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On the Oscillation of Impulsive Partial Fractional Differential Equations

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***Abstract*—In this paper, we investigate the oscillation properties of a class of impulsive partial fractional differential equations with several delays. Some sucient conditions for oscillation of the solutions are obtained by employing integral transformation technique and differential inequality method, and an example is given to illustrate the main result.**

***Keywords*—oscillation; impulsive; delays; partial fractional differential equations**

I. Introduction

It is well known that many important mathematical models are described by differential equations containing fractional order derivatives. The theories of fractional differential equations and their applications have been investigated extensively, we can find numerous applications in viscoelasticity, electrochemistry, control, porousmedia, electromagnetic, etc. see[1-6]. The oscillatory behavior of ordinary differential equations, partial differential equations and impulsive partial differential equations have been investigated by many papers in the past, see [7-11]. The oscillatory behavior of various classes of fractional ordinary differential equations and fractional partial differential equations have been investigated by many authors in recent years, see[12-20].

However, to the best of our knowledge very little is known regarding the oscillatory behavior of impulsive partial fractional differential equations with several delays up to now. Recently, A. Raheem and Md. Maqbul studied the oscillation behavior of solutions of a class of impulsive partial fractional differential equations with forcing term subject to Robin and Dirichlet boundary conditions by using differential inequality method in [21].

The purpose of this paper is to study the oscillation properties of the solutions to a class of impulsive partial fractional differential equations with several delays

$$\begin{cases} D_{+,t}^{1+\alpha}u(t,x)+b(t)D_{+,t}^{\alpha}u(t,x)=a(t)h(u(t,x))\Delta u(t,x)- \\ \quad \sum_{i=1}^{n}q_i(t,x)f_i(u(t-\delta_i,x))-g(t,x), t\neq t_k, (t,x)\in R_+\times\Omega\equiv G, \\ D_{+,t}^{\alpha}u(t_k^+,x)-D_{+,t}^{\alpha}u(t_k^-,x)=\sigma(t_k,x)D_{+,t}^{\alpha}u(t_k,x), t=t_k, k=1,2,\cdots, \end{cases} \tag{1}$$

with the boundary condition

$$\frac{\partial u(t,x)}{\partial n}=w(t,x,u(t,x)),(t,x)\in R_+\times\partial\Omega, t\neq t_k. \tag{2}$$

Where $\alpha\in(0,1)$ is a constant; $D_{+,t}^{\alpha}$ is the Riemann-Liouville fractional derivative of order α of $u(t,x)$ with respect to t; Ω is a boundary domain in R^n with a smooth boundary $\partial\Omega$ and $\overline{\Omega}=\Omega\cup\partial\Omega$; $R_+=(0,+\infty)$, Δ is Laplacian operator, and n is the unit exterior normal vector to $\partial\Omega$; $b(t)$, $a(t)\in PC[R_+,R_+]$, PC denote the class of functions which are piecewise continuous in t with discontinuities of first kind only at $t=t_k$; $\delta_i=const.>0$. The solution $u(t,x)$ of the problem (1) - (2) and $D_{+,t}^{\alpha}u(t,x)$ are piecewise continuous with discontinuities of first kind only at $t=t_k$, and left continuous at $t=t_k, k=1,2,\cdots$.

Throughout this paper, we assume that the following conditions hold:

(H1): $f_i:R\to R$ is a continuous function such that $f_i(u)/u\geq k_i>0$, for all $u\neq 0$; and k_i is a positive constant.

(H2): $q_i(t,x)\in PC[R_+\times\overline{\Omega},R_+]$, and $q_i(t)=\min_{1\leq i\leq n}\min_{x\in\overline{\Omega}}q_i(t,x)$.

(H3): $g(t,x)\in PC[R_+\times\overline{\Omega},R]$.

(H4): $h(u)\in C(R,R)$; $uh'(u)\geq 0$; $w(t,x,u)$ is a piecewise continous function, such that $uw(t,x,u)h(u)\leq 0$.

(H5) $\sigma:R_+\times\overline{\Omega}\to R_+$ such that $\sigma(t_k,x)\leq\alpha_k$.

(H6): The given numbers $0<t_1<\cdots<t_k<\cdots$, are such that $\lim_{k\to\infty}t_k=+\infty$.

