Hermite-hadamard Type Inequalities for Harmonic-arithmetically Extended s-ε-Convex Functions

Ying Zheng*
Academic Periodicals Agency, Inner Mongolia University for Nationalities, Tongliao 028043, China
*Corresponding author

***Abstract*—In the paper, the authors introduce a new concept of harmonic-arithmetically extended s-ε-convex functions and establish some inequalities of Hermite-Hadamard type for this class of functions.**

Keywords—harmonic-arithmetically extended s-ε-convex function; Hermite-Hadamard type integral inequalities

I. Introduction

Definition 1.1

A function $f: I \subseteq R=(-\infty,+\infty) \to R$ is said to be convex function if

$$f(tx+(1-t)y) \le tf(x)+(1-t)f(y), \quad (1)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Definition 1.2[1] Let X is a real linear space, $D \subseteq X$ is convex set, and $\varepsilon \ge 0$. A function $f: D \to R$ is said to be ε-convex function on D if

$$f(tx+(1-t)y) \le tf(x)+(1-t)f(y)+\varepsilon, \quad (2)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

The concept of s-convex function was introduced in article [2]:

Definition 1.3[2] Let $s \in (0,1]$ be a real number. A function $f: I \subseteq R_0=[0,\infty) \to R$ is said to be s-convex if

$$f(tx+(1-t)y) \le t^s f(x)+(1-t)^s f(y), \quad (3)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

The concept of generalized s-convex function was introduced in article [3]:

Definition 1.4 [3] For some $s \in [-1,1]$

A function $f: I \subseteq R \to R$ is said to be extended s-convex function if

$$f(tx+(1-t)y) \le t^s f(x)+(1-t)^s f(y) \quad (4)$$

holds for all $x, y \in I$ and $t \in (0,1)$.

Definition 1.5[4] Let $m \in (0,1]$. A function $f:(0,b] \to R$ is said to be harmonic-arithmetically m-convex function if

$$f\left(\left(\frac{t}{x}+m\frac{1-t}{y}\right)^{-1}\right) \le tf(x)+m(1-t)f(y), \quad (5)$$

holds for all $x, y \in (0,b]$ and $t \in [0,1]$.

Definition 1.6[5] For some $s \in [-1,1]$, a function $f: I \subseteq R_+=(0,+\infty) \to R$ is said to be harmonic-arithmetically extended s-convex if

$$f\left(\left(\frac{t}{x}+\frac{(1-t)}{y}\right)^{-1}\right) \le t^s f(x)+(1-t)^s f(y), \quad (6)$$

holds for all $x, y \in I$ and $t \in (0,1)$. If the inequality (6) is reversed, then f is said to be harmonic-arithmetically extended s-concave function.Study of convex functions and the Hermite-Hadamard type integral inequalities have always been a very active research topic. First, in article [5], S. S. Dragomir give the Hermite-Hadamard type integral inequalities of convex functions, as follows:

Theorem 1.1([6. Theorem2.2 and 2.3]). Let $f: I \subseteq R \to R$ be a differentiable on I and $a,b\in I^{o}$, with $a<b$.

(i) If $|f'|$ is convex on $[a,b]$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\le\frac{(b-a)(|f'(a)|+|f'(b)|)}{8}; \quad (7)$$

(ii) If $|f'|^{p/(p-1)}$ is convex on $[a,b]$ for $p>1$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\le\frac{b-a}{2(p+1)^{1/p}}\left(\frac{|f'(a)|^{p/(p-1)}+|f'(b)|^{p/(p-1)}}{2}\right)^{(p-1)/p} \quad (8)$$

In article [7], U.S. Kirmaci proved the following inequality is established:

Theorem 1.2([7. Theorem2.3 and 2.4]) Let $f:I\subseteq R\to R$ be a differentiable on I and $a,b\in I^{o}$ with $a<b$. If $|f'|^{p}$ is convex on $[a,b]$ for $p>1$, then

