Journal of Nonlinear Mathematical Physics Volume 12, Number 1 (2005), 53–72 Article

An L_2 Norm Trajectory-Based Local Linearization for Low Order Systems

Fethi BELKHOUCHE

EECS department, Tulane University
New Orleans, LA, USA
E-mail: belkhouf@eecs.tulane.edu

Received November 24, 2003; Accepted April 27, 2004

Abstract

This paper presents a linear transformation for low order nonlinear autonomous differential equations. The procedure consists of a trajectory-based local linearization, which approximates the nonlinear system in the neighborhood of its equilibria. The approximation is possible even in the non-hyperbolic case which is of a particular interest. The linear system is derived using an L_2 norm optimization and the method can be used to approximate the derivative at the equilibrium position. Unlike the classical linearization, the L_2 norm linearization depends on the initial state and has the same order as the nonlinearity. Simulation results show good agreement of the suggested method with the nonlinear system.

1 Introduction

This paper deals with low order systems of nonlinear autonomous ordinary differential equations (ODEs) of the following form

$$\begin{aligned} &\frac{dx}{dt} = F\left(x\left(t\right)\right) \\ &x\left(t_0\right) = x_0 \end{aligned} \tag{1.1}$$

System (1.1) is an initial value problem, since the solutions are dependent on the initial state. In the n-dimensional case, $x \in I\!R^n, F : I\!R^n \longrightarrow I\!R^n$. The state of the system at any time is given by n real variables $x_1(t), x_2(t), ..., x_n(t)$, and the phase velocity of the system at point $x = [x_1, x_2, ..., x_n]^T$ is defined by the vector fields $F(x) = [f_1(x), f_2(x), ..., f_n(x)]^T$.

It is of course well-known that the dynamics of nonlinear systems are much more complicated than linear systems, and many problems that are completely solved for linear systems are still open problems for nonlinear systems. Contrary to linear systems, nonlinear systems have qualitatively different regions of operation.

In general, linearization methods use the property of breaking the nonlinear system into many operating regions and establish local approximations. The classical linearization is based on Fréchet derivative at the equilibrium point. This method is the best approximation of the nonlinear system in the neighborhood of the equilibrium. This is true under the

assumption of hyperbolicity. For non-hyperbolic equilibria, Hartman-Grobman theorem states that the nonlinear system and its linearized version are not equivalent. Another drawback for the classical linearization of nonlinear ODEs is that the system may contain highly nonlinear terms. In this case the approximation error becomes important in the neighborhood of the initial state, due to the fact that the classical linearization is a first order approximation.

In order to overcome open problems in nonlinear systems theory, other linearization methods with different approaches and different goals have been proposed. Jordan et al. ([2] and [3]) used a least square approach to derive the linear system when considering the asymptotic behavior. Benouaz and Arino [4] suggested using a similar procedure to study the asymptotic stability. Terrell [6] used a linearization along trajectory to determine the local observability of nonlinear differential-algebraic equations.

New results on the linearization of the state equations in the critical case are carried out in [5] and [7]. In [7] the study is restricted to the linearization of second order differential equations, where an exact linearization method is discussed.

In this paper, the task is to associate an equivalent linear system to the nonlinear system. The linear system is determined using an optimization technique. The method is applied to the approximation of the nonlinear autonomous vector field and its derivative at the equilibrium point. We also show that the linear equivalent system depends on the initial state, which makes the main difference with the classical linearization. The construction of local approximate normal forms from the method is also discussed. Unlike some linearization methods such as feedback linearization or that suggested in [7]. The method discussed in this paper is not an exact linearization. It is based on the minimization of the error in an optimal way, while exact linearizations make use of a variable transformation that is nonlinear in general. Exact linearization methods present the advantage of being "exact". However it is worth noting that not all nonlinear vector fields are exactly linearizable. Furthermore in some situations it is difficult to find the appropriate nonlinear variable transformation. On the other side, approximate linearizations can be numerically implemented, where different numerical techniques can be used to calculate the approximation.

2 L_2 norm optimal linearization of non-linear ODEs

Consider the initial value problem given by equation (1.1); we define the equivalent linear system by

$$\begin{aligned} &\frac{dx}{dt} = Ax\left(t\right) \\ &x\left(t_0\right) = x_0 \end{aligned} \tag{2.1}$$

where A is a time independent matrix to be determined in order to minimize the residual error vector

$$e = F\left(x\right) - Ax \tag{2.2}$$

for a particular solution $x(t; t_0, x_0)$ of the nonlinear system. There exist several possibilities for the choice of the minimization criterion. In this paper we use an L_2 norm minimization.

One of the main advantages of using the L_2 norm criterion is the existence of closed form solution to the approximation. Let us define the following cost function

$$J(A) = \int_{x_0}^{x_f} \|e\|^2 \, dx \tag{2.3}$$

The unknown matrix A is obtained by the minimization of $J(A)$, where x_0 and x_f are the initial and the final states, respectively. Matrix A approximates the nonlinear system along the particular trajectory $x(t; t_0, x_0)$ that starts from x_0 at time t_0 and goes to x_f when $t = t_f$. Thus matrix A varies for different initial and final states.

Note that the cost function (2.3) fails when the nonlinear vector field depends explicitly on time. Our approach is different from other approaches that use least squares methods ([2], [3] and [4]) where the cost function was defined as

$$J(A) = \int_{0}^{+\infty} \|e\|^2 \, dt \tag{2.4}$$

As a result, the determination of matrix A requires the knowledge of the nonlinear solution $x(t; t_0, x_0)$. Furthermore, the convergence of the method requires the spectrum of $DF(x)$ (DF denotes the Jacobian matrix of F) near the origin to be negative, i.e., the method does not converge near saddles or repellers. We will see that the procedure presented here overcome this problem by considering the backward evolution of the nonlinear solution, and does not need the knowledge of the solution $x(t; t_0, x_0)$ for the computation of the linear system. We assume without loss of generality that the equilibrium of the nonlinear vector fields is situated at the origin. We restrict our study to the scalar and the two-dimensional cases. The generalization of the method to higher order systems is straightforward but requires the use of numerical integration methods.

2.1 Scalar case

Consider the case where the nonlinear system is given by the following scalar nonlinear autonomous ODE

$$\begin{aligned} &\frac{dx}{dt} = f(x(t)) \\ &x(t_0) = x_0 \end{aligned} \tag{2.5}$$

In this case $x \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$, with $f(0) = 0$. We assume that f satisfies all conditions for which the solution for system (2.5) exists and is unique.

