fluid as the combination of a compressional wave associated with the arbitrary function $f\big(x - (\delta_F + u)t\big)$ and an expansion wave associated with the arbitrary function $g\big(x + (\delta_F - u)t\big)$. Since $\nabla\times\vec{v} = 0$, the fluid is irrotational. This particular case ($\kappa = 2$) provides an example of the MHD generator principle, *i.e.* conversion of mechanical energy into electromagnetic energy. The fluid flows with a velocity $\vec{v} = u\,\vec{e}_1$ and the magnetic field $\vec{H}$ lies along the direction $\vec{e}_3$. This results in an electric current $\vec{J}$ which takes the form

$$\vec{J} = \nabla\times\vec{H} = \frac{H_o}{8(2A_o + H_o^2)}\big[f(r) - g(s)\big]\left[\frac{dg}{ds} - \frac{df}{dr}\right]\vec{e}_2\,. \tag{3.61}$$

Finally, if the double wave FF propagates along the magnetic field $\vec{H}$, we obtain a double acoustic wave in hydrodynamics [17].

3.5 Double magnetoacoustic entropic waves $\boldsymbol{FE_1}$.

We assume that the wave vector $\vec{\lambda}^F$ of the fast magnetoacoustic wave F and the wave vector $\vec{\lambda}^{E_1}$ of the entropic wave E_1 are orthogonal to the magnetic field $\vec{H}$. We study the following two cases.

1). Consider the one-dimensional case of a superposition of an entropic wave E_1 and a magnetoacoustic wave F propagating in opposite directions that are perpendicular to $\vec{H} = \big(H_1, H_2, 0\big)$. We assume that their phase velocities are $\lambda_o^E = -w$ and $\lambda_o^F = \delta_F - w$, respectively. We have

$$\lambda^F = \big(\delta_F - w, 0, 0, 1\big)\,, \qquad \lambda^{E_1} = \big(-w, 0, 0, 1\big)\,, \tag{3.62}$$

where

$$\delta_F = \varepsilon\left[\frac{\kappa p}{\rho} + \frac{|\vec{H}|^2}{\rho}\right]^{\frac{1}{2}}, \qquad \varepsilon = \pm 1\,. \tag{3.63}$$

The vector fields, (3.9), take the forms

$$X_1 = \frac{\partial}{\partial x}\,, \qquad X_2 = \frac{\partial}{\partial y}\,. \tag{3.64}$$

We assume that the unknown functions $\mathrm{u} = \big(\rho, p, \vec{v}, \vec{H}\big)$ are some functions of the Riemann invariants $s = z + (\delta_F - w)t$ and $r = z - wt$. Substituting these functions into MHD equations (1.1)-(1.5) we obtain

$$\begin{aligned} \rho &= \rho(s)\,, \quad p = A_o\,\rho^\kappa\,, \qquad \vec{v} = \vec{\alpha}(r)\times\vec{e}_3 + w(\rho)\,\vec{e}_3\,, \\ \vec{H} &= H_o\,\rho(s)\big[\cos\varphi(r)\,\vec{e}_1 + \sin\varphi(r)\,\vec{e}_2\big]\,, \end{aligned} \tag{3.65}$$

where A_o and H_o are arbitrary constants; ρ, φ and $\vec{\alpha}$ are arbitrary functions of their arguments. The Riemann invariants s and r are given by

$$s = z - \left(w(\rho) - \varepsilon \left[\kappa A_o \rho^{(\kappa-1)} + H_o^2 \rho \right]^{\frac{1}{2}} \right) t\,, \qquad r = z - w(\rho) t\,, \quad \varepsilon = \pm 1\,. \tag{3.66}$$

The function w has the form

$$w(\rho) = 2 \begin{cases} \sqrt{A_o}\left[\sqrt{\beta_o \rho + 1} - \operatorname{arc\,tanh}[\sqrt{\beta_o \rho + 1}]\right] & \text{for } \kappa = 1, \\ \dfrac{\sqrt{\kappa A_o}}{(\kappa-1)} \sqrt{\rho^{\kappa-1} + \beta_o \rho} \left[1 + \dfrac{(\kappa-2)\sqrt{\beta_o}}{\sqrt{\rho^{\kappa-2} + \beta_o}} \right] {}_2F_1\left(a_1, b_1, c_1; \dfrac{-\rho^{\kappa-2}}{\beta_o} \right) & \text{for } \kappa \neq 1, 2. \end{cases} \tag{3.67}$$