(H7): At the moments of impulsive the following relation is satisfied

$$D_{+,t}^{\alpha}u(t_k^-,x)=D_{+,t}^{\alpha}u(t_k,x).$$

For the sake of convenience, in this paper, we denote

$$U(t)=\int_{\Omega}u(t,x)dx, G(t)=\int_{\Omega}g(t,x)dx. \tag{3}$$

II. Preliminaries

A. Definition 2.1

A nonzero solution $u(t,x)$ of the problem (1)-(2) is said to be nonoscillatory in the domain G if there exists a number $t_0\ge 0$ such that $u(t,x)$ has a constant sign for $(t,x)\in[t_0,+\infty)\times\Omega$. Otherwise, it is said to be oscillatory.

B. Definition 2.2

The Riemann-Liouville fractional partial derivative of order $\alpha>0$ with respect to t of a function $u(t,x)$ on the half-axis R_+ is defined by

$$D_{+,t}^{\alpha}u(t,x)=\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_0^t(t-v)^{-\alpha}u(v,x)dv, t>0, \tag{4}$$

where $\alpha\in(0,1)$; Γ is gamma function, and $\Gamma(\alpha)=\int_0^{+\infty}s^{\alpha-1}e^{-s}ds$.

C. Definition 2.3

The Riemann-Liouville fractional derivative of order $\alpha>0$ of a function $x:R_+\to R$ on the half-axis R_+ is defined by

$$D_{+,t}^{\alpha}x(t)=\frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)}\frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}}\int_0^t(t-v)^{\lceil\alpha\rceil-\alpha-1}x(v)dv, t>0, \tag{5}$$

where $\alpha\in(0,1)$; Γ is gamma function, $\lceil\alpha\rceil$ is the ceiling function of α .

D. Lemma 2.4[3]

Let $0<\alpha<1$, $m\in N$ and $D=d/dx$. If the fractional derivatives $(D_+^{\alpha}y)(x)$ and $(D_+^{m+\alpha}y)(x)$ exist, then

$$\left(D^m D_+^{\alpha}y\right)(x)=\left(D_+^{m+\alpha}y\right)(x) \tag{6}$$

III. Main results

A. Theorem 3.1

If impulsive fractional differential inequality

$$D_{+,t}^{1+\alpha}U(t)+b(t)D_{+,t}^{\alpha}U(t)\le -G(t), \tag{7}$$

$$D_{+,t}^{\alpha}U(t^+)\le(1+\alpha_k)D_{+,t}^{\alpha}U(t), k=1,2,\cdots, \tag{8}$$

has no eventually positive solutions and impulsive fractional differential inequality

$$D_{+,t}^{1+\alpha}U(t)+b(t)D_{+,t}^{\alpha}U(t)\ge -G(t), \tag{9}$$

$$D_{+,t}^{\alpha}U(t^+)\ge(1+\alpha_k)D_{+,t}^{\alpha}U(t), k=1,2,\cdots, \tag{10}$$

has no eventually negative solutions, then every nonzero solution $u(t,x)$ of the problem (1) and (2) is oscillatory in the domain G .

Proof. Suppose to contrary that $u(t,x)$ be a nonzero solution of the problem (1) and (2) which is nonoscillatory in the domain G . Without loss of generality, we assume that $u(t,x)$ is an eventually positive solution of problem (1) and (2) in the domain G , then there exists a $t_0\ge 0$ such that $u(t,x)>0$, $u(t-\delta_i)>0$, for $(t,x)\in[t_0,+\infty)\times\Omega$.

Case 1: $t\ne t_k$. Integrating the first equation of problem (1) with respect to x over the domain Ω , we have

$$\begin{aligned}&\int_{\Omega}D_{+,t}^{1+\alpha}u(t,x)dx+b(t)\int_{\Omega}D_{+,t}^{\alpha}u(t,x)dx=\\&a(t)\int_{\Omega}h(u(t,x))\Delta u(t,x)dx-\int_{\Omega}\sum_{i=1}q_i(t,x)f_i(u(t-\delta_i,x))dx\\&-\int_{\Omega}g(t,x)dx\end{aligned} \tag{11}$$