Consider a particular solution $x(t; t_0, x_0)$ of equation (2.5). The task is to find a linear system for which the solution approximates $x(t; t_0, x_0)$. The linear system has the following form

$$\begin{aligned} &\frac{dx}{dt} = ax(t) \\ &x(t_0) = x_0 \end{aligned} \tag{2.6}$$

where a is a real number to be determined. Consider a solution of the nonlinear system which goes from x_0 to x_f in the time interval $[t_0, t_f]$. We suggest determining the linear system by the minimization of the cost function given by

$$J(a) = \int_{x_0}^{x_f} |f(x) - ax|^2 \, dx \tag{2.7}$$

In order to obtain the minimum error, the derivative of $J(a)$ with respect to a is set equal to zero

$$\frac{\partial J}{\partial a} = 0 \tag{2.8}$$

from which we obtain

$$a = \frac{\int_{x_0}^{x_f} f(x)\, x dx}{\int_{x_0}^{x_f} x^2 dx} = \frac{3}{x_f^3 - x_0^3} \int_{x_0}^{x_f} f(x)\, x dx \tag{2.9}$$

The real number a in equation (2.9) is defined as the L_2 norm linearization of equation (2.5) along the trajectory that starts from x_0 and goes to x_f. In fact a represents a sort of average value along the trajectory and the line of slope a approximates the area under $f(x)$ in the interval $x \in [x_0, x_f]$. As shown in equation (2.9) , the linear approximation is a function of the initial and final states. Of course a does not exist when $x_f = x_0$.

2.2 Asymptotic approximation near the origin

Equation (2.9) is established for a finite time. In this section we are interested in the approximation of the nonlinear system when time goes to infinity. There exist two cases

1. The trajectory starting from x_0 tends to its equilibrium position when time tends to infinity. The equilibrium point is an attractor for this particular trajectory, since the equilibrium point is situated at the origin, equation (2.9) becomes

$$a = \frac{3}{x_0^3} \int_0^{x_0} f(x)\, x dx \tag{2.10}$$

2. The trajectory starting from x_0 tends to infinity when time tends to infinity, the equilibrium point is a repeller for this particular trajectory. The L_2 norm linearization is computed by considering the backward evolution of the solution. The linear system obtained here is given by equation (2.10) also.

When x_0 is small, a is defined as the L_2 norm linearization of equation (2.5) near the origin. In this case a represents a trajectory-based local approximation of the nonlinear system near the origin.

Under the assumption that the nonlinear vector fields is continuously differentiable, it is possible to write (2.10) as a function of the successive derivatives of the nonlinear vector field. By using a simple integration by parts we get

$$a = \frac{3}{x_0^3} \left[\frac{x_0^2}{2} f(x_0) - \int_0^{x_0} \frac{x^2}{2} f^{(1)}(x)\, dx \right] \tag{2.11}$$

where $f^{(1)}$ is the first derivative of f. If we use twice the integration by parts, we get

$$a = \frac{3}{x_0^3} \left[\frac{x_0^2}{2} f(x_0) - \frac{x_0^3}{6} f^{(1)}(x_0) + \int_0^{x_0} \frac{x^3}{6} f^{(2)}(x)\, dx \right] \tag{2.12}$$

In general for $N \geq 0$, we get

$$a = \frac{3}{x_0^3}\left[\sum_{n=0}^{N}\frac{(-x_0)^{n+2}}{(n+2)!}f^{(n)}(x_0) - \int_0^{x_0}\frac{(-x)^{N+2}}{(N+2)!}f^{(N+1)}(x)\,dx\right] \tag{2.13}$$

where $f^{(n)}$ is the $(n)^{th}$ derivative of f. It is worth to note that the first terms in equations (2.11), (2.12) and (2.13) are not dominant unless the successive derivatives of the nonlinear vector field are equal to zero.

3 Properties of the approximation

3.1 Case when f is a linear vector field

In this case f is a linear vector field, i.e., $f(x) = bx$, b is a real number. From equation (2.10) we get $a = b$, which means that the transformation of a linear system is the system itself.

3.2 Relationship with the derivative of the nonlinear vector field

Let $a(x_0)$ be the L_2 norm linearization of the nonlinear vector field in the neighborhood of the origin which is an equilibrium point for f, an important property is the relationship of $a(x_0)$ with the derivative at the origin. By considering the limit of $a(x_0)$ when the initial state x_0 tends to the origin, we obtain

$$\lim a(x_0) = f'(0)\,; x_0 \to 0 \tag{3.1}$$

This can be proven using L'Hopital's rule in equation (2.10). If x_0 is chosen very small, $a(x_0)$ can be seen as a perturbation of $f'(0)$, thus $a(x_0)$ enables us to approximate the derivative of the nonlinear function at the origin.

3.3 Case when f is a polynomial function of degree N

The L_2 norm linearization is written under the form of power series in the initial state when the nonlinear vector field has a polynomial form. Assume that $f(x)$ is polynomial of degree N with the origin as an equilibrium point

$$f(x) = \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \cdots + \alpha_N x^N = \sum_{n=1}^{N}\alpha_n x^n \tag{3.2}$$

For the L_2 norm linearization, equation (2.10) gives

$$a(x_0) = \alpha_1 + \frac{3}{4}\alpha_2 x_0 + \frac{3}{5}\alpha_3 x_0^2 + \cdots + \frac{3}{N+2}\alpha_N x_0^{N-1} \tag{3.3}$$

$$= \sum_{n=1}^{N}\frac{3}{n+2}\alpha_n x_0^{n-1} \tag{3.4}$$

$$= f'(0) + \tilde{a}(x_0) \tag{3.5}$$

where $\tilde{a}(x_0)$ represents higher order terms. Clearly $f'(0) = \alpha_1$. The linearization is written under the form of a polynomial of degree $N-1$ in the initial state. The linearized system is the following

$$\begin{array}{l} \frac{dx}{dt} = \left[\sum_{n=1}^{N} \frac{3}{n+2}\alpha_n x_0^{n-1}\right] x \\ x(t_0) = x_0 \end{array} \tag{3.6}$$

which has the same order (N) as the nonlinear function. Furthermore, the number of terms depending on x_0 in $a(x_0)$ is the same as the number of non-linearities in the nonlinear vector field. The classical linearization method is a first order approximation, this results in a significant drawback in terms of the error due to the approximation, where the quality of the classical linearization degrades for highly nonlinear systems. For highly nonlinear systems, it is more convenient to include more terms in the approximation. The L_2 norm linearization uses this property, since it has the same order as the nonlinear function, and all nonlinear terms are taken into account. For this reason the method seems to be more accurate for the approximation of highly nonlinear vector fields.

