Longitudinal Vibrations of Seismic Disturbance Vertical Bar

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***Abstract-* Longitudinal vibrations of vertical bar of harmonic and random disturbance are considered. Mathematical models of free, forced harmonic and random vibrations are described. The methods of d'Alembert, separation of variables and finite differences are used. To determine the eigenvalues of boundary value problem, a high-precision and tabular analytical method is proposed. The eigenfunctions, amplitude-frequency characteristics of kinematically excited harmonic and random vibrations are discussed. The analogies and connections between the solutions of deterministic and stochastic problems are identified. A number of conclusions are made that contribute to the theoretical and methodological foundations of the interdisciplinary interaction of research in the structural mechanics and seismology.**

Keywords-: harmonic and random vibrations, continually discrete system, spectral density of random process, discrete line spectrum of dispersions.

I. INTRODUCTION

The tasks of the theory of seismic resistance are among the most complex modern problems of the construction design [1]. Currently, both standards and studies mainly focus on the transverse vibrations from the horizontal effects of earthquake on buildings and structures. Meanwhile, longitudinal vibrations cause a certain danger to vertical building structures located near the earthquake epicenter [2, 3]. In [3], it is stated that “the vertical component dominants over other vibrational components in the epicentral area”. It is known that during the Gazli earthquake of 1976 (Uzbekistan, the USSR), the vertical accelerations of the ground surface had greatly exceeded the horizontal ones. In the Spitak earthquake, at the main push, the ground acceleration in horizontal and vertical directions was amounted to 0.21 g, 0.15 g, respectively [4].

The study of longitudinal vibrations is also relevant when we talk about man-made impact on bases and foundations. They arise from traffic, machinery and equipment being operated at close distances from the building structures and can transmit significant kinematic and dynamic effects. At the same time the longitudinal vibrations of vertical bar structures are little studied and the number of written works is insignificant [5].

Both seismic and man-made disturbances of stochastic nature can be described for the preliminary calculations in the form of a harmonic process $u_0(t) = A_0 e^{i\Omega t}$. Such a replacement becomes adequate if the disturbances are narrowband random processes of particular frequency. In support of such a deterministic analysis of seismic impact, it can be argued that the understanding of deterministic representation of seismic impact and the response from it allows a good understanding of the nature of the structure. The results obtained below confirm this assumption.

These structures are often continual discrete systems, consisting of the areas with the distributed mass and concentrated masses. Publications [6–8] are devoted to this topic.

Let us consider a vertical homogeneous bar (Fig. 1) with a uniform cross-section, with the length sections l, made of material with the elastic modulus E, with the material density ρ, cross-sectional areas S_j, carrying the discrete masses M_j at different height levels and based on an elastic foundation with a coefficient stiffness c and mass foundation M_0. The dynamic displacement of the sections is described by the function u(x, t). The static displacements constituting a small part of total deviations are not considered in this work.

II. Free vibrations

It is known that free longitudinal vibrations of bars are described by partial differential equation of hyperbolic type with respect to the displacement of the section in the longitudinal direction u(x, t),

$$\ddot{u} - a^2 u'' = 0, \quad a^2 = E / \rho, \quad x \in (0,l), \quad t > -\infty. \tag{1}$$

The point above the symbol corresponds to the time derivative, the superscript strokes correspond to the differentiation with respect to the argument x. The design model gives the boundary conditions arising from the conditions of fixing the ends of the bar