By using Green's formula, combing boundary condition (2) and assumption (H4), we obtain

$$\begin{aligned}&\int_{\Omega}h(u)\Delta u(t,x)dx=\int_{\partial\Omega}h(u)\frac{\partial u(t,x)}{\partial n}dx-\int_{\Omega}h'(u)|gradu|^2dx\\&=\int_{\partial\Omega}h(u)w(t,x,u)dx-\int_{\Omega}h'(u)|gradu|^2dx\le 0\end{aligned} \tag{12}$$

According to assumption (H1) and (H2), we have

$$\begin{aligned}&\int_{\Omega}\sum_{i=1}^{n}q_i(t,x)f_i(u(t-\delta_i,x))dx\\&\ge\sum_{i=1}^{n}k_iq_i(t)\int_{\Omega}u(t-\delta_i,x)dx\ge 0, t\ge t_0.\end{aligned} \tag{13}$$

And according to Lemma 2.4, combing (11)-(13) , we can easily obtain

$$D_{+,t}^{1+\alpha}U(t)+b(t)D_{+,t}^{\alpha}U(t)\le -\sum_{i=1}^{n}k_i q_i(t)\int_{\Omega}u(t-\delta_i,x)dx$$
$$\le -\sum_{i=1}^{n}k_i q_i(t)\int_{\Omega}u(t-\delta_i,x)dx-G(t)\le -G(t),t\ge t_0. \tag{14}$$

Case 2: $t=t_k$.Integrating the second equation of problem (1) with respect to x over the domain Ω, and according to assumption (H5), we have

$$D_{+,t}^{\alpha}U\left(t_k^{+}\right)=D_{+,t}^{\alpha}\int_{\Omega}u\left(t_k^{+},x\right)dx$$
$$\le (1+\alpha_k)D_{+,t}^{\alpha}\int_{\Omega}u\left(t_k,x\right)dx=(1+\alpha_k)D_{+,t}^{\alpha}U\left(t_k\right). \tag{15}$$

Thus impulsive fractional differential inequality (14) and (15) imply that the function $U(t)=\int_{\Omega}u(t,x)dx$ is an eventually positive solution of the fractional impulsive differential inequality (7) and (8) which contradicts the conditions of the theorem.

On the other hand, if $u(t,x)$ is an eventually negative solution of the problem (1) and (2) in the domain G, then using the similar method, we can easily obtain that $U(t)=\int_{\Omega}u(t,x)dx$ is an eventually negative solution of the fractional impulsive differential inequality (9) and (10) which contradicts the conditions of the theorem. The proof is completed.

B. Lemma 3.2[4]

Assume that

$$\omega'(t)\le g_1(t)\omega(t)+g_2(t),t\ne t_k,t\ge t_0,$$
$$\omega(t_k^{+})\le (1+a_k)\omega(t_k),k=1,2,\cdots, \tag{16}$$

where $0<t_1<\cdots<t_k<\cdots$ and $\lim_{k\to\infty}=+\infty$; $\omega\in PC^1[R_+,R]$; $g_1,g_2\in[R_+,R]$ and a_k are constants. Then

$$\omega(t)\le\omega(t_0)\prod_{t_0<t_l<t}(1+a_k)\exp\left(\int_{t_0}^{t}g_1(s)ds\right)$$
$$+\int_{t_0}^{t}\prod_{s<t_l<t}(1+a_k)\exp\left(\int_{s}^{t}g_1(\sigma)d\sigma\right)g_2(s)ds \tag{17}$$

C. Lemma 3.3[4]

If $U(t)$ is a solution of the impulsive fractional differential inequality (7) and (8) (or (9) and (10)), and

$$E(t)=\int_0^t(\tau-t)U(\tau)d\tau\text{, φορ }\alpha\in(0,1)\text{, ανδ }t>0,$$

then

$$E'(t)=\Gamma(1-\alpha)(D_{+,t}^{\alpha}U)(t)\text{, φορ }\alpha\in(0,1)\text{, ανδ }t>0.$$

