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Modulus-based Matrix Multi-splitting Multi-Parameter Methods for a Class of Weakly Nonlinear Complementarity Problem

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***Abstract*—The modulus-based matrix multi-splitting multi-parameter methods for solving a class of weakly nonlinear complementarity problem are presented. The weaker convergence conditions are analyzed. The numerical results show that the new methods are efficient.**

***Keywords*—weakly nonlinear complementarity problem; modulus-based matrix multi-splitting; multi-parameter**

I. Introduction

In this paper, we consider the following weakly nonlinear complementarity problem: for $N \in R^{n\times n}$, $q \in R^n$, finding a pair of vectors u, $w \in R^n$ such that

$$w = Nu + q + m(u) \geq 0,\ w \geq 0,\ u^T\left(Nu + q + m(u)\right) = 0,$$

where the nonlinear term $m(u) = (\ m_1(u_1)\ ,\ m_2(u_2),\ \ldots,\ m_i(u_i)\ )^T$, and $m_i(u_i)$ is a real function of a domain in real numbers, and satisfies that $\frac{\partial m_i}{\partial u_i} \geq 0$.

The complementarity problem has been widely used in the research of operation research, optimal control, engineering problems, economics and mechanics, etc. A series of methods is proposed to solve the linear complementarity problem; see [2,3,4,5,9]. By splitting the coefficient matrix, the modulus-based matrix splitting iterative methods are proposed see [6,7,8]. Hong [8] and Li [11] proposed a class of modulus-based iterative methods for solving the implicit complementarity problems. Zhang [13] summarized the development of the modulus-based matrix splitting iterative methods. Li [10] proposed the multi-splitting iterative methods for solving a class of nonlinear complementarity problems. Zheng [16,17] developed the accelerated modulus-based matrix splitting iteration method for the linear complementarity problem. Xia and Li [12] applied the modulus-based matrix splitting iterative methods to solving the nonlinear complementarity problem. Zhang[14] presented modulus-based synchronous multi-splitting multi-parameters methods for linear complementarity problems, and obtained the weaker convergence conditions of the new method. Recently, Zhang [15] further studied modulus-based synchronous block multi-splitting multi-parameter methods for linear complementarity problems, and provide the wider convergence results and the optimal relaxation parame-ters.

In this paper, we present the modulus-based multi-splitting multi-parameter accelerated overrelaxation iterative methods (MSMMAOR) to solving the weakly nonlinear complement-arity problem (1), and give the weaker convergence conditions.

The rest of the paper is as follows. We give some necessary notations and lemmas in Section 2, and propose the modulus-based synchronous multi-splitting multi-parameter accelerated overrelaxation iterative methods for solving the weakly nonlinear complementarity in Section 3. The convergence of the new methods is proved in Section 4. The numerical results are shown in Section 5, and some concluding remarks are given in Section 6.

II. Preliminaries

Given the matrix $N = (n_{ij}) \in R^{n\times n}$, its comparison matrix $\langle N\rangle = (\langle n_{ij}\rangle) \in R^{n\times n}$ is defined by

$$\langle n_{ij}\rangle = \begin{cases} |n_{ij}|, & i = j \\ -|n_{ij}|, & i \neq j \end{cases},\ i, j = 1, 2, \ldots, n.$$

A matrix N is called a Z-matrix if all of its off-diagonal entries are nonpositive. The set of all Z-matrix is denoted as $Z^{n\times n}$. A nonsingular matrix called an M-matrix if N is a Z-matrix and $N^{-1} \geq 0$. A matrix is an H-matrix if its comparison matrix $\langle N\rangle$ is an M-matrix, and it is an H_+-matrix if it is an H-matrix with positive diagonal entries. Let N, $L \in R^{n\times n}$ be M-matrices, $\Omega \in R^{n\times n}$ be a positive matrix, and $C\ \in R^{n\times n}$. Then, $N \leq L$ implies $L^{-1} \leq N^{-1}$, and $N \leq C \leq \Omega$ implies that C is an M-matrix. Let $N = Q - W$ is called a splitting of the matrix N if Q is nonsingular. The splitting $N = Q - W$ is called an H- compatible splitting if $\langle N\rangle = \langle Q\rangle - |W|$.

Lemma2.1 ([1])

Let $N \in Z^{n\times n}$ which has all positive diagonal entries. N is an M- matrix if and only if $\rho\left(D^{-1}B\right)<1$, where $D = diag\left(N\right)$, $B = D - N$.