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\le\frac{b-a}{4}\left(\frac{4}{p+1}\right)^{1/p}(|f'(a)|+|f'(b)|); \quad (9)$$

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\le\frac{b-a}{16}\left(\frac{4}{p+1}\right)^{1/p}\left[\left(|f'(a)|^{p/(p-1)}+3|f'(b)|^{p/(p-1)}\right)^{(p-1)/p}\right.$$
$$\left.+\left(3|f'(a)|^{p/(p-1)}+|f'(b)|^{p/(p-1)}\right)^{(p-1)/p}\right]. \quad (10)$$

In article [8] gives the following Hermite-Hadamard type integral inequality of s-convex function:

Theorem 1.3[8] Let $f:I\subseteq R_0\to R$ be differentiable on I^{o} and $a,b\in I$ with $a<b$. If $|f'|^{q}$ is s-convex on $[a,b]$ for some fixed $s\in(0,1]$ and $q\ge1$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|$$
$$\le\frac{b-a}{2}\left(\frac{1}{2}\right)^{1-1/q}\left[\frac{2+1/2^{s}}{(s+1)(s+2)}\right]^{1/q}\left[|f'(a)|^{q}+|f'(b)|^{q}\right]. \quad (11)$$

The main purpose of this paper is to introduce the concept of "harmonic-arithmetically extended s-ε- convex functions" to establish some new Hermite-Hadamard type inequalities for these classes of functions.

II. DEFINITION AND LEMMA

Now we introduce the concept of harmonic -arithmetically extended s-ε- convex functions :

Definition 2.1

For some $s\in[-1,1]$ and $\varepsilon\ge0$, a function $f:I\subseteq R_{+}=(0,+\infty)\to R$ is said to be harmonic-arithmetically extended s-ε- convex if

$$f\left(\left(\frac{t}{x}+\frac{(1-t)}{y}\right)^{-1}\right)\le t^{s}f(x)+(1-t)^{s}f(y), \quad (12)$$

holds for all $x,y\in I$ and $t\in(0,1)$. If the inequality (12) is reversed, then f is said to be harmonic-arithmetically extended s-ε- concave function.

Lemma 2.1 Let $f:I\subseteq R_{+}\to R$ be a differentiable function on I with $a,b\in I$, $a<b$. If $f'\in L_1([a,b])$, then

$$f(H(a,b))-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathrm{d}x$$
$$=\frac{b-a}{4ab}\int_0^1\left[(1-t)\left(ta^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-2}f'\left(\left(ta^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-1}\right)\right.$$
$$\left.-(1-t)\left(tb^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-2}f'\left(\left(tb^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-1}\right)\right]\mathrm{d}t, \quad (13)$$

where. $H(a,b)=\dfrac{2ab}{a+b}$

Proof. Let $x=\left(ta^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-1}$ for $t\in[0,1]$, then

$$\int_0^1(1-t)\left(ta^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-2}f'\left(\left(ta^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-1}\right)\mathrm{d}t$$
$$=\frac{2ab}{b-a}f(H(a,b))-\left(\frac{2ab}{b-a}\right)^{2}\int_a^{H(a,b)}\frac{f(x)}{x^2}\mathrm{d}x.$$

Similarly, letting $x=\left(tb^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-1}$ for all $t\in[0,1]$, gives

$$\int_0^1(1-t)\left(tb^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-2}f'\left(\left(tb^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-1}\right)\mathrm{d}t$$
$$=\frac{2ab}{b-a}f\left([H(a,b)]^{-1}\right)-\left(\frac{2ab}{b-a}\right)^{2}\int_{H(a,b)}^{b}\frac{f(x)}{x^2}\mathrm{d}x.$$

Adding these two equalities leads to Lemma 2.1.