3.4 The case of non-hyperbolic equilibria

For the scalar case, a nonlinear hyperbolic equilibrium point is defined by an equilibrium point x_{eq} for which $f'(x_{eq}) = 0$. The classical linearization fails in this case, and there is no equivalence between the nonlinear system and its classical linearized system. We assume that the nonlinear vector field presents a non-hyperbolic equilibrium at the origin. Let us write f under the following form

$$f(x) = f'(0)\, x + g(x) \tag{3.7}$$

with $g(0) = 0$, $g'(0) = 0$. For the case of the non-hyperbolic equilibrium at the origin, we have $f'(0) = 0$. In this case, equation (2.10) gives for the L_2 norm approximation

$$a(x_0) = \frac{3}{x_0^3}\int_0^{x_0} g(x)\, x dx \tag{3.8}$$

$a(x_0)$ is a function of higher order terms. Unlike the classical linearization, the approximation given by $a(x_0)$ does not vanish when the origin is a non-hyperbolic equilibrium point, and the L_2 norm linearization can approximate the nonlinear system in the non-hyperbolic case, which is of extreme importance in dynamical systems theory. Furthermore, it is possible to establish a relationship between the linearization $a(x_0)$ represented by equation (3.8) and the asymptotic behavior of the equilibrium.

3.5 Case when f is a symmetric function

When the nonlinear vector fields presents some symmetries, the linearization also presents symmetries. From the formula of $a(x_0)$ it can be seen that:

- (i) $a(x_0)$ presents an even symmetry i.e., $a(-x_0) = a(x_0)$ when the nonlinear vector field presents an odd symmetry.
- (ii) $a(x_0)$ presents an odd symmetry i.e., $a(-x_0) = -a(x_0)$ when the nonlinear vector field presents an even symmetry.

3.6 Case when the equilibrium is not situated at the origin

In the previous sections, the equilibrium point was assumed to be situated at the origin. The method is not restricted to this case and can be applied to approximate nonlinear vector fields in the neighborhood of an equilibrium point situated at x_{eq}. From equation (2.10), we have

$$a = \frac{\int_0^{y_0} f\left(y + x_{eq}\right) y dy}{\int_0^{y_0} y^2 dy} \tag{3.9}$$

where y_0 is chosen near the origin. If we put $y_0 = x_0 - x_{eq}$, then

$$a = \frac{3}{\left(x_0 - x_{eq}\right)^3} \int_0^{x_0 - x_{eq}} f\left(y + x_{eq}\right) y dy \tag{3.10}$$

and x_0 is close to the equilibrium point x_{eq}. The real number a given by (3.9) or (3.10) approximates the nonlinear vector field near the equilibrium point situated at x_{eq}.

The derivative at any equilibrium point of the nonlinear vector field can be computed using the L_2 norm linearization. This can be accomplished by considering the limit when the initial state tends to the equilibrium, that is when $x_0 \to x_{eq}$ in equation (3.10) or $y_0 \to 0$ in equation (3.9). By using L'Hopital's rule, we get

$$\lim a\left(x_0\right) = f'\left(x_{eq}\right); x_0 \to x_{eq} \tag{3.11}$$

3.7 Error due to the approximation and initial state

The error due to the approximation is smaller when the final state is close to the initial state (approximation on short time intervals), this can be seen from the following inequality

$$\int_{x_0}^{x_f} \left|f\left(x\right) - ax\right| dx \leq \left(x_f - x_0\right) \sup \left|f - ax\right| \tag{3.12}$$

Similarly if we consider the approximation near the origin, inequality (3.12) becomes

$$\int_0^{x_0} \left|f\left(x\right) - ax\right| dx \leq x_0 \sup \left|f - ax\right| \tag{3.13}$$

This shows that the approximation near the origin is better for small x_0.

4 Approximation in the two-dimensional case

In dynamical systems theory, the two-dimensional case is of an extreme importance. Simplification of higher order systems and reduction to normal forms lead usually to two-dimensional systems. This is the case of normal forms for elementary bifurcations for example.

In this section we consider systems of equation (1.1), where $x \in I\!R^2, F : I\!R^2 \longrightarrow I\!R^2$. We assume that F satisfies all conditions for which the initial value problem has a unique solution. Let us put for the initial states $x\left(t_0\right) = x_0 = \left[x_{01}, x_{02}\right]^T$, and $x_{f1} = x_1\left(t_f; t_0, x_{01}\right)$ and $x_{f2} = x_2\left(t_f; t_0, x_{02}\right)$ for the final states. The equivalent linear system is given by

equation (2.1) where A is 2×2 matrix. Matrix A is obtained by the minimization of the following cost functions

$$\begin{array}{l} J_1(a_{11}, a_{12}) = \int_{x_{01}}^{x_{f1}} \int_{x_{02}}^{x_{f2}} |f_1(x_1, x_2) - a_{11}x_1 - a_{12}x_2|^2 \, dx_2 dx_1 \\ J_2(a_{21}, a_{22}) = \int_{x_{01}}^{x_{f1}} \int_{x_{02}}^{x_{f2}} |f_2(x_1, x_2) - a_{21}x_1 - a_{22}x_2|^2 \, dx_2 dx_1 \end{array} \tag{4.1}$$

where f_1 and f_2 are the components of F. After the differentiation of $J_1(a_{11}, a_{12})$ and $J_2(a_{21}, a_{22})$ with respect to a_{11}, a_{12} and a_{21}, a_{22}, respectively, we get in matrix form

$$A = \left[\int_{x_{01}}^{x_{f1}} \int_{x_{02}}^{x_{f2}} [F(x)] [x]^T \, dx_2 dx_1 \right] \left[\int_{x_{01}}^{x_{f1}} \int_{x_{02}}^{x_{f2}} [x] [x]^T \, dx_2 dx_1 \right]^{-1} \tag{4.2}$$

where $[x]^T$ is the transpose of the state vector. Similarly to the scalar case, it is possible to consider the asymptotic behavior. We consider the following cases

1. The spectrum of the Jacobian matrix of $F(x)$ at the initial state (denoted by $DF(x_0)$) is negative, the trajectory starting from x_0 is asymptotically stable and tends to its equilibrium position. For an equilibrium point situated at the origin, equation (4.2) becomes

$$A = \left[\int_0^{x_{01}} \int_0^{x_{02}} [F(x)] [x]^T \, dx_2 dx_1 \right] \left[\int_0^{x_{01}} \int_0^{x_{02}} [x] [x]^T \, dx_2 dx_1 \right]^{-1} \tag{4.3}$$

Since the trajectory is asymptotically stable, matrix A presents negative spectrum.

2. The spectrum of $DF(x_0)$ is positive or presents a saddle structure, matrix A is obtained by considering the backward evolution of the nonlinear solution.

It turns out that equation (4.3) is valid for all cases without dependence on the spectrum of $DF(x_0)$.