Lemma2.2 ([1])

Let $N \in R^{n\times n}$ be an H- matrix, $D = diag\left(N\right)$ be the diagonal matrix of N, and $B = D - N$. Then, the following statements hold true:

1. N is nonsingular;
2. $N^{-1} \le \langle N\rangle^{-1}$;
3. $|D|$ is nonsingular and $\rho\left(D^{-1}B\right)<1$.

Lemma2.3 (Perron-Frobenius Theorem)

Let $N \in R^{n\times n}$ be a irreducible nonnegative matrix, then

1. The spectral radius $\rho\left(N\right)$ is a positive real eigenvalue of matrix N;
2. The eigenvector x of $\rho\left(N\right)$ is a positive vector;
3. $\rho\left(N\right)$ is a simple eigenvalue of N;
4. If any element of matrix N is increased, $\rho\left(N\right)$ increases.

Lemma 2.4 ([12])

Let $N = Q - W$ be a splitting of the matrix $N \in R^{n\times n}$, and Ω be a positive diagonal matrix. Then, for the nonlinear complementarity problem (1), the following statements hold true:

(a) If $\left(u, w\right)$ is a solution of the nonlinear complementarity problem (1), then $x = \frac{\gamma}{2}\left(u - \Omega^{-1}\cdot\ w\right)$ satisfies the implicit fixed-point equation

$$\left(\Omega + Q\right)x = Wx + \left(\Omega - N\right)|x| - \gamma\left[q + m\left(\frac{1}{\gamma}\left(|x| + x\right)\right)\right]. \quad (2)$$

(b)If x satisfies the implicit fixed-point equation (2), then

$$u = \frac{1}{\gamma}\left(|x| + x\right), \quad w = \frac{1}{\gamma}\Omega\left(|x| - x\right)$$

are solutions of the weakly nonlinear complementarity problem (1).

III. Modulus-based Multi-splitting Multiparameter Accelerated Over-relaxation Iterative Method

Let ℓ $\left(\ell \le n\right)$ be a given positive integer and $N \in R^{n\times n}$. Let L_k be a strictly lower-triangular matrix and $U_k = D - L_k - N$ for $k = 1, 2, \ldots, \ell$, where $D = diag\left(N\right)$. Note that U_k has zero diagonal but needs not be upper triangular. Let the weighting matrices $E_k \in R^{n\times n}$ $\left(k = 1, \ldots, \ell\right)$ be nonnegative diagonal matrices satisfying $\sum_{k=1}^{\ell} E_k = I$ (the identity matrix). Then, the collection of triples $\left(D - L_k, U_k, E_k\right)$ $\left(k = 1, 2, \cdots, \ell\right)$ is called a triangular multi-splitting of the matrix N.

Let $N = Q_k - W_k$ be a multi-splitting of $N \in R^{n\times n}$,

$$\begin{cases} Q_k = \frac{1}{\alpha}\left(D - \beta L_k\right) \\ W_k = \frac{1}{\alpha}\left[\left(1-\alpha\right)D + \left(\alpha - \beta\right)L_k + \alpha U_k\right] \end{cases} \quad k = 1, 2, \ldots, \ell,$$

then from the system of equations (2), we get

$$\begin{aligned} \left(\alpha\Omega + D - \beta L_k\right)x &= \left[\left(1-\alpha\right)D + \left(\alpha-\beta\right)L_k + \alpha U_k\right]x \\ &+ \alpha\left(\Omega - N\right)|x| - \alpha\gamma\left[q + m\left(\frac{1}{\gamma}\left(|x| + x\right)\right)\right] \end{aligned} \quad (3)$$

The following is the MSMAOR method for solving the weakly nonlinear complementarity problem(1).

***Method 3.1*(MSMAOR method)**

Step 1: Given $x^{(0)} \in R^n$, and $\varepsilon > 0$, set $m := 0$;

Step 2: On ℓ different processors corresponding fixed point system, respectively.

$$\begin{aligned} \left(\alpha\Omega + D - \beta L_k\right)x^{(m+1,k)} &= \left[\left(1-\alpha\right)D + \left(\alpha-\beta\right)L_k + \alpha U_k\right]x^{(m)} \\ &+ \alpha\left(\Omega - N\right)\left|x^{(m)}\right| - \alpha\gamma\left[q + m\left(\frac{1}{\gamma}\left(\left|x^{(m)}\right| + x^{(m)}\right)\right)\right]; \end{aligned}$$

Step 3: Let

$$x^{(m+1)} = \sum_{k=1}^{\ell} E_k x^{(m+1,k)}, \text{ ανδ } u^{(m+1)} = \frac{1}{\gamma}\left(\left|x^{(m+1)}\right| + x^{(m+1)}\right).$$

If $RES\left(u^{(m+1)}\right) < \varepsilon$, then stop; otherwise, set $m := m+1$, and return to Step 2.