Lemma 2.1 Let $s > -1.u,v > 0$, with $u \neq v$, then

$$H(u,v) \triangleq \int_0^1 (1-t)\left(tu^{-1}+(1-t)v^{-1}\right)^{-2} dt = \frac{u^2 v[v(\ln v-\ln u)-(v-u)]}{(v-u)^2},$$

$$S(u,v,s) \triangleq \int_0^1 (1-t)t^s\left(tu+(1-t)v\right)^2 dt = \frac{(s+1)(s+2)u^2+4(s+1)uv+6v^2}{(s+1)(s+2)(s+3)(s+4)},$$

$$T(u,v,s) \triangleq \int_0^1 (1-t)^{s+1}\left(tu+(1-t)v\right)^2 dt = \frac{2u^2+2(s+2)uv+(s+2)(s+3)v^2}{(s+2)(s+3)(s+4)}.$$

III. Main Results

Theorem 3.1 Let $s \in (-1,1]$ and $\varepsilon \geq 0$, $f: I \subseteq R_+ \to R$ be differentiable on I, $a,b \in I$ with $a<b$, $f' \in L_1([a,b])$,

If $|f'|^q$ is harmonic-arithmetically extended s-ε- convex on $[a,b]$ for $q \geq 1$, then

$$\left| f(H(a,b)) - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \right| \leq \frac{b-a}{4ab}\Big\{[H(a,H(a,b))]^{(q-1)/q}$$
$$\times\left[S(a,H(a,b);s)\left|f'(a)\right|^q + T(a,H(a,b);s)\left|f'(H(a,b))\right|^q + \varepsilon H(a,H(a,b))\right]^{1/q}$$
$$+[H(b,H(a,b))]^{(q-1)/q}$$
$$\times\left[S(b,H(a,b);s)\left|f'(b)\right|^q + T(b,H(a,b);s)\left|f'(H(a,b))\right|^q + \varepsilon H(b,H(a,b))\right]^{1/q}\Big\}, \quad (14)$$

where $H(u,v), S(u,v;s), T(u,v;s)$ is defined by lemma 2.2.

Proof. Since Lemma 2.1 and Hölder inequality, then

$$\left| f(H(a,b)) - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \right|$$
$$\leq \frac{b-a}{4ab}\Big\{\left[\int_0^1 (1-t)\left(ta^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-2}dt\right]^{(q-1)/q}$$
$$\times\left[\int_0^1 (1-t)\left(ta^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-2}\left|f'\left([ta^{-1}+(1-t)[H(a,b)]^{-1}]^{-1}\right)\right|^q dt\right]^{1/q}$$
$$+\left[\int_0^1 (1-t)\left(tb^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-2}dt\right]^{(q-1)/q}$$
$$\times\left[\int_0^1 (1-t)\left(tb^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-2}\left|f'\left([tb^{-1}+(1-t)[H(a,b)]^{-1}]^{-1}\right)\right|^q dt\right]^{1/q}\Big\}. \quad (15)$$

Using Lemma2.2, then

$$\int_0^1 (1-t)\left(ta^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-2}dt = H(a,H(a,b)),$$
$$\int_0^1 (1-t)\left(tb^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-2}dt = H(b,H(a,b)).$$

By the harmonic-arithmetically extended s-ε- convexity of function $|f'|^q$ on $[a,b]$, fundamental inequality and Lemma 2.2, then following (16) and (17)

$$\int_0^1 (1-t)\left(ta^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-2}\left|f'\left([ta^{-1}+(1-t)[H(a,b)]^{-1}]^{-1}\right)\right|^q dt$$
$$\leq \int_0^1 (1-t)\left(ta^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-2}\left(t^s\left|f'(a)\right|^q+(1-t)^s\left|f'(H(a,b))\right|^q+\varepsilon\right)dt$$
$$\leq \int_0^1 (1-t)\left(ta+(1-t)[H(a,b)]\right)^2\left(t^s\left|f'(a)\right|^q+(1-t)^s\left|f'(H(a,b))\right|^q\right)dt+\varepsilon H(a,H(a,b)) \quad (16)$$
$$= S(a,H(a,b);s)\left|f'(a)\right|^q + T(a,H(a,b);s)\left|f'(H(a,b))\right|^q + \varepsilon H(a,H(a,b)),$$