Matrix A in equation (4.3) exists and is unique if the matrix given by

$$D = \left[\int_0^{x_{01}} \int_0^{x_{02}} [x] [x]^T \, dx_2 dx_1 \right] \tag{4.4}$$

is invertible. After integration, we get for matrix D

$$D = \begin{bmatrix} \frac{1}{3}x_{01}^3 x_{02} & \frac{1}{4}x_{01}^2 x_{02}^2 \\ \frac{1}{4}x_{01}^2 x_{02}^2 & \frac{1}{3}x_{02}^3 x_{01} \end{bmatrix} \tag{4.5}$$

Since $det(D) = \frac{7}{144}x_{01}^4 x_{02}^4$, matrix D is invertible when $x_{01} \neq 0$ and $x_{02} \neq 0$. Note that the procedure fails in general when $x(t; t_0, x_0)$ is a periodic function.

It is also worth noting that it is possible to write the elements of A explicitly as a function of the initial states and the nonlinear vector field. Consider the general case where the nonlinear system is written under the following form

$$\begin{array}{l} \frac{dx_1}{dt} = b_{11}x_1 + b_{12}x_2 + g_1(x_1, x_2) \\ \frac{dx_2}{dt} = b_{21}x_1 + b_{22}x_2 + g_2(x_1, x_2) \\ x(t_0) = x_0 \end{array} \tag{4.6}$$

where $[g_1, g_2]^T = G : I\!R^2 \longrightarrow I\!R^2$, G is the nonlinear part of F, with $G(0) = 0$, $DG(0) = 0$. We put

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \tag{4.7}$$

Matrix B is simply the Jacobian matrix of F at the origin. Simple calculations allow us to obtain

$$\begin{aligned}
a_{11} &= b_{11} - \tfrac{12}{7x_{02}^3x_{01}^3}\left(-4\Gamma_{11}x_{02}^2 + 3\Gamma_{12}x_{01}x_{02}\right)\\
a_{12} &= b_{12} + \tfrac{12}{7x_{02}^3x_{01}^3}\left(-3\Gamma_{11}x_{02}x_{01} + 4\Gamma_{12}x_{01}^2\right)\\
a_{21} &= b_{21} - \tfrac{12}{7x_{02}^3x_{01}^3}\left(-4\Gamma_{21}x_{02}^2 + 3\Gamma_{22}x_{01}x_{02}\right)\\
a_{22} &= b_{22} + \tfrac{12}{7x_{02}^3x_{01}^3}\left(-3\Gamma_{21}x_{01}x_{02} + 4\Gamma_{22}x_{01}^2\right)
\end{aligned} \tag{4.8}$$

where

$$\begin{aligned}
\Gamma_{11} &= \int_0^{x_{01}}\int_0^{x_{02}} g_1(x_1,x_2)\,x_1dx_2dx_1\\
\Gamma_{12} &= \int_0^{x_{01}}\int_0^{x_{02}} g_1(x_1,x_2)\,x_2dx_2dx_1\\
\Gamma_{21} &= \int_0^{x_{01}}\int_0^{x_{02}} g_2(x_1,x_2)\,x_1dx_2dx_1\\
\Gamma_{22} &= \int_0^{x_{01}}\int_0^{x_{02}} g_2(x_1,x_2)\,x_2dx_2dx_1
\end{aligned} \tag{4.9}$$

System (4.8) can be written as follows

$$A = B + \tilde{A}(x_0) \tag{4.10}$$

It is easy to see that matrix B is recovered in the L_2 norm linearization matrix. The other terms which are included in $\tilde{A}$ result from the non-linearity. These terms make the difference with the classical linearization, since they do not exist in the Jacobian matrix. Furthermore, the elements of the j^{th} row ($j = 1, 2$) of the L_2 norm linearization matrix depend on the j^{th} component of the nonlinear vector field, this means that if the j^{th} component of the nonlinear vector field is linear, then the j^{th} row in the L_2 norm linearization matrix is the same as the j^{th} row in the Jacobian matrix of the nonlinear vector field at the origin.

5 An extension of the scalar case

It is possible to generalize the linearization method described for the scalar case to some special two-dimensional systems. Let us consider a particular case where equation (1.1) can be written under the following form

$$\begin{aligned}
&\frac{dx_1}{dt} = f_1(x_1,x_2) = \hat{f}_{11}(x_1) + \hat{f}_{12}(x_2)\\
&\frac{dx_2}{dt} = f_2(x_1,x_2) = \hat{f}_{21}(x_1) + \hat{f}_{22}(x_2)\\
&x(t_0) = x_0
\end{aligned} \tag{5.1}$$

where $\hat{f}_{11}$ and $\hat{f}_{21}$ are nonlinear functions in x_1, $\hat{f}_{12}$ and $\hat{f}_{22}$ are nonlinear functions in x_2, with $\hat{f}_{ij}(0) = 0$. We assume that the initial states are the same i.e., $x_{01} = x_{02} = \tilde{x}_0$. This is not a restriction since it can be accomplished by a simple shift of coordinates, and since

the linearization depends on the initial state, another shift of coordinates can be used to recover the initial $[x_{01}, x_{02}]^T$. By identification with the scalar case, we get

$$\begin{aligned} a_{11} &= \frac{3}{\tilde{x}_0^3}\int_0^{\tilde{x}_0} \hat{f}_{11}(x_1)\, x_1 dx_1 \\ a_{12} &= \frac{3}{\tilde{x}_0^3}\int_0^{\tilde{x}_0} \hat{f}_{12}(x_2)\, x_2 dx_2 \\ a_{21} &= \frac{3}{\tilde{x}_0^3}\int_0^{\tilde{x}_0} \hat{f}_{21}(x_1)\, x_1 dx_1 \\ a_{22} &= \frac{3}{\tilde{x}_0^3}\int_0^{\tilde{x}_0} \hat{f}_{22}(x_2)\, x_2 dx_2 \end{aligned} \tag{5.2}$$

which can be written in matrix form as follows

$$A = \frac{3}{\tilde{x}_0^3}\int_0^{\tilde{x}_0} \left[\hat{F}(x)\right][\Psi] \tag{5.3}$$

where $3/\tilde{x}_0^3$ is a scalar, $\left[\hat{F}(x)\right]$ and $[\Psi]$ are 2×2 matrices given by

$$\hat{F} = \begin{bmatrix} \hat{f}_{11} & \hat{f}_{12} \\ \hat{f}_{21} & \hat{f}_{22} \end{bmatrix} \text{ and } \Psi = \begin{bmatrix} x_1 dx_1 & 0 \\ 0 & x_2 dx_2 \end{bmatrix}$$

Equation (5.3) is an extension of the scalar case, it presents a simpler version of the approximation. For the existence of matrix A in equation (5.3) it is necessary that $\tilde{x}_0 \neq 0$.