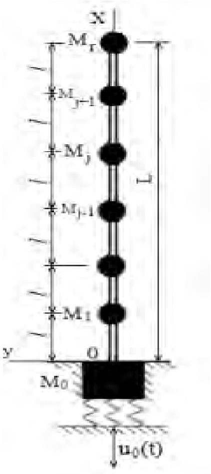

Fig. 1. Design model

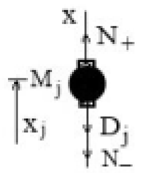

Fig. 2. Area connections

$$b_1 u'(0,t) - M_0\, \ddot{u}(0,t) - c u(0,t) = 0, \quad b_j = ES_j. \tag{2}$$

$$b_r u'(l, t) + M_r \ddot{u}(l,t) = 0, \quad t > -\infty. \tag{3}$$

where, 1, j, r in the subscripts are the numbers of the corresponding areas. Additional conditions are still required to ensure the area connections of the bar separated by concentrated masses. Let's find them making up the equation of motion of the given masses basing on the d'Alembert principle (Fig. 2)

$$N_+ - N_- - D_j = 0, \tag{4}$$

consisting in the fact that internal and inertial forces must be balanced. N_+, N_- are the longitudinal forces in the upper and lower cross-sections, D_j is the d'Alembert inertia force. Let's replace the forces in (4) with their expressions and write the following:

$$b_{j+1}\, u'_+(x_j,t) - b_j\, u'_-(x_j,t) - M_j\, \ddot{u}(x_j,t) = 0, \; j = 1, 2, \ldots r-1. \tag{5}$$

The initial conditions for free vibrations are not required; kinematic disturbances of the base are not taken into account $u_0(t) \equiv 0$.

Equations (1) - (3), (5) form a mathematical model of free vibrations. We will write the solution of the problem using the method of separation of variables as a product

$$u(x,t) = X(x)\, e^{i\omega t}, \tag{6}$$

where, X (x), ω is eigenform and the vibration frequency. Substitution of (6) into (1) - (3), (5) gives

$$\omega^2 X + a^2 X'' = 0, \quad x \in (0, l). \tag{7}$$

$$b_1 X'(0) + \omega^2 M_0 X(0) - cX(0) = 0, \; b_r X'(l) - \omega^2 M_r X(l) = 0. \tag{8}$$

$$b_{j+1} X'_+ - b_j X'_- + \omega^2 M_j X = 0, \; j = 1, 2, ..r-1. \tag{9}$$

The boundary value problem (7) - (9) is then solved using the finite difference method [9]. This choice is justified by the fact that the algorithms and programs obtained in this case easily adapt to vibrations of bars of variable cross-section, to forced vibrations from the dynamic and random loads. To this end, instead of the continuous domain of definition of the variable x let's introduce a discrete domain L_h in the form of nodes of a uniform grid with h step

$$L_h = [x_i : \; x_i = (i-1)h, i = 1, \; 2, \ldots, n\,], \; h = L/(n-1),$$

where, n is the number of grid nodes. Let's replace the values of the function and derivatives with the approximate well-known finite-difference values at the grid nodes with the accuracy of $O(h^2)$. For the equation (7), the central differential derivatives at the internal points i = 2, 3, ... n-1 will be used, for the end points i = 1, n is one-sided differential derivatives. At the points with coordinates of concentrated masses there are discontinuities of the first derivative of the function X(x) of the first kind, i.e. it is not smooth. Therefore, one-sided derivatives for X '(x) should be applied here. As a result, a system of algebraic equations was obtained in matrix-vector form

$$B(\omega)X = 0. \tag{10}$$

where, B is a square matrix of order n, X = [X1, X2, ..., Xn] ^{T} is a transposed vector the components of which are the bar displacements in the grid nodes.

The system of equations (10) has a trivial solution, which corresponds to a static problem and is not of great interest. Non-zero solutions can exist if the determinant of A is zero, i.e.

$$\det B(\omega) = 0. \tag{11}$$

The characteristic equation (11) is an algebraic equation relative to ω. Its roots form a numerical set of cardinality n. When determining the elements of this set, one can do without composing this equation, taking advantage of the possibility of constructing a high-precision diagram ω - detB (), for example, using Matlab computing complex. The abscissas of the intersection of the diagram of the axis ω determine the vector of eigen frequencies Ω.