D. Theorem 3.4

If for some $\tau_2>0$,

$$\int_{\tau_2}^{\infty}\exp(-b(\sigma)d\sigma)ds=\infty, \tag{18}$$

further, for some $\tau_1\ge 0$,

$$\limsup_{t\to\infty}\frac{\int_{\tau_1}^{t}\prod_{s<t_l<t}(1+\alpha_k)\exp\left(-\int_s^t b(\sigma)d\sigma\right)G(s)ds}{\prod_{\tau_1<t_l<t}(1+\alpha_k)\exp(-\int_{\tau_1}^{t}b(s))ds}=\infty, \tag{19}$$

and

$$\liminf_{t\to\infty}\frac{\int_{\tau_1}^{t}\prod_{s<t_l<t}(1+\alpha_k)\exp\left(-\int_s^t b(\sigma)d\sigma\right)G(s)ds}{\prod_{\tau_1<t_l<t}(1+\alpha_k)\exp(-\int_{\tau_1}^{t}b(s))ds}=-\infty, \tag{20}$$

then each nonzero solution of problem (1) and (2) is oscillatory in the domain G.

Proof. To prove the theorem, it is sufficient to prove that the impulsive fractional differential inequality (7) and (8) has no eventually positive solutions, and the impulsive fractional differential inequality (9) and (10) has no eventually negative solutions. Suppose to contrary that the impulsive fractional differential inequality (7) and (8) has an eventually positive solution $U(t)$, then there exists $\tau_1\ge 0$ such that $U(t)>0$, $U(t-\delta_i)>0,G(t)>0,t\ge\tau_1$.

Let

$$v(t,x)=\exp(\int_{t_0}^{t}b(s)ds).$$

By using Lemma 2.4 and the impulsive fractional differential inequality (7), we have

$$[(D_{+,\ t}^{\alpha}U)v(t)]'=(D_{+,\ t}^{\alpha}U)'(t)v(t)+b(t)(D_{+,\ t}^{\alpha}U)(t)v(t)$$
$$\le -G(t)v(t)<0$$

Thus $(D_{+,\ t}^{\alpha}U)(t)v(t)$ is strictly decreasing for $t\ge\tau_1$ and is eventually of constant sign. Since $v(t)>0$, for $t\ge\tau_1$, we know that $(D_{+,\ t}^{\alpha}U)(t)$ is eventually of constant sign. Then, $(D_{+,\ t}^{\alpha}U)(t)>0$ for $t\ge\tau_1$. Otherwise, $(D_{+,\ t}^{\alpha}U)(t)<0$ for $t\ge\tau_1$, there exists $\tau_2 \in [\tau_1, \infty)$, such that $(D^{\alpha}_{+,\ t}U)(\tau_2)v(\tau_2) < 0$. Since $(D^{\alpha}_{+,\ t}U)(t)v(t)$ is strictly decreasing for $t \ge \tau_1$, so $(D^{\alpha}_{+,\ t}U)(t)v(t) < (D^{\alpha}_{+,\ t}U)(\tau_2)v(\tau_2) = C < 0$ for $t \ge \tau_2$. From Lemma 3.3 , we have

$$\frac{E'(t)}{\Gamma(1-\alpha)} = (D^{\alpha}_{+,\ t}U)(t) < C\exp(-\int_{t_0}^{t} b(s)ds), t \ge \tau_2.$$

Integrating from τ_2 to t, we have

$$E(t) < E(\tau_2) + \Gamma(1-\alpha)C\int_{\tau_2}^{t}\exp(-\int_{t_0}^{t} b(\sigma)d\sigma)ds.$$

Letting $t \to \infty$, and using condition (18), we get $\lim\limits_{t\to\infty} E(t) = -\infty$, which is controdiction to the fact that $E(t) > 0$. Hence $(D^{\alpha}_{+,\ t}U)(t) > 0$ for $t \ge \tau_1$. Let $\omega(t) = (D^{\alpha}_{+,\ t}U)(t)$. From Lemma 2.4 and impulsive fractional inequality (7) and (8), we have

$$\omega'(t) \le -b(t)\omega(t) - G(t), t \ge t_0, t \ne t_k,$$
$$\omega(t_k^+) \le (1+\alpha_k)\omega(t_k). k = 1,2,\cdots.$$

By using Lemma 3.2, we have

$$\omega(t) \le \omega(\tau_1)\prod_{\tau_1<t_l<t}(1+\alpha_k)\exp\left(-\int_{\tau_1}^{t} b(s)ds\right) - \int_{\tau_1}^{t}\prod_{s<t_l<t}(1+\alpha_k)\exp\left(-\int_{s}^{t} b(\sigma)d\sigma\right)G(s)ds$$

by using the condition (19) and taking $t \to \infty$, it follows from above equation

$$\liminf_{t\to\infty}\frac{\int_{\tau_1}^{t}\prod_{s<t_l<t}(1+\alpha_k)\exp\left(-\int_{s}^{t} b(\sigma)d\sigma\right)G(s)ds}{\prod_{\tau_1<t_l<t}(1+\alpha_k)\exp(-\int_{\tau_1}^{t} b(s))ds} = -\infty,$$

which contradicts $\omega(t) > 0$. The proof is completed.