It is worth noting that if the nonlinear vector field depends on external parameters, then the linear system will be written as a function of these external parameters. This approach can be used in the case where the nonlinear system presents bifurcations, especially for the control of bifurcations.

The approximations obtained in (4.2) and (5.3) are different. In general, equation (4.2) includes more terms than (5.3). Consider for example the following system which contains only one non-linearity

$$\begin{aligned} &\tfrac{dx_1}{dt} = b_{11}x_1 + b_{12}x_2 \\ &\tfrac{dx_2}{dt} = b_{21}x_1 + b_{22}x_2 + x_2^3 \\ &x_1(t_0) = x_2(t_0) = x_0 \end{aligned} \tag{5.4}$$

Using equations (4.2) and (5.3), we get the following systems, respectively

$$\begin{aligned} &\tfrac{dx_1}{dt} = b_{11}x_1 + b_{12}x_2; \tfrac{dx_2}{dt} = b_{21}x_1 + \left[b_{22} + \tfrac{39}{70}x_0^2\right] x_2 \\ &x_1(t_0) = x_2(t_0) = x_0 \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} &\tfrac{dx_1}{dt} = b_{11}x_1 + b_{12}x_2; \tfrac{dx_2}{dt} = b_{21}x_1 + \left[b_{22} + \tfrac{42}{70}x_0^2\right] x_2 \\ &x_1(t_0) = x_2(t_0) = x_0 \end{aligned} \tag{5.6}$$

Equations (5.5) and (5.6) are slightly different. Note that the properties of the approximation in the scalar case are also satisfied in the two-dimensional case. For example, the two-dimensional L_2 norm linearization tends to the Jacobian matrix of the nonlinear vector fields at the origin when $x_0 \to 0$.

6 Normal forms from the linearization

In this section, we discuss the construction of linear normal forms from the linearization method. Normal forms play an important role in dynamical systems theory, where considerable simplifications can be achieved. In general, normal forms are obtained using variable transformations which are nonlinear functions of the state variables [8]. In the general case, the suggested method does not allow to directly derive normal forms for nonlinear systems. However it's possible to derive normal forms from the approximation (after linearization) by using variable transformations. An algorithm is suggested for this purpose. Since the linearization discussed here is an approximate linearization with local aspect, the normal forms obtained from the method are approximate normal forms with local aspect. A comparison with normal forms obtained using the classical linearization is discussed. Here we consider two normal forms

1. Jordan normal form

$$\frac{dz}{dt} = \Lambda z \tag{6.1}$$

where Λ is a diagonal matrix

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{6.2}$$

Under this normal form the solutions are decoupled.

2. Controller normal form

$$\frac{dz}{dt} = Cz \tag{6.3}$$

with

$$C = \begin{bmatrix} 0 & 1 \\ c_1 & c_2 \end{bmatrix} \tag{6.4}$$

We associate systems (6.1) and (6.3) with an initial state $z(t_0) = z_0 = [z_{01}, z_{02}]^T$. We first consider some special cases.

6.1 Special cases

There exist two special cases for which linear normal forms can be deduced from the linearization without variable transformation. These cases are the following

(i) First case: the nonlinear system has the following form

$$\begin{array}{l} \frac{dx_1}{dt} = b_{11}x_1 + g_1(x_1) \\ \frac{dx_2}{dt} = b_{22}x_2 + g_2(x_2) \end{array} \tag{6.5}$$

with $g_1(0) = 0 = g_2(0)$ and $g_1'(0) = 0 = g_2'(0)$. This is a decoupled system, where the solutions for $x_1(t)$ and $x_2(t)$ are independent from each other. Clearly, Jordan

normal form can be obtained from the approximation directly by using equation (2.10). In this case the normal form is

$$\begin{array}{l} \lambda_1 = b_{11} + \frac{3}{x_{01}^3} \int_0^{x_{01}} g_1(x_1)\, x_1 dx_1 \\ \lambda_2 = b_{22} + \frac{3}{x_{02}^3} \int_0^{x_{02}} g_2(x_2)\, x_2 dx_2 \end{array} \tag{6.6}$$

For simplicity we write equation (6.6) as follows

$$\begin{array}{l} \lambda_1 = b_{11} + \tilde{a}_{11}(x_{01}) \\ \lambda_2 = b_{22} + \tilde{a}_{22}(x_{02}) \end{array} \tag{6.7}$$

(*ii*) Second case: the nonlinear system has the following form

$$\begin{array}{l} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = b_{21}x_1 + b_{22}x_2 + g_2(x_1, x_2) \end{array} \tag{6.8}$$

with $g_2(0,0) = 0$ and $\frac{\partial g_2(0,0)}{\partial x_1} = 0 = \frac{\partial g_2(0,0)}{\partial x_2}$. In this case, the controller normal form can be obtained directly from the linearization. By the minimization of the cost function J_2, we get for the normal form

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_{21} \\ b_{22} \end{bmatrix} + [D]^{-1} \begin{bmatrix} \int_0^{x_{01}} \int_0^{x_{02}} g_2(x_1, x_2)\, x_1 dx_2 dx_1 \\ \int_0^{x_{01}} \int_0^{x_{02}} g_2(x_1, x_2)\, x_2 dx_2 dx_1 \end{bmatrix} \tag{6.9}$$

where matrix D is given by (4.4). Similarly to equation (6.6), we can write system (6.9) as

$$\begin{array}{l} c_1 = b_{21} + \tilde{a}_{21}(x_{01}, x_{02}) \\ c_2 = b_{22} + \tilde{a}_{22}(x_{01}, x_{02}) \end{array} \tag{6.10}$$

Equations (6.5) and (6.8) can be seen as nonlinear normal forms, for which we associate linear normal forms. For these two particular cases, we have $z_0 = x_0$ (since no variable transformation is considered). Recall that for the linearization near the origin, the initial state x_{01} and x_{02} are chosen small. An important remark about equations (6.7) and (6.10) is that the normal forms depend on the initial states, where (λ_1, λ_2) and (c_1, c_2) are represented as the sum of two terms, namely b_{ij} which represent the linear part of the nonlinear vector and $\tilde{a}_{ij}$ which represent higher order terms. It is clear that normal forms obtained using the classical linearization include only the b_{ij} terms.

6.2 General case

Both normal forms can be obtained from matrix A using linear transformations. We suggest the following algorithm

1. Compute the L_2 norm linearization of the nonlinear system. The obtained system is $\frac{dx}{dt} = Ax = \left[B + \tilde{A}(x_0)\right] x$. As is already mentioned, the classical linearization is given by $\frac{dx}{dt} = DF(0)\, x = Bx$.