Example 1. Let Fig. 1 show a bar of a standard steel pipe with length of the sections l = 3 m, with the number of sections r = 4, diameter D = 102 mm and wall thicknesses δ = [3.2; 2.8; 2.2; 1.8] mm descending upwards. The bar rests on a foundation with the mass M_0=10,000 kg and a foundation with a stiffness coefficient c = 10^6 N/m. The discrete masses are given by the vector M = [2000; 4000; 1000; 3000] kg.

The results obtained in the Matlab computing environment are shown in Fig. 3. The first three eigenvalues read from the monitor screen are equal to = [9.9; 70.9; 132.7] sec^{-1}.

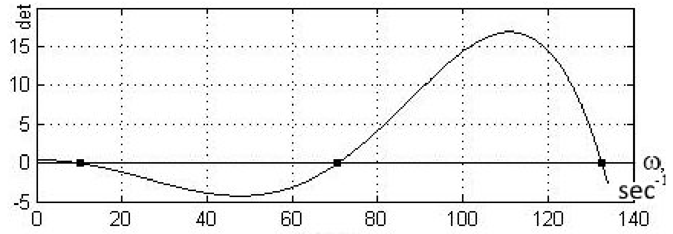

Fig. 3. Eigenvalues

The seismic kinematic impact on the bases of building structures are random processes with a continuous non-uniform frequency spectrum. The dominant frequencies are in the region of small values: 0 ... 30 c^{-1}. In this case, the first eigen frequency falls into this area, which can lead to dangerous resonant vibrations. Therefore, it is necessary to take measures for the detuning of eigen frequencies from the dominant seismic frequencies. This can be done in various ways, for example, by increasing the rigidity of the structure, reducing the mass, etc.

The second and the third frequencies do not create danger for seismic stability of the structure, since at these frequencies the spectral density of random process of disturbances is very small.

Next, the task is to find the eigenvectors Y_k (k = 1, 2, 3, ... n) of matrix B, representing the eigenmodes of vibrations. They can be determined by the well-known methods of linear matrix algebra. It should be noted that the determinant of matrix B is zero and, therefore, the eigenvectors can be calculated only up to a term. In this case, one of the nonzero components of the vector Y_k can be taken to be an arbitrary number, for example, one. Then, on substituting one and excluding one of the elements of the vector Y_k, the equation (10) is converted to

$$D(\Omega_m)Y_m = d_m. \quad (12)$$

where, D is a square matrix of the order n-1; Y_m, d_m– (n-1) are dimensional vectors obtained by substituting one into equation (10).

It is quite natural that the determinant of matrix D is not zero, Y_m is easily calculated, an arbitrary unit is added to it and the desired eigenfunction Y_k is formed.

According to the example given above, using such algorithm and the system (12) it is possible to obtain the eigenvibration shown in Fig. 4.

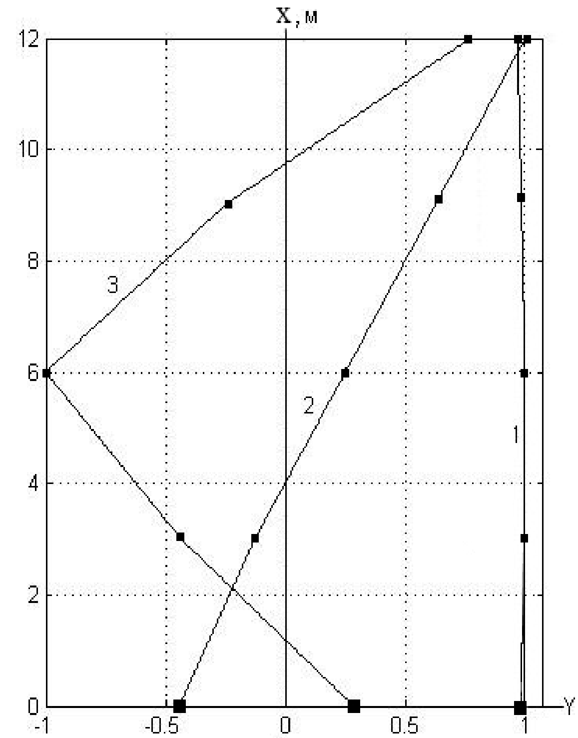

Fig. 4. Eigenfunctions

The analysis of the curves shows that the a priori assumption that the eigenfunctions are not smooth is justified. The derivative X'(x) in the places of concentration of discrete masses has discontinuities of the first kind.