On the other hand, suppose to contrary that the impulsive fractional differential inequality (9) and (10) has an eventually negative solution $U(t)$, then there exists $\tau_1 \ge 0$ such that $U(t) < 0$, $U(t-\delta_i) < 0$, $G(t) < 0, t \ge \tau_1$. Then using similar methods, according the condition (20) and taking $t \to \infty$, we can obtain

$$\limsup_{t\to\infty}\frac{\int_{\tau_1}^{t}\prod_{s<t_l<t}(1+\alpha_k)\exp\left(-\int_{s}^{t} b(\sigma)d\sigma\right)G(s)ds}{\prod_{\tau_1<t_l<t}(1+\alpha_k)\exp(-\int_{\tau_1}^{t} b(s))ds} = \infty,$$

which contradicts $\omega(t) < 0$. The proof is completed.

IV. Example

In this section, we discuss an illustrative example.

Example 4.1 Consider the following problem

$$\begin{cases} D^{\frac{7}{4}}_{+,t}u(t,x) + \frac{1}{t}D^{\frac{3}{4}}_{+,t}u(t,x) = e^{-t}u^2(t,x)\Delta u(t,x) - \\ \quad (1+t^2+x^2)u(t-\frac{2\pi}{3},x)e^{\left[u(t-\frac{2\pi}{3},x)\right]^2} - t\cos x, t \ne t_k, \\ D^{\frac{3}{4}}_{+,t}u(t_k^+,x) - D^{\frac{3}{4}}_{+,t}u(t_k^-,x) = t_k^{-3}\cos x D^{\frac{3}{4}}_{+,t}u(t_k,x), t = t_k. \end{cases} \quad (21)$$

with the boundary condition

$$\frac{\partial u(t,x)}{\partial n} = w(t,x,u(t,x)) = -u^3(t,x), (t,x) \in R_+ \times \partial\Omega, t \ne t_k. \quad (22)$$

Here

$\alpha = \frac{3}{4}, \Omega = (0, \frac{\pi}{2}), n = 1$. $b(t) = \frac{1}{t}, a(t) = e^{-t}, h(u) = u^2$,

$$q_1(t,x) = 1+t^2+x^2,\ f_1(u) = ue^{u^2},\ \delta_1 = \frac{2\pi}{3},\ g(t,x) = t\cos x,$$
$$\sigma(t_k,x) = t_k^{-3}\cos x,\ \alpha_k = k_k^{-3},\ (t,x) \in R_+ \times \left(0, \frac{\pi}{2}\right).$$

We can easily see that

$$\int_{\tau_2}^{\infty}\exp(-b(\sigma)d\sigma)ds = \int_{\tau_2}^{\infty}\exp(-\int_{t_0}^{t}\frac{1}{\sigma}b(\sigma)d\sigma)ds = \int_{\tau_2}^{\infty}\frac{t_0}{t}dt = \infty,$$

$$\limsup_{t\to\infty}\frac{\int_{\tau_1}^{t}\prod_{s<t_l<t}(1+\alpha_k)\exp\left(-\int_{s}^{t} b(\sigma)d\sigma\right)G(s)ds}{\prod_{\tau_1<t_l<t}(1+\alpha_k)\exp(-\int_{\tau_1}^{t} b(s))ds} = \infty,$$

and

$$\liminf_{t\to\infty} \frac{\int_{\tau_1}^{t} \prod_{s<t_l<t} (1+\alpha_k) \exp\left(-\int_s^t b(\sigma)d\sigma\right) G(s)ds}{\prod_{\tau_1<t_l<t} (1+\alpha_k) \exp(-\int_{\tau_1}^{t} b(s))ds} = -\infty.$$

The conditions of Theorem 3.4 are satisfied, thus every solution of the problem (21)-(22) oscillates.

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