2. Using a linear transformation $x = Pz$, calculate the normal form based on matrix A. P is the similarity matrix. Similarity is an equivalence relation. Different standard techniques can be used for this purpose. The system under normal form is the following

$$\frac{dz}{dt} = P^{-1}APz \tag{6.11}$$

Matrix $P^{-1}AP$ can be either equal to matrix Λ or C. The transformation P determines the type of the normal form.
For Jordan normal form
Assume that matrix A has independent eigenvectors, then it is diagonalizable, which means that there exists an invertible matrix P such that $\Lambda = P^{-1}AP$, where matrix P is given by [9]

$$P = [v_1|v_2] \tag{6.12}$$

v_1 and v_2 are the eigenvectors of A.
For controller normal form
Assume that the eigenvalues of A are different, then the controller normal form is given by $C = P^{-1}AP$. Matrices C and A are invariant under similarity transformations [10]. Thus matrices C and A have the same eigenvalues, Let λ_1 and λ_2 be the eigenvalues of A, c_1 and c_2 are determined by solving the characteristic equation for C and using λ_1 and λ_2.

Remark. When matrix A has repeated eigenvalues with both algebraic and geometric multiplicity equal to 2, matrix Λ has a modified form

$$\Lambda = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{bmatrix} \tag{6.13}$$

In a similar way, if the eigenvalues of A are complex (in 2-D, the eigenvalues will be complex conjugate), then matrix Λ has the following form

$$\Lambda = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \tag{6.14}$$

with $\lambda_{1,2} = \alpha \mp \beta$.

Matrix $P^{-1}AP$ can be written as follows

$$P^{-1}AP = P^{-1}\left(B + \tilde{A}(x_0)\right)P \tag{6.15}$$

$$= P^{-1}BP + P^{-1}\tilde{A}(x_0)P \tag{6.16}$$

This transformation includes two terms. The first term $P^{-1}BP$ represents the linear normal form obtained from the classical linearization. The second term $P^{-1}\tilde{A}(x_0)P$ results from higher order terms in the nonlinear vector field. Thus the normal form obtained from the L_2 norm linearization can be seen as a generalization of the normal form obtained using the classical linearization. This was also shown in the special cases.

The construction of normal forms based on the L_2 norm linearization method is illustrated in the following examples.

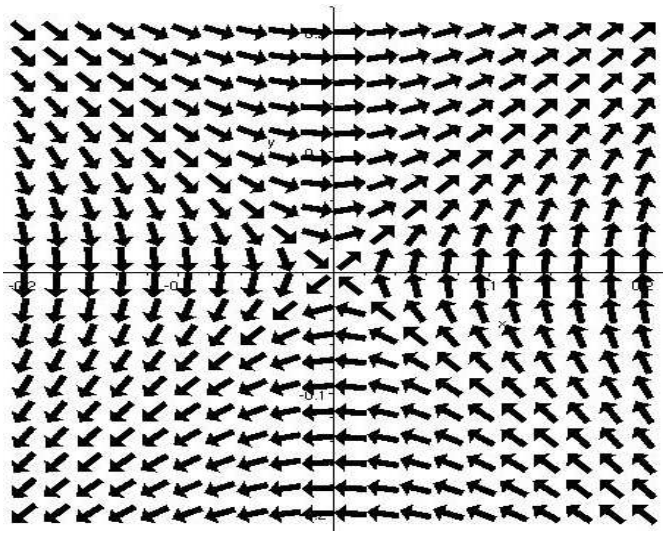

Figure 1. Vector field for Lienard equation near the origin (system 6.18)

Example. Local normal forms for Lienard equation
Consider Lienard equation, which arises in the study of nonlinear mechanics

$$\begin{array}{l} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = -h_1(x_1)\, x_2 - h_2(x_1) \end{array} \tag{6.17}$$

This system is already under nonlinear controller normal form. Our aim to show the equivalence with the linear form. For $h_1(x_1) = x_1^2, h_2(x_1) = -x_1$, the system becomes

$$\begin{array}{l} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = -x_1^2 x_2 + x_1 \end{array} \tag{6.18}$$

and the L_2 norm linearization is given by

$$A = \left[\begin{array}{cc} 0 & 1 \\ -\frac{2}{7} x_{02} x_{01} + 1 & -\frac{5}{42} x_{01}^2 \end{array} \right] \tag{6.19}$$

Matrix A represents also the linear controller normal form for the system. For an initial state equal to 1, we get

$$C = \left[\begin{array}{cc} 0 & 1 \\ -\frac{2}{7} + 1 & -\frac{5}{42} \end{array} \right], \Lambda = \left[\begin{array}{cc} 0.7877 & 0 \\ 0 & -0.9068 \end{array} \right] \tag{6.20}$$

A comparison between the nonlinear normal vector field and the linear normal vector field is shown in figures 1 and 2. Clearly both systems have a saddle structure. This can be also seen from matrix Λ.

Example. The following system represents a non-hyperbolic equilibrium point at the origin

$$\begin{array}{l} \frac{dx_1}{dt} = -x_1 - x_2 \\ \frac{dx_2}{dt} = -x_1 - x_2 - x_2^3 \end{array} \tag{6.21}$$

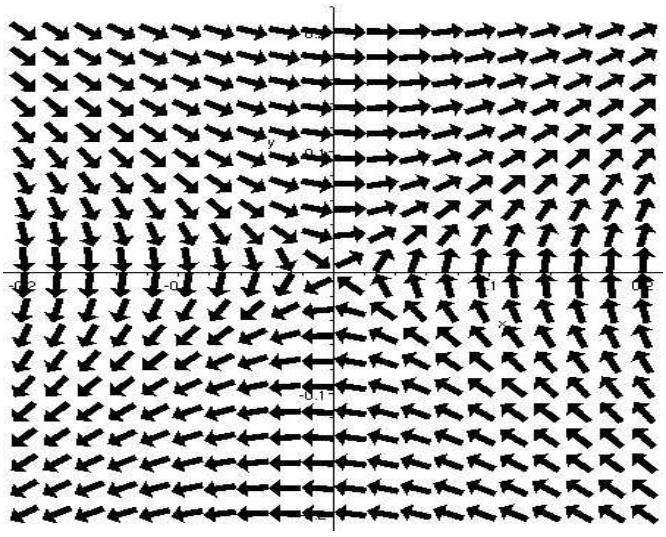

Figure 2. Vector field for the linear normal form of Lienard equation 6.18, for $x_{01} = x_{02} = 1$

We get for matrix A

$$A = \begin{bmatrix} -1 & -1 \\ -1 & -1 - \frac{3}{5}x_{02}^2 \end{bmatrix} \tag{6.22}$$

Normal forms obtained from matrix A for an initial state equal to 1 are as follows

$$C = \begin{bmatrix} 0 & 1 \\ -0.6 & -2.6 \end{bmatrix}, \Lambda = \begin{bmatrix} -2.3440 & 0 \\ 0 & -0.2560 \end{bmatrix} \tag{6.23}$$

In the case of non-hyperbolic equilibrium point, there is no equivalence between the classical linearized system and the nonlinear system, thus it is not possible to construct normal forms from the classical linearization. However this is possible using the L_2 norm linearization. The similarity between the nonlinear vector field and the L_2 linearized vector field is illustrated in figures 3 and 4.