The Lorentz force related to the double wave FE_1 is

$$\vec{F}_m = -\frac{1}{2} \nabla \left[|\vec{H}|^2 \right] = -H_o^2\, \rho \frac{d\rho}{ds}\, \vec{e}_3\,, \tag{3.68}$$

and represents the contribution of the fast magnetoacoustic wave F. The gradient of hydrodynamic pressure takes the form

$$\nabla p = \kappa A_o\, \rho^{(\kappa-1)}\, \frac{d\rho}{ds}\, \vec{e}_3\,. \tag{3.69}$$

It follows that the magnetic field lines undergo compressions and expansions along the direction of the wave vector $\vec{\lambda}^F$. On the other hand the entropic wave E_1 contributes to the flow vorticity

$$\nabla \times \vec{v} = \vec{e}_3 \times \left(\frac{d\vec{\alpha}}{dr} \times \vec{e}_3 \right). \tag{3.70}$$

Since $\vec{F}_m$ is a conservative force, the circulation (2.21) of fluid is preserved.

In the special case $\kappa = 2$ the solution (3.65) is reduced to the following expressions

$$\begin{aligned} \rho &= \rho(s)\,, \qquad p = A(r)\rho^2\,, \qquad \vec{v} = \vec{\alpha}(r) \times \vec{e}_3 + 2\varepsilon \sqrt{C_2 \rho(s)}\, \vec{e}_3\,, \qquad \varepsilon = \pm 1\,, \\ \vec{H} &= \rho(s)\, \mathcal{H}(r) \left[\cos\varphi(r)\, \vec{e}_1 + \sin\varphi(r)\, \vec{e}_2 \right], \end{aligned} \tag{3.71}$$

where C_2 in an arbitrary constant. The arbitrary functions $A(r)$ and $\mathcal{H}(r)$ satisfy the algebraic relation

$$2A(r) + \mathcal{H}^2(r) = C_2\,. \tag{3.72}$$

Thus the entropic wave E_1 has an influence on the Lorentz force, but does not affect the equilibrium between the hydrodynamic and magnetic pressures.

2). We now consider the case for which both wave vectors $\vec{\lambda}^F$ and $\vec{\lambda}^{E_1}$ are orthogonal to the magnetic field $\vec{H} = \big(0, 0, H\big)$

$$\lambda^F = \big(\delta_F - u, 1, 0, 0 \big)\,, \qquad \lambda^{E_1} = \big(-v, 0, 1, 0 \big)\,, \tag{3.73}$$

where

$$\delta_F = \varepsilon \left[\frac{\kappa p}{\rho} + \frac{H^2}{\rho} \right]^{\frac{1}{2}}, \qquad \varepsilon = \pm 1\,. \tag{3.74}$$

In this case the vector fields, (3.9), take the forms

$$X_1 = \frac{1}{v}\frac{\partial}{\partial t} + \frac{(u-\delta_F)}{v}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \qquad X_2 = \frac{\partial}{\partial z}. \tag{3.75}$$

The solutions associated with the vector fields, (3.75), exist and for $\kappa = 2$ they are given by

$$\rho = \rho(s), \;\; p = A(r)\rho^2, \;\; \vec{v} = \left[b(r) - 2\varepsilon\sqrt{C_2\rho(s)}\right]\vec{e}_1 + v_o\vec{e}_2 + w(r)\vec{e}_3, \;\; \vec{H} = \rho(s)\mathcal{H}(r)\vec{e}_3, \tag{3.76}$$

where $\varepsilon = \pm 1$, C_2 and v_o are arbitrary constants; ρ, b and w are arbitrary functions of their arguments. The Riemann invariants are

$$s = x + \left(3\varepsilon\sqrt{C_2\rho(s)} - b(r)\right)t, \qquad r = y - v_o t, \qquad \varepsilon = \pm 1. \tag{3.77}$$

