Advances in Engineering Research, volume 120
International Forum on Energy, Environment Science and Materials (IFEESM 2017)

About the Torsional Constant for thin-walled rod with open cross-section

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Key words: open thin-walled rod, torsional constant, finite element method

Abstract: This paper presents theoretically the torsional inertia moment for open thin-walled rod, based on the theory of thin-walled shell. In the deduction, the rigid contour hypothesis is adopted and the cross-sectional warp is taken into account. With the above torsional inertia moment obtained and using the similar method for closed beam elements in Reference [1], the stiffness matrix of thin-walled beam element with open cross-section is derived with the warping of cross-section considered. Other than the traditional stiffness matrix of beam element having twelve DOFs, i.e. six for each node, this stiffness matrix is corresponding to fourteen DOFs, i.e. seven for each node. This matrix could be used for the finite element analysis of open thin-walled rod, with section warping considered.

Introduction

In building structures, the thin-walled rod with open cross-section is widely used. It is usually analyzed by theoretical solution or finite element method (FEM) [1-3] for simple structure or complicated structure respectively. Either way, there is an important or even critical property of the cross-section needing to be determined first. It is the torsional inertia moment for open thin-walled cross-section.

In this paper, the torsional inertia moment for open thin-walled rod is derived theoretically based on the theory of thin-walled shell. In the deduction, the rigid contour hypothesis is adopted and the cross-sectional warp is taken into account. With the above torsional inertia moment obtained and using the similar method for closed beam elements in Reference [1], the stiffness matrix of thin-walled beam element with open cross-section is derived with the warping of cross-section considered. Other than the traditional stiffness matrix of beam element having twelve DOFs, i.e. six for each node, this stiffness matrix is corresponding to fourteen DOFs, i.e. seven for each node. This matrix could be used for the finite element analysis of open thin-walled rod, with section warping considered.

The torsional constant for thin-walled rod with open cross-section

Figure 1 gives an illustration of the cross section of an open thin-walled rod. There are three sets of coordinate system, i.e. Cartesian coordinate system located at the centroid of cross-section, contour coordinate system whose original location is moving along the midcourt line of the thin-walled cross section, and coordinate system located at the shear center, whose axes are parallel to that of contour coordinate system. Detailedly speaking, select arbitrary point P at the midcourt line C as the original point, and denote the vectors along the wall thickness orientation and the tangential midcourt line as *n* and *s* respectively, then the contour coordinate system is obtained. Offset the original point of contour coordinate system to the shear point, and denote the axes parallel to *n* and *s* as $\tilde{n}$ and $\tilde{s}$ respectively, then another coordinate system is obtained.

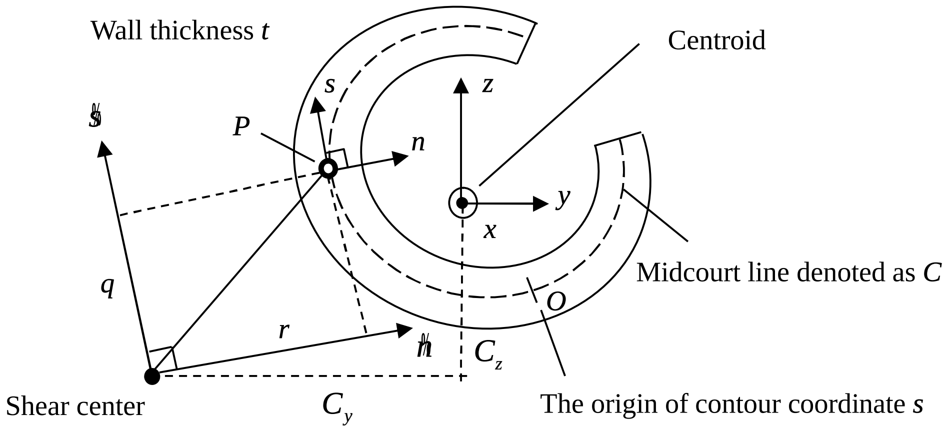

Figure 1 Cross section of an open thin-walled rod

According to Reference [5], the warping function of the cross-section is as following:

$$w(n,s)=\overline{w}+\overline{\overline{w}} \tag{1}$$

Where $\overline{w}$ denotes the contour warping function, equaling the sectorial area of the midcourt line, $\overline{\overline{w}}$ denotes the thickness warping function, see the following Equation (2):

$$\overline{w}=\int_0^s r_c(s)ds+\overline{w}_0\,,\quad \overline{\overline{w}}=-nq_c(s) \tag{2}$$