It can be seen that the continental part of the structure insignificantly affects free vibrations, which is explained by its small mass in comparison with discrete masses.

The diagram of the first form of vibrations shows that the displacements of all sections are almost identical. Consequently, the dynamic deformations and stresses at low disturbance frequencies will be insignificant.

The vibrations in the second form occur in such a way that the lower and upper parts move in the opposite directions to the demarcation point at a height of 4 meters. Similar phenomena are observed in the third form vibrations, but with three areas or zones of alternating phases of displacements. The discrete mass M_2, significantly exceeding the others, dominates in the definition of this mode of vibrations. Similar phenomena are observed in vibrations of the third form, but with three areas or zones of displacement. It is noticeable that the foundation with a large mass M_0 has a significant impact on the mode of oscillation.

Additional consideration of higher eigenvalues and functions showed that these spectral pairs are divided into two sets. The first set contains pairs formed by discrete masses, and their number is equal to the number of such masses. The second set of infinite, but calculated power corresponds to vibrations of continual part of the structure. Such a clear distinction occurs in the cases considered in this work, when discrete masses are significantly superior to the masses of continual segments. The study of this issue with free vibrations of mechanical systems of mixed nature, where the distributed and concentrated masses are comparable in magnitude is worthy of attention.

III. Kinematically Excited Harmonic Vibrations

The kinematic disturbances affecting the elastic foundation are presented as a harmonic function (Fig. 1)

$$u_0(t) = A_0\, e^{i\Omega t}.$$

Then, the deterministic mathematical model of forced longitudinal vibrations can be represented using the main hyperbolic equation in partial differential

$$\ddot{u} - a^2 u'' = 0, \quad a^2 = E/\rho, \quad \epsilon(0, l), \quad t > -\infty \tag{13}$$

and boundary conditions

$$b_1\, u'(0,t) - M_0\, \ddot{u}((0,t) - cu(0,t) + cu_0(t) = 0. \tag{14}$$

$$b_r u'(l, t) + M_r \bar{u}(l, t) = 0, \quad b_j = EA_j, \quad t > -\infty. \tag{15}$$

The solution of the problem (13)-(15) will be writen using the method of separation of variables as the product

$$u(x,t) = X(x)\, e^{i\Omega t}, \tag{16}$$

where, X(x) is the eigenfunction; Ω is the vibration frequency. Substitution of (16) into (13)-(15) gives

$$\Omega^2 X + a^2 X'' = 0, \quad x\epsilon\ (0, l). \tag{17}$$

$$b_1 X'(0) + \Omega^2 M_0 X(0) - cX(0) + cA_0 = 0. \tag{18}$$

$$b_{j+1} X'_{+} - b_j X'_{-} + \Omega^2 M_j X = 0, j=1,2, \ldots r-1. \tag{19}$$

$$b_r X'(l) - \Omega^2 M_r X(l) = 0. \tag{20}$$

The problem (17)-(20) is then solved using the finite difference method. Then, instead of the equation (17) the following algebraic equations is obtained

$$X_{i-1} + \mu X_i + X_{i+1} = 0, \quad i = 2, 3, \ldots, n-1, \quad \mu = -2 + (h\Omega/a)^2. \tag{21}$$

At the same time the additional conditions (18 - 20) take the form

$$\beta X_1 + 4X_2 - X_3 = d_1, \ \beta = -3 + 2h(M_0\Omega^2 - c)/b_1, d_1 = -2hcA_0/b_1. \tag{22}$$

$$-b_j X_{k-2} + 4b_j X_{k-1} + \varepsilon_j X_k + 4b_{j+1} X_{k+1} - b_{j+1} X_{k+2} = 0, \ j=1, 2, \ldots r - 1. \tag{23}$$

$$\varepsilon_j = -3b_{j+1} - 3b_j + 2h\Omega^2 M_j.$$

$$X_{n-2} - 4X_{n-1} + \gamma X_n = 0, \quad \gamma = 3 - 2h\Omega^2 M_r / b_r. \tag{24}$$