In conclusion, the L_2 norm linearization allows us to construct approximate linear local normal forms, where numerical tools can be used to accomplish this task. These normal forms can be seen as a generalization of the normal forms obtained from the classical linearization, where higher order terms are taken into account. Also the L_2 norm linearization allows to obtain linear normal forms near a non-hyperbolic equilibrium point.

The method can be implemented numerically by integrating the nonlinear function. In the two-dimensional case, it is possible to numerically implement the method using a matrix language such as Matlab. Furthermore, it is possible to calculate analytically the linear approximation using symbolic languages such as Maple. By using a numerical algorithm, the method can be generalized easily to higher dimensional systems.

7 Numerical example

In this section we test the method and elaborate a comparison with the classical linearization. We consider a model for Laser emission.

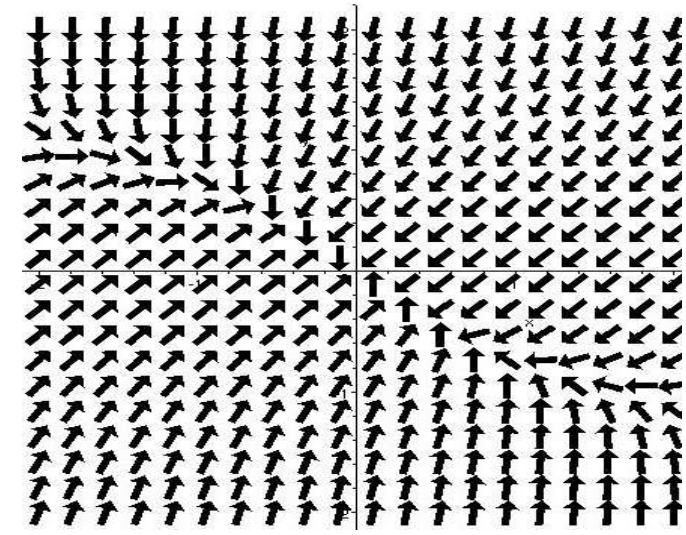

Figure 3. Vector field for system 6.21 near the origin

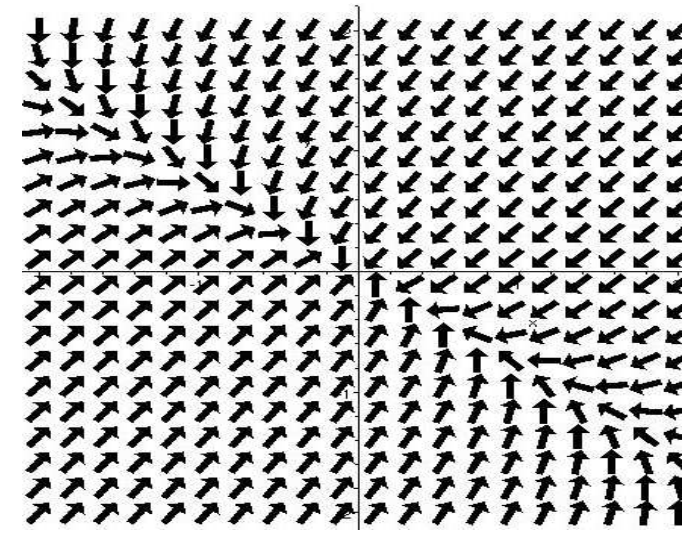

Figure 4. Linear vector field for system 6.21, for $x_{02} = 1$

7.1 Simplified Laser model

We consider a simplified model for Laser emission [1]. The state variable is the number of photons $p(t)$ in the Laser field, and its rate of change is given by the equation

$$\frac{dp}{dt} = gain - loss$$

$$\frac{dp}{dt} = GN(t)\,p - Kp \tag{7.1}$$

where G is the gain coefficient, $N(t)$ is the number of excited atoms, K is the inverse of the life time constant of photons in the laser. The actual number of the excited atoms depends on the number of photons in the Laser

$$N(t) = N_0 - \mu p(t) \tag{7.2}$$

Equation (7.1) becomes

$$\begin{array}{l} \dfrac{dp}{dt} = (GN_0 - K)\,p - \mu G p^2 \\ p(0) = p_0 \end{array} \tag{7.3}$$

We consider two cases. The first case corresponds to $N_0 < \dfrac{K}{G}$, in this case the Laser acts like a lamp, and the second case corresponds to $N_0 = \dfrac{K}{G}$ which represents a bifurcation point.

7.1.1 First case: $N_0 < \dfrac{K}{G}$

For system (7.3) the classical and the L_2 norm linearizations are respectively given by

$$\frac{dp}{dt} = (GN_0 - K)\,p; p(0) = p_0 \tag{7.4}$$

and

$$\frac{dp}{dt} = \left(GN_0 - K - \frac{3}{4}p_0\mu G\right)p; p(0) = p_0 \tag{7.5}$$

A comparison between (7.4) and (7.5) shows that the L_2 norm linearization includes more terms than the classical linearization. For the numerical computations we take $GN_0 - K = -3$, and $\mu G = 1$. Figure 5 shows the solutions, both approximations are in good agreement with the nonlinear system. Since it is difficult to express the error in closed form, it is natural to use numerical computations, figure 6 shows the plot for the relative error

$$error(t) = \frac{|\tilde{x}(t) - x(t)|}{|\tilde{x}(t)|}$$

as a function of time, where $\tilde{x}(t)$ is the nonlinear solution and $x(t)$ is the linear solution. From figure 6 we have the following remarks

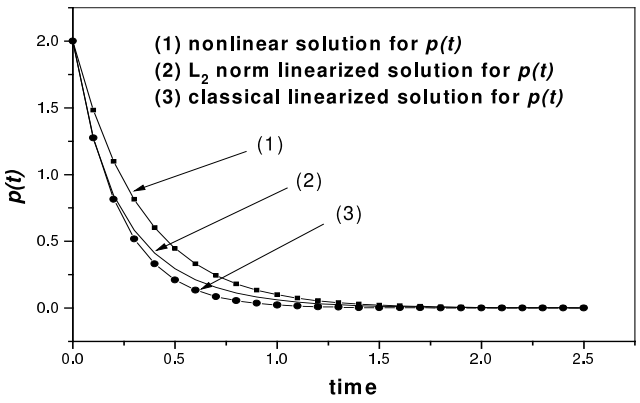

Figure 5. Solution for $p(t)$ when $N_0 < \dfrac{K}{G}$

1. The maximum error is about 2.5% for the L_2 norm linearization and 7.5% for the classical linearization.

2. The curves representing the error intersect at a critical point t_c, for $t < t_c$ the L_2 norm linearization presents better approximation, for $t > t_c$ the classical linearization is better.