Where $\overline{w}_0$ denotes the contour warping function at s=0; r_c, q_c denotes *r* and *q* respectively corresponding to the midcourt line

The following Equation (3) could be obtained from Equation (1) and (2):

$$\frac{\partial w}{\partial s}=r_c(s)-n\,,\quad \frac{\partial w}{\partial n}=-q_c(s) \tag{3}$$

The following equations could be obtained for pure torsional problem at cylindrical coordinate system *xns*:

$$u_x=-w(n,s)c\,,\quad u_n=-q(s)q_x\,,\quad u_s=r(s)q_x \tag{4}$$

The shearing strain is as follow:

$$\begin{cases} g_{xn}=\dfrac{\partial u_x}{\partial n}+\dfrac{\partial u_n}{\partial x}=\left(-q_c+q\right)q_{x,x} \\ g_{xs}=\dfrac{\partial u_x}{\partial s}+\dfrac{\partial u_s}{\partial x}=\left(r(s)-(r_c(s)-n)\right)q_{x,x} \end{cases} \tag{5}$$

Observing Figure 1, the following equation could be obtained:

$$r(s) = r_c(s) + n, \quad q(s) = q_c(s) \tag{6}$$

Substitute Equation (6) into (5), the following Equation (7) could be derived:

$$g_{xn} = 0, \quad g_{xs} = 2nq_{x,x} \tag{7}$$

The potential variational formula for thin-walled rod is as follow [1-3]:

$$d\Pi = \int_l \int_A G g_{xs} d g_{xs} dA dl = \int_l GJ q_{x,x} d q_{x,x} dl \tag{8}$$

Where J denotes the torsional inertia moment, see the following equation for details:

$$J = \int_A (2n)^2 dA = \int_{s_1}^{s_2} \int_{-t/2}^{t/2} (2n)^2 dn ds = \int_{s_1}^{s_2} \frac{1}{3} t^3 ds \tag{9}$$

It is obviously that J denotes St.Venant torsional constant.

Deducing the above equations in rectangular coordinate system, then

$$J' = \int_A [(z - C_z + w_{,y})^2 + (y - C_y - w_{,z})^2] dA \tag{10}$$

Considering the fact that torsional inertia moment is the inherent nature of the cross-section and has nothing to do with the coordinate system, the following equation could be derived:

$$J = J' = \int_A [(z - C_z + w_{,y})^2 + (y - C_y - w_{,z})^2] dA \tag{11}$$

The stiffness matrix of thin-walled beam element with open cross-section

Use the similar method for closed beam elements, see Reference [1], together with the above Equation (11), the linear stiffness matrix of open thin-walled beam element could be derived as follow:

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}^{11} & \mathbf{K}^{12} \\ \mathbf{K}^{21} & \mathbf{K}^{22} \end{bmatrix} \tag{12}$$

$$\mathbf{K}^{11} = \begin{bmatrix} \frac{EA}{l} & 0 & 0 & 0 & 0 & 0 & 0 \\ & \frac{12EI_z}{l^3} & 0 & 0 & 0 & \frac{6EI_z}{l^2} & 0 \\ & & \frac{12EI_y}{l^3} & 0 & -\frac{6EI_y}{l^2} & 0 & 0 \\ & & & \frac{6GJ}{5l} + \frac{12EI_w}{l^3} & 0 & 0 & \frac{GJ}{10} + \frac{6EI_w}{l^2} \\ & & & & \frac{4EI_y}{l} & 0 & 0 \\ & & \text{sym} & & & \frac{4EI_z}{l} & 0 \\ & & & & & & \frac{2GJl}{15} + \frac{4EI_w}{l} \end{bmatrix} \tag{13}$$