The system of equations (21)-(24) is rewritten in a matrix-vector form resulting in the following:

$$BX = d, \tag{25}$$

where, B (Ω), d is the matrix of coefficients and the vector of the right side:

$$B = \begin{pmatrix} \beta & 4 & 1 & & & & & & \\ 1 & \mu & 1 & & & & & & \\ & 1 & \mu & 1 & & & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & -b_j & 4b_j & \varepsilon_j & b_{j+1} & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & & 1 & \mu & 1 & \\ & & & & & & 1 & \mu & 1 \\ & & & & & & 1 & -4 & \gamma \end{pmatrix} \quad d = \begin{pmatrix} -2hcA_0/b_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The zero elements of the matrix are not shown.

Example 2. Let us assume that Fig. 1 shows a bar of a standard steel pipe with the length of the sections l = 3 m, number of the sections r = 4, diameter D = 102 mm and wall thicknesses δ = [3,2; 2.8; 2.2; 1.8] mm descending upwards. The bar rests on a foundation with the mass M_0 =10,000 kg and a foundation with a stiffness coefficient c = 10^6 N/m. The vector of discrete masses M = [2000; 4000; 1000; 3000] kg and the amplitude of harmonic kinematic perturbations A_0 = 10 cm are given.

An initial fragment of the eigenfrequency spectrum was obtained for a given bar.

$$\omega = [\ 9.9; \quad 70.9; \quad 132.7\ ]\ \text{sec}^{-1}. \tag{26}$$

The solution results of the system of equations (25) are represented by the forms of forced harmonic vibrations X(x) in Fig. 5, constructed with the disturbance frequency values

$$\Omega = [\ 5; \quad 69; \quad 132.5\ ]\ \text{sec}^{-1}.$$

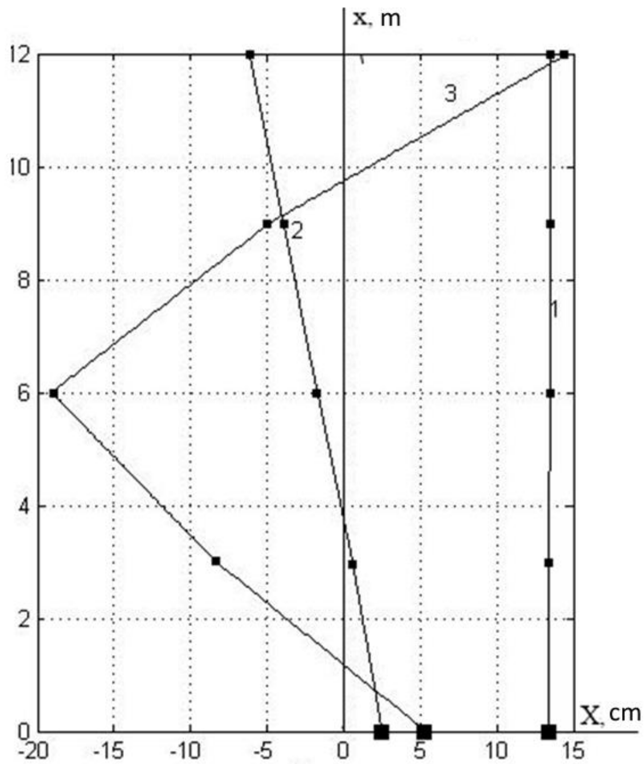

Fig. 5. Form of harmonic vibrations

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The nature of the curves significantly depends on the location of Ω within the spectrum of eigenfrequencies (26). Basinf on the analysis of the curves shown in Fig. 5 it can be stated that at the disturbance frequencies Ω <ω1 (curve 1) the curved axis of the bar is a straight line, the deviations of the bar with concentrated masses along the entire length are in-phase. With the frequency of disturbances Ω> ω1, the movements of the ends of the bar become opposite (curve 2), i.e. they are in phase opposition. As Ω approaches ω3, the shape of the curve is close to the third form of free vibrations (curve 3). A strong influence of the heaviest masses M_0 and M_2 is shown in Fig. 4.