$$\int_0^1 (1-t)\left(tb^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-2}\left|f'\left([tb^{-1}+(1-t)[H(a,b)]^{-1}]^{-1}\right)\right|^q dt$$
$$\leq \int_0^1 (1-t)\left(tb^{-1}+(1-t)[H(a,b)]^{-1}\right)^{-2}\left(t^s\left|f'(b)\right|^q+(1-t)^s\left|f'(H(a,b))\right|^q+\varepsilon\right)dt$$
$$\leq \int_0^1 (1-t)\left(tb+(1-t)[H(a,b)]\right)^2\left(t^s\left|f'(b)\right|^q+(1-t)^s\left|f'(H(a,b))\right|^q\right)dt+\varepsilon H(b,H(a,b))$$
$$= S(b,H(a,b);s)\left|f'(b)\right|^q + T(b,H(a,b);s)\left|f'(H(a,b))\right|^q + \varepsilon H(b,H(a,b)). \quad (17)$$

A combination of (15) to (17) gives the required inequality (14).

Corollary 3.1.1 Under the assumptions of Theorem 3.1,when $q=1$,we have

$$\left| f(H(a,b)) - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \right| \leq \frac{b-a}{4ab}\Big\{S(a,H(a,b);s)\left|f'(a)\right|$$
$$+S(b,H(a,b);s)\left|f'(b)\right| + \left[T(a,H(a,b);s)+T(b,H(a,b);s)\right]\left|f'(H(a,b))\right|$$
$$+\varepsilon\left[H(a,H(a,b))+H(b,H(a,b))\right]\Big\}.$$

Corollary 3.1.2 Under the assumptions of Theorem 3.1,if $q=1$ and $s=1$,then

$$\left| f(H(a,b)) - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \right| \leq \frac{b-a}{240ab}$$
$$\times\Big\{\left[6a^2+8aH(a,b)+6[H(a,b)]^2\right]\left|f'(a)\right| + \left[6b^2+8bH(a,b)+6[H(a,b)]^2\right]\left|f'(b)\right|$$
$$+2\left[a^2+b^2+3(a+b)H(a,b)+12[H(a,b)]^2\right]\left|f'(H(a,b))\right|$$
$$+60\varepsilon\left[H(a,H(a,b))+H(b,H(a,b))\right]\Big\}.$$

Theorem 3.1bb Let $s \in (-1,1]$ and $\varepsilon \geq 0$, $f: I \subseteq R_+ \to R$ be differentiable on I, $a,b \in I$ with

$a<b$, $f'\in L_1([a,b])$, If $|f'|^q$ is harmonic-arithmetically extended s-ε- convex on $[a,b]$ for $q>1$, then

$$\begin{aligned}&\left|f(H(a,b))-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathrm{d}x\right|\le\frac{b-a}{4ab}\left[\frac{(q-1)}{2(2q+1)(2q+1)}\right]^{1-1/q}\\&\times\left\{\left[\frac{(q-1)a^{\frac{2(2q-1)}{q-1}}+[(3q-1)H(a,b)-2(2q-1)a][H(a,b)]^{\frac{3q-1}{q-1}}}{[H(a,b)-a]^2}\right]^{1-1/q}\right.\\&\times\left(\frac{1}{(s+1)(s+2)}|f'(a)|^q+\frac{1}{s+2}|f'(H(a,b))|^q\right)^{1/q}\\&+\left[\frac{(q-1)b^{\frac{2(2q-1)}{q-1}}+[(3q-1)H(a,b)-2(2q-1)b][H(a,b)]^{\frac{3q-1}{q-1}}}{[b-H(a,b)]^2}\right]^{1-1/q}\\&\left.\times\left(\frac{1}{(s+1)(s+2)}|f'(b)|^q+\frac{1}{(s+1)(s+2)}|f'(H(a,b))|^q\right)^{1/q}\right\}.\end{aligned}\tag{18}$$