Let $\alpha = \alpha_k$, $\beta = \beta_k$, the following is MSMMAOR method for solving weakly nonlinear complementarity problem.

***Method 3.2*(MSMMAOR method)**

Step 1: Given $x^{(0)} \in R^n$, and $\varepsilon > 0$, set $m := 0$;

Step 2: On ℓ different processors corresponding fixed point system, respectively.

$$\left(\alpha_k \Omega + D - \beta_k L_k\right) x^{(m+1,k)} = \Big[\left(1-\alpha_k\right) D + \left(\alpha_k - \beta_k\right) L_k + \alpha_k U_k\Big] x^{(m)} + \alpha_k (\Omega - N)\left|x^{(m)}\right| - \alpha_k \gamma \left[q + m\left(\frac{1}{\gamma}\left(\left|x^{(m)}\right| + x^{(m)}\right)\right)\right];$$

Step 3: Let

$$x^{(m+1)} = \sum_{k=1}^{\ell} E_k x^{(m+1,k)} \text{, αvδ } u^{(m+1)} = \frac{1}{\gamma}\left(\left|x^{(m+1)}\right| + x^{(m+1)}\right).$$

If $RES\left(u^{(m+1)}\right) < \varepsilon$, then stop; otherwise, set $m := m+1$, and return to Step 2.

IV. CONVERGENCE

In this section, we prove the convergence of MSMAOR and MSMMAOR. We assume that $\left(u^*, w^*\right)$ is a solution of the nonlinear complementarity problem (1). By Lemma 2.4, we know that $x^* = \frac{\gamma}{2}\left(u^* - \Omega^{-1} w^*\right)$ satisfies the implicit fixed-point equation (2).

Let N be an H_+- matrix, with the notations $D = diag(N)$ and $B = D - N$, by Lemma 2.2, we see that $\rho\left(\left|D^{-1}\right|\left|B\right|\right) < 1$. Let $J = D^{-1}|B|$ and $J_\varepsilon = J + \varepsilon e e^T$, where $e = (1, \ldots, 1)^T \in R^n$ and $\varepsilon > 0$. Then, $\langle N \rangle = D(I - J)$, and by Lemma 2.3, there exists a positive vector such that $J_\varepsilon v_\varepsilon = \rho_\varepsilon v_\varepsilon$.

Theorem 4.1

Let $N \in R^{n \times n}$ be an H_+- matrix, with $D = diag(N)$ and $B = D - N$. Let $\left(D - L_k, U_k, E_k\right)$ $\left(k = 1, 2, \cdots, \ell\right)$ be a triangular multi-splitting of the matrix N , and $N = D - L_k - U_k$ satisfies $\langle N \rangle = D - \left|L_k\right| - \left|U_k\right|$. Assume that $\Omega \ge D + M_n$ is a positive diagonal matrix, γ is a positive constant, $m(u)$ satisfies that $\frac{\partial m_i}{\partial u_i} \ge 0$, and $M_n = diag\left(\frac{\partial m_1}{\partial u_1}, \frac{\partial m_2}{\partial u_2}, \cdots, \frac{\partial m_n}{\partial u_n}\right)$. Then, the iteration sequence $\left\{u^{(m)}\right\}_{m=1}^{\infty} \subset R_+^n$ generated by MSMAOR iteration method converges to theunique solution u^* of the problem (1) for any initial vector $x^{(0)} \in R^n$, provided that the relaxation parameters α and β satisfy

$$0 < \beta \le \alpha < \frac{1}{\rho\left(D^{-1}|B|\right)}.$$

Proof

By Lemma 2.4 , Lemma 2.2, and Lemma 2.3,

$$\left|x^{(m+1)} - x^*\right| \le L_{MSMAOR}(\alpha, \ \beta)\left|x^{(m)} - x^*\right|$$

where

$$L_{MSMAOR} = \sum_{k=1}^{\ell} E_k\left(\alpha\Omega + D - \beta\left|L_k\right|\right)^{-1}\Big[\left|1-\alpha\right| D + (\alpha - \beta)\left|L_k\right| + \alpha\left|U_k\right|\Big] + \sum_{k=1}^{\ell} E_k\left(\alpha\Omega + D - \beta\left|L_k\right|\right)^{-1}\Big[\alpha\left|Mn\right| + \alpha(\Omega - D - Mn) + \alpha|B|\Big]$$

As $0 < \alpha < \frac{1}{\rho\left(D^{-1}|B|\right)}$, We have $L_{MSMAOR}(\alpha, \ \beta) < 1$. This completes the proof.