In all cases, the lines X(x) are similar in shape to the eigenfunctions shown in Fig. 4. At these frequencies, the influence of the continual areas is almost imperceptible. As a result of additional calculations, the curvilinearity in the forms of forced vibrations appears at high frequencies of disturbances that are not relevant for seismic vibrations.

IV. FORCED RANDOM VIBRATIONS

Let's now suppose that the kinematic disturbance $u_0(t)$ is a stationary random centered process. At the steady state of vibration, the output function $u(x,t)$ will be a centered space-time random field, stationary, centered in time and non-uniform in the spatial coordinate. Let's find the dispersion of displacements $D_u(x)$ of the bar cross-sections.

When calculating the dispersions, let's use an algorithm and a program developed for harmonic vibrations in part III. We will now consider the issue in more detailed way. It is known that the continuous random centered stationary process $u_0(t)$, given above, can be represented as a Fourier series [10]

$$u_0(t) = \sum_{i=1}^{\infty}(W_i \cos\omega_i t + V_i \sin\omega_i t). \tag{27}$$

where, W_i and V_i are uncorrelated random variables. Simple transformations, taking into account the relationship between the dispersion and correlation function of a stationary random process and the uncorrelated nature of the coefficients of the series (27), give

$$D[u_0(t)] = \sum_{i=1}^{\infty} D_i. \tag{28}$$

This result is shown graphically in Fig. 6 in the form of the so-called discrete line spectrum of dispersions. It turns out that there is a binary correspondence between dispersions and frequencies, i.e. a certain frequency is assigned to each dispersion. It is easy to see that the limiting transition from discrete values ω_i to a continuum leads to a continuous spectral density function. In this case, the dispersion of random process of the input process $u_0(t)$ seems to be an improper integral

$$D_0 = \int_{-\infty}^{\infty} S(\omega)d\omega. \tag{29}$$

Now an elementary dispersion $S(\omega)d\omega$ is assigned for each discrete frequency hashed in Fig. 7.

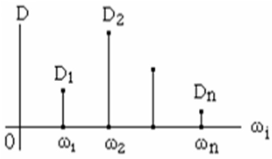

Fig. 6. Point spectrum

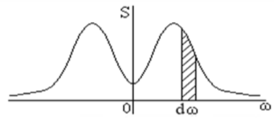

Fig. 7. Spectral dencisity

If to take the elementary dispersion instead of the amplitude of the input harmonic process A_0 (part III), then the output of the problem will be the elementary dispersion of deviations $dD_u(\omega_k,x)$. A subsequent summing up over ω_k gives the dispersion of deviations $D_u(x)$.

It has to be taken into account that the dispersion of input disturbances is distributed on the frequency axis ω according to Fig. 7 and put a continuous spectral density in accordance with discrete, calculated at an infinite countable set of positive frequencies

$$S = \{S_1, S_2, S_3, \dots\},\ Si = S(\omega_i),\quad \omega_i > 0.$$

The entire variance of the input process is represented as the sum of the elementary dispersions of the harmonics that make up the random process.

$$D_0 \approx 2h_\omega \sum_k S_k. \tag{30}$$

where, ω is a fine pitch of the frequency axis breakdown; factor 2 takes into account the left negative semi-axis of frequencies.

Further calculations are carried out in the same way as it was described in the Example 2, namely, instead of A_0, the elementary dispersion of disturbances $h_\omega S_k$ is inserted into the algorithm and the program. The results are summarized as the dispersion of an output displacement process.

$$D_u(x_i) = \sum_k D_k.$$

Example 3. The examples given above are used.