3. For both approximations, the error tends to zero uniformly when time goes to infinity.

Figure 7 represents the error as a function of the state $p(t)$ and shows that the L_2 norm linearization gives better approximation for a wide range of $p(t)$ especially when $p(t)$ is close to p_0. The classical linearization gives better approximation when $p(t)$ is close to the origin, this confirms that the classical linearization is the best linear approximation near the origin.

7.1.2 Second case: $N_0 = \dfrac{K}{G}$

The origin is a non-hyperbolic point for the nonlinear system. Thus the classical linearization fails and the L_2 norm linearization gives the following system

$$\frac{dp}{dt} = \left[-\frac{3}{4}p_0\mu G\right] p; p(0) = p_0 \tag{7.6}$$

Since the origin is non-hyperbolic, the classical linearization does not reflect the behavior of the nonlinear system. As shown in figure 8, the L_2 norm linearization presents good approximation for the nonlinear system.

8 Conclusion

In this paper, we have presented an approximation for low order nonlinear autonomous ordinary differential equations. We used a simple optimization approach to derive the linear system. The main difference of the method with the classical linearization is its

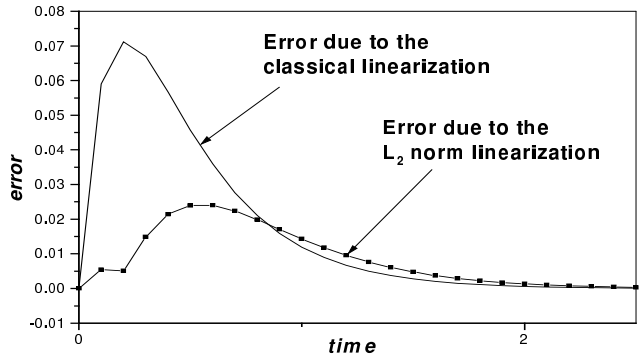

Figure 6. Error due to the approximations as a function of time

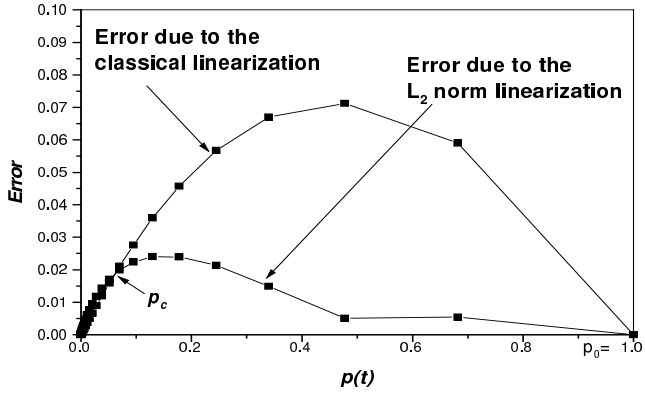

Figure 7. Error due to the approximations as a function of the solution

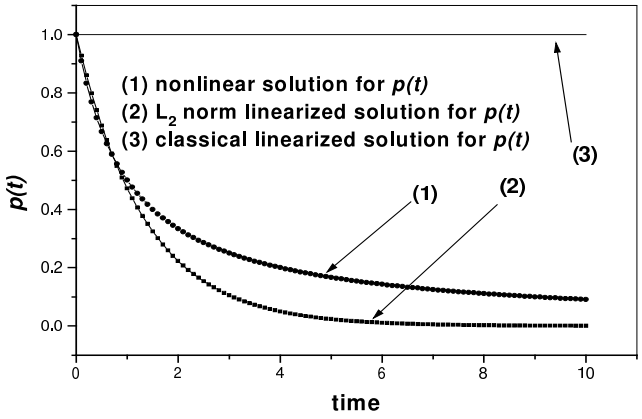

Figure 8. Solution for $p\,(t)$ when $N_0 = \dfrac{K}{G}$

dependence on the initial state; this makes the order of the method the same as the nonlinearity and allows the approximation of the nonlinear vector fields near a non-hyperbolic equilibrium. The method can be used to approximate the derivative at any equilibrium point. The construction of linear approximate normal forms using the method is also discussed. Simulation shows good agreement of the the linear solution with the nonlinear solution. Furthermore, a numerical comparison with the classical linearization shows that the L_2 norm linerization presents better approximation in a large neighborhood of the initial state.

References

[1] Strogatz H S, Nonlinear dynamics and chaos, with application to physics, biology, chemistry, and engineering, Addison-Wesley, 1995.

[2] Jordan A, Benmouna M, Bensenane A and Borucki A, Optimal Linearization Method Applied to the Resolution on Non-linear State Equations, RAIRO-AP II **21**, (1987), 175-185.

[3] Jordan A, Benmouna M, Bensenane A, and Borucki A, Optimal Linearization of Nonlinear State Equations. RAIRO-AP II **21**, (1987) 263-271.

[4] Benouaz T and Arino O, Determination of the Stability of a Nonlinear Ordinary Differential Equation by Least Square Approximation, Computational Procedure, Appl. Math. and Comp Sci. **5**, (1995), 33-48.

[5] Shi J, Global Topological Linearization in Critical Case, Nonlinear Analysis, **43**, (2001) 509-525.

[6] Terrell W J, Local Observability of Nonlinear Differential-Algebraic Equations DAEs From the Linearization Along Trajectory, IEEE Transaction on Autom. Contr. **46** (2001), 1947-1950.

[7] Wafo Soh C and Mohamed F M, Linearization Criteria for a System of Second-order Ordinary Differential Equations, Int. J. of Non-Linear Mech. **4**, 671-677.

[8] Wiggins S, Introduction to Applied Nonlinear Dynamical Systems and Chaos, Springer Verlag, New York, 1999.

[9] Dorsey J, Continuous and Discrete Control Systems, McGraw-Hill, New York, 2002.

[10] Dickinson B W, Matrices and Linear Algebra, in The Control Handbook, Editor: Levine W S, CRC Press, 1996.