Hence the functions $A(r)$ and $\mathcal{H}(r)$ satisfy the algebraic relation

$$2A(r) + \mathcal{H}^2(r) = C_2. \tag{3.78}$$

The Lorentz force takes the form

$$\vec{F}_m = -\frac{1}{2}\nabla\left[|\vec{H}|^2\right] = -\mathcal{H}^2(r)\rho\frac{d\rho}{ds}\,\vec{e}_1, \tag{3.79}$$

which is the result of both waves' contribution, but the entropic wave E_1 does not disturb the equilibrium between hydrodynamic and magnetic pressures (see (3.76) and (3.78)). The compressions and expansions of the fluid must take place in the direction of the fast magnetoacoustic wave propagation $\vec{\lambda}^F$, otherwise the flow would be incompressible along the directions $\vec{e}_2$ and $\vec{e}_3$. The fluid vorticity is given by

$$\nabla\times\vec{v} = \frac{dw}{dr}\,\vec{e}_1 - \frac{db}{dr}\,\vec{e}_3. \tag{3.80}$$

This indicates that the vorticity depends only on the entropic waves E_1. Kelvin's Theorem ensures that the circulation (2.21) of the fluid is preserved.

4 Concluding remarks.

In this paper we develop the conditional symmetry method as a tool for recovering the solutions of hyperbolic systems in the form of Riemann waves and their superpositions. This idea was firstly outlined in [1] where, among others, the symmetry criterion for the existence of group invariant rank-k solutions was derived. The progress made here consists in adapting the conditional symmetry method procedure to the solutions based on Riemann invariants. Its most important element is the introduction of the requirement that the graph of the solution be invariant under vector fields, (3.9), with the orthogonality property (3.6). When this takes place, the appropriate change of variable, (3.13), allows us to rectify the vector fields (3.9) and, consequently, determine the invariance conditions (3.15) which are to be imposed on the initial system of equations (1.6).

Riemann waves and their superpositions have been studied so far only in the context of the generalized method of characteristics [19]. This approach requires imposing certain conditions on vector fields γ_s and λ^s of the waves entering into an interaction. In order for us to comment on its relation to the conditional symmetry method we outline here the basic assumptions of GMC.

A form of solution, called a Riemann k-wave, is postulated for which the matrix of the tangent mapping du is given by

$$\frac{\partial u^\alpha}{\partial x^\mu}(\mathrm{x}) = \sum_{s=1}^{k} \xi^s\,(\mathrm{x})\gamma_s^\alpha\,(\mathrm{u})\lambda_\mu^s\,(\mathrm{u})\,, \tag{4.1}$$

where $\xi^s \neq 0$ are treated as arbitrary functions of x and we assume that the vector fields $\gamma_1, \dots, \gamma_k$ are linearly independent. We assume also that commutators of all vector fields γ_l and γ_s are linear combinations of these fields

$$[\gamma_l, \gamma_s] \ \in\ \mathrm{span}\,\{\gamma_l, \gamma_s\}\,, \qquad \forall\, l \neq s = 1, \dots, k\,. \tag{4.2}$$

If these conditions are satisfied, then, due to the homogeneity of wave relation (1.12), we may change the lengths of the vectors γ_l and γ_s, so that the commutators of these vector fields vanish, *i.e.*

$$[\gamma_l, \gamma_s] = 0\,, \qquad \forall\, l \neq s = 1, \dots, k\,. \tag{4.3}$$

The vector fields $\gamma_1, \dots, \gamma_k$ form an abelian distribution on the space U. There exists a parametrization of the integral surface $\mathcal{S}$ in U tangent to these fields,

$$\mathcal{S}\ :\quad \mathrm{u} = f\left(r^1, \dots, r^k\right), \tag{4.4}$$

such that

$$\frac{\partial f}{\partial r^s} = \gamma_s\,, \qquad \forall\, s \neq l = 1, \dots, k\,. \tag{4.5}$$