Let's take a random *process with hidden periodicity* (*characteristic frequency*) as input kinematic seismic effects. Its spectral density is presented is follows:

$$S(\omega) = \frac{2\alpha\theta^2\sigma^2}{\pi[(\omega^2-\theta^2)^2 + 4\alpha^2\omega^2]};\quad \theta^2 = \alpha^2 + \beta^2.$$

where, α is a broadband parameter, β is a characteristic frequency, σ is a standard deviation. Let us find the dependence of the dispersion of displacements $D_u(x)$ of random vibrations on the characteristic frequency of disturbances. In this connection the broadband ratio is relatively small. Then, the random disturbances are narrowband and close to the harmonic ones. As a consequence, the random vibrations of the bar should also be close to the harmonic vibrations discussed above. To this end, having previous parameters, supplemented by the parameters of broadband and root-mean-square deviation,

$$\alpha = 0.1 \text{ sec}^{-1}; \qquad \sigma = 10 \text{ cm},$$

calculations are carried out at the increasing values of characteristic frequency β that coincide with the frequencies of deterministic disturbances Ω_k in the Example 2 given above

$$\beta = [\,5;\quad 69;\;\; 132.5\,] \text{ sec}^{-1}.$$

To make the comparison with the amplitudes of harmonic vibrations easier, the calculated dispersions are replaced by the standard deviations $\sigma_u(x)$ and are presented in the form of diagrams in Fig. 8. Taking into account the fact that the standard deviations of a random process can take only positive values, it can be stated that the curves are identical in both qualitative and quantitative terms with the corresponding amplitude diagrams in Fig. 5. The reason is that for small values of the bandwidth parameter, as in the case of ($\alpha << \beta$), the disturbing process is close to the harmonic one. As a consequence, all the conclusions drawn from Fig. 5 for the harmonic vibrations, remain valid, however, in terms of the characteristic frequency and standard deviations. The assumption of a close connection between deterministic and stochastic vibrations is thus confirmed.

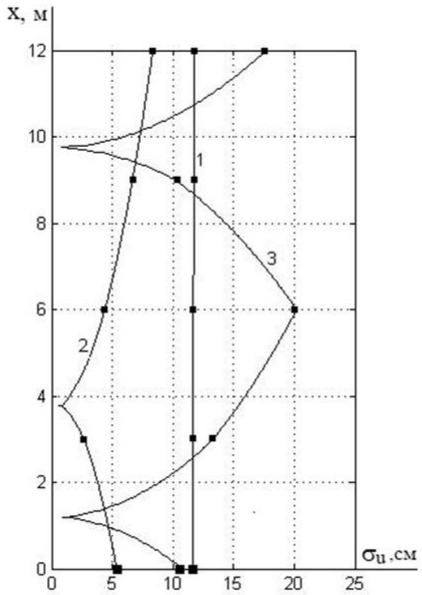

Fig. 8. Root-mean-square deviation

V. Conclusion

1. In the epicentral area of earthquakes, the longitudinal vibrations caused by seismic disturbances may cause a serious danger to the strength and stability of vertical bars.

2. The spectral pairs of the eigenvalues and functions of free vibrations are divided into two sets. The first set contains the pairs formed by discrete masses, and their number is equal to the number of such masses. The second set of infinite, but calculated power corresponds to the vibrations of the continual part of the structure.

3. The preliminary consideration of the boundary problem of forced harmonic vibrations greatly simplifies the solution of complex seismic stochastic boundary value problems.

4. The finite difference numerical method allows us to create universal algorithms and computer programs that easily adapt to a variety of complex problems related to vibrations of vertical bars: free, forced vibrations, vibrations of rods of variable cross-section, combinations of transverse and longitudinal vibrations, etc.

5. A simple and effective method for determining the variance, which is the most important parameter of random vibrations, has been developed and implemented.

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