For MSMMAOR, we give the following convergence result.

Theorem 4.2

Let $N \in R^{n \times n}$ be an H_+- matrix, with $D = diag(N)$ and $B = D - N$. Let $\left(D - L_k, U_k, E_k\right)$ $\left(k = 1, 2, \cdots, \ell\right)$ be a triangular multi-splitting of the matrix N , and $N = D - L_k - U_k$ satisfies $\langle N \rangle = D - \left|L_k\right| - \left|U_k\right|$. Assume that $\Omega \ge \max\left\{\frac{\beta_k\text{-}1}{\alpha_k} D, D\right\} + M_n$ is a positive diagonal matrix, γ is a positive constant, $m(u)$ satisfies that $\frac{\partial m_i}{\partial u_i} \ge 0$, and $M_n = diag\left(\frac{\partial m_1}{\partial u_1}, \frac{\partial m_2}{\partial u_2}, \cdots, \frac{\partial m_n}{\partial u_n}\right)$. Then,

the iteration sequence $\left\{u^{(m)}\right\}_{m=1}^{\infty} \subset R_+^n$ generated by the MSMMAOR iteration method converges to the unique solution u^* of the problem(1) for any initial vector $x^{(0)} \in R^n$, provided the relaxation parameters α_k and β_k $\left(k=1,2,\ldots,\ell\right)$ satisfy

$$0<\beta_k \le \alpha_k \le 1 \text{ or}$$

$$0<\beta_k<\frac{1}{\rho\left(D^{-1}|B|\right)},\ 1<\alpha_k<\frac{1}{\rho\left(D^{-1}|B|\right)}.$$

V. NUMERICAL RESULTS

In this section, we use an example to test the numerical effectiveness of Method1. Let $RES\left(\quad \min\left(z,w\right)\right) \le 10^{-5}$, $n \in Z^+$ be the order of the matrix N, d denote the iteration steps, and CPU denote the iteration time.

Example 5.1.

Consider the following nonlinear complementarity problem (1). I is a unit matrix.Let $N \in \mathbb{R}^{n\times n}$ is an H_+- matrix,

$$N=\begin{pmatrix} B & -0.5I & O & \cdots & O & O \\ -1.5I & B & -0.5I & \cdots & O & O \\ O & -1.5I & B & \cdots & O & O \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ O & O & \cdots & \cdots & B & -0.5I \\ O & O & \cdots & \cdots & -1.5I & B \end{pmatrix}$$

where

$$B=\begin{pmatrix} 4 & -1 & O & \cdots & O & O \\ -1 & 4 & -1 & \cdots & O & O \\ O & -1 & 4 & \cdots & O & O \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ O & O & \cdots & \cdots & 4 & -1 \\ O & O & \cdots & \cdots & -1 & 4 \end{pmatrix},$$

and $q=\left(1,-1,\ldots 1,-1\right)^T$, $m(u)=\left(u_1^3,u_2^3,\ldots,u_{n-1}^3,u_n^3\right)^T$.

The numerical results of MSMAOR method are shown in TABLE I, and the numerical results of MSMMAOR method are shown in TABLE II.

TABLE I. THE NUMERICAL RESULTS OF MSMAOR METHOD FOR EXAMPLE 5.1

MSMAOR		n=2500		n=3600	
(α, β)	ℓ	IT	CPU	IT	CPU
(1.07, 1.01)	1	15	0.289	15	0.605
	2	13	0.131	17	0.360
	3	18	0.096	18	0.273
	4	17	0.071	18	0.195

TABLE II. THE NUMERICAL RESULTS OF MSMMAOR METHOD FOR EXAMPLE 5.1

MSMMAOR		n=2500		n=3600	
(α_k, β_k)	ℓ	IT	CPU	IT	CPU
(1, 0)	1	22	0.451	22	0.984
(0.9, 0.8) (1, 1)	2	14	0.121	14	0.353
(0.9, 0.8) (1, 1) (0.9, 0.9)	3	14	0.072	14	0.189
(0.9, 0.8) (0.9, 0.9) (1, 1) (1, 0)	4	14	0.054	14	0.112

From Tables I and II, we observe that the convergence conditions of Theorem 4.2 are weaker than Theorem 4.1 and MSMMAOR is better than MSMAOR.

VI. CONCLUSION

In this paper, we extend MSMMAOR to solve a class of weakly nonlinear complementarity problems, analyze its convergence and present its weaker convergent conditions than MSMAOR. The numerical results confirm the theory results and show that MSMMAOR is better than MSMAOR.

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