The wave vectors $\lambda^s\,(\mathrm{u})$ become functions of the parameters $r^1, \dots, r^k$. Consequently the differential of (4.4) gives

$$\mathrm{du} = \sum_{s=1}^{k} \frac{\partial f}{\partial r^s} dr^s, \qquad dr^s = \sum_{\mu=1}^{p} \frac{\partial r^s}{\partial x^\mu}\, dx^\mu\,, \tag{4.6}$$

which, together with the assumption (4.1), leads to a system of exterior forms

$$dr^s = \xi^s\,(\mathrm{x})\lambda_\mu^s\left(r^1, \dots, r^k\right) dx^\mu\,. \tag{4.7}$$

It has been shown ([20], [21]) that the system (4.7) has solutions if the following conditions are satisfied

$$\frac{\partial \lambda^s}{\partial r^l} \ \in\ \mathrm{span}\,\{\lambda^s, \lambda^l\}\,, \qquad \forall\, s \neq l = 1, \dots, k\,. \tag{4.8}$$

Conditions (4.3) and (4.8) ensure that the set of solutions of the system (1.6), subjected to (4.1), depends on k arbitrary functions of one variable. It has been proved [21] that all solutions, *i.e.* the general integral, of the system (4.7), under conditions (4.8), can be obtained by solving, with respect to the variables $r^1, \dots, r^k$, the system in implicit form

$$\lambda^s_\mu \left(r^1, \dots, r^k\right) x^\mu = \psi^s \left(r^1, \dots, r^k\right), \tag{4.9}$$

where ψ^s are arbitrary functionally independent differentiable functions of k variables $r^1, \dots, r^k$. The solutions of (4.9) are constant on $(p-k)$-dimensional hyperplanes perpendicular to wave vectors λ^s.

As we can see, both methods discussed here exploit the invariance properties of the original system of equations. In the GMC they have the purely geometric character (4.4), whereas in the case of CSM we make use of the symmetry group properties (3.9). The two approaches describe two different facets of the same geometric object.

There are, however, two basic differences between the generalized method of characteristics and our approach. Riemann multiple waves defined by (4.1) constitute a more limited class of solutions than the rank-k solutions postulated by the conditional symmetry method. This difference results from the fact that the scalar functions ξ^s in (4.1) (describing the profiles of simple waves) are substituted in our case (in expression (3.3)) with a k by k matrix Φ which allows for much broader range of initial data.

The second difference consists in the fact that the restrictions (4.3) and (4.8) on the vector fields γ_s and λ^s, ensuring the solvability of the problem by the generalized method of characteristics, are not necessary in the conditional symmetry method. This makes possible for us to consider more complex configurations of simple waves entering into an interaction.

In order to verify the efficiency of our approach we have used it for constructing rank-2 solutions of MHD equations. The results obtained in this work coincide to a great degree with the ones obtained earlier by the means of the generalized method of characteristics. All our rank-2 solutions are in fact Riemann double waves (in the sense of definition (4.1)). This is probably largely due to the fact that, to facilitate computations, we have introduced additional simplifications, assuming some specific configurations of the wave vectors λ^s included in superpositions. Most of the solutions obtained here are already known [6]. Some, however, namely solutions (3.50)-(3.52), (3.57)-(3.58) and (3.65)-(3.67) are new. It is worth mentioning that, at least in the case of MHD equations, our method proved to be much easier to implement than GMC. In particular the invariance condition (3.15) provides us with a set of coordinates very convenient for integration of the reduced system (3.17).

Finally it has to be said that for many physical systems, *e.g.* nonlinear field equations, fluid membrane equations, there have been very few, if any, known examples of multiple-wave solutions. The version of the conditional symmetry method proposed here offers a new, and it seems promising, way to deal with this problem.

Acknowledgements

This work has been partially supported by research grants from NSERC of Canada and FCAR du Québec.

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Table 1. Double Riemann wave solutions described by MHD equations, + indicates that double wave solutions exist; − indicates that double wave solutions do not exist, $\varepsilon = \pm 1$.

Waves	E	A^ε	F^ε	S^ε
E	+	+	+	−
A^ε	+	+	−	−
F^ε	+	−	+	−
S^ε	−	−	−	−