Proof. Since Lemma 2.1 and Hölder inequality, then

$$\begin{aligned}&\left|f(H(a,b))-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathrm{d}x\right|\\&\le\frac{b-a}{4ab}\left\{\left(\int_0^1(1-t)[ta^{-1}+(1-t)[H(a,b)]^{-1}]^{-2q/(q-1)}\mathrm{d}t\right)^{1-1/q}\right.\\&\times\left(\int_0^1(1-t)\left|f'\left([ta^{-1}+(1-t)[H(a,b)]^{-1}]^{-1}\right)\right|^q\mathrm{d}t\right)^{1/q}\\&+\left(\int_0^1(1-t)[tb^{-1}+(1-t)[H(a,b)]^{-1}]^{-2q/(q-1)}\mathrm{d}t\right)^{1-1/q}\\&\left.\times\left(\int_0^1(1-t)\left|f'\left([tb^{-1}+(1-t)[H(a,b)]^{-1}]^{-1}\right)\right|^q\mathrm{d}t\right)^{1/q}\right\}.\end{aligned}\tag{19}$$

From the harmonic-arithmetically extended s-ε- convexity of function $|f'|^q$, we obtain

$$\begin{aligned}&\int_0^1(1-t)\left|f'\left([ta^{-1}+(1-t)[H(a,b)]^{-1}]^{-1}\right)\right|^q\mathrm{d}t\\&\le\int_0^1(1-t)\left(t^s|f'(a)|^q+(1-t)^s|f'(H(a,b))|^q\right)\mathrm{d}t\\&=\frac{1}{(s+1)(s+2)}|f'(a)|^q+\frac{1}{s+2}|f'(H(a,b))|^q,\end{aligned}\tag{20}$$

$$\begin{aligned}&\int_0^1(1-t)\left|f'\left([tb^{-1}+(1-t)[H(a,b)]^{-1}]^{-1}\right)\right|^q\mathrm{d}t\\&=\int_0^1(1-t)\left(t^s|f'(b)|^q+(1-t)^s|f'(H(a,b))|^q\right)\mathrm{d}t\end{aligned}$$

$$\begin{aligned}&\int_0^1(1-t)\left|f'\left([ta^{-1}+(1-t)[H(a,b)]^{-1}]^{-1}\right)\right|^q\mathrm{d}t\\&=\frac{1}{(s+1)(s+2)}|f'(b)|^q+\frac{1}{s+2}|f'(H(a,b))|^q.\end{aligned}\tag{21}$$

$$\begin{aligned}&\int_0^1(1-t)[ta^{-1}+(1-t)[H(a,b)]^{-1}]^{-2q/(q-1)}\mathrm{d}t\\&\le\int_0^1(1-t)[ta+(1-t)H(a,b)]^{2q/(q-1)}\mathrm{d}t\\&=\frac{(q-1)\left[(q-1)a^{\frac{2(2q-1)}{q-1}}+[(3q-1)H(a,b)-2(2q-1)a][H(a,b)]^{\frac{3q-1}{q-1}}\right]}{2(2q+1)(2q+1)[H(a,b)-a]^2}\end{aligned}\tag{22}$$

and

$$\begin{aligned}&\int_0^1(1-t)[tb^{-1}+(1-t)[H(a,b)]^{-1}]^{-2q/(q-1)}\mathrm{d}t\\&\le\int_0^1(1-t)[tb+(1-t)H(a,b)]^{2q/(q-1)}\mathrm{d}t\\&=\frac{(q-1)\left[(q-1)b^{\frac{2(2q-1)}{q-1}}+[(3q-1)H(a,b)-2(2q-1)b][H(a,b)]^{\frac{3q-1}{q-1}}\right]}{2(2q+1)(2q+1)[b-H(a,b)]^2}.\end{aligned}\tag{23}$$

A combination of (19) to (23) gives the required inequality (18).

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