$$\mathbf{K}^{12}=\begin{bmatrix} -\frac{EA}{l} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{12EI_z}{l^3} & 0 & 0 & 0 & \frac{6EI_z}{l^2} & 0 \\ 0 & 0 & -\frac{12EI_y}{l^3} & 0 & -\frac{6EI_y}{l^2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{6GJ}{5l}-\frac{12EI_w}{l^3} & 0 & 0 & \frac{GJ}{10}+\frac{6EI_w}{l^2} \\ 0 & 0 & \frac{6EI_y}{l^2} & 0 & \frac{2EI_y}{l} & 0 & 0 \\ 0 & -\frac{6EI_z}{l^2} & 0 & 0 & 0 & \frac{2EI_z}{l} & 0 \\ 0 & 0 & 0 & -\frac{GJ}{10}-\frac{6EI_w}{l^2} & 0 & 0 & -\frac{GJl}{30}+\frac{2EI_w}{l} \end{bmatrix} \tag{14}$$

$$\mathbf{K}^{21}=\begin{bmatrix} -\frac{EA}{l} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{12EI_z}{l^3} & 0 & 0 & 0 & -\frac{6EI_z}{l^2} & 0 \\ 0 & 0 & -\frac{12EI_y}{l^3} & 0 & \frac{6EI_y}{l^2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{6GJ}{5l}-\frac{12EI_w}{l^3} & 0 & 0 & -\frac{GJ}{10}-\frac{6EI_w}{l^2} \\ 0 & 0 & -\frac{6EI_y}{l^2} & 0 & \frac{2EI_y}{l} & 0 & 0 \\ 0 & \frac{6EI_z}{l^2} & 0 & 0 & 0 & \frac{2EI_z}{l} & 0 \\ 0 & 0 & 0 & \frac{GJ}{10}+\frac{6EI_w}{l^2} & 0 & 0 & -\frac{GJl}{30}+\frac{2EI_w}{l} \end{bmatrix} \tag{15}$$

$$\mathbf{K}^{22}=\begin{bmatrix} \frac{EA}{l} & 0 & 0 & 0 & 0 & 0 & 0 \\ & \frac{12EI_z}{l^3} & 0 & 0 & 0 & -\frac{6EI_z}{l^2} & 0 \\ & & \frac{12EI_y}{l^3} & 0 & \frac{6EI_y}{l^2} & 0 & 0 \\ & & & \frac{6GJ}{5l}+\frac{12EI_w}{l^3} & 0 & 0 & -\frac{GJ}{10}-\frac{6EI_w}{l^2} \\ & & & & \frac{4EI_y}{l} & 0 & 0 \\ & & \text{sym} & & & \frac{4EI_z}{l} & 0 \\ & & & & & & \frac{2GJl}{15}+\frac{4EI_w}{l} \end{bmatrix} \tag{16}$$

Where I_y and I_z denotes the principal moment of inertia according to axis v and z respectively;

J denotes the torsional moment of inertia for open thin-walled rod, see Equation (11); I_w denotes the cross section's warping moment of inertia, see the following equation:

$$I_w = \int_A w^2 dA \tag{17}$$

Where ω denotes the warping function of cross-section [4,5,6]

Summary

In this paper, the torsional inertia moment for open thin-walled rod is derived theoretically based on the theory of thin-walled shell. In the deduction, the rigid contour hypothesis is adopted and the cross-sectional warp is taken into account. With the above torsional inertia moment obtained and using the similar method for closed beam elements in Reference [1], the stiffness matrix of thin-walled beam element with open cross-section is derived with the warping of cross-section considered. Other than the traditional stiffness matrix of beam element having twelve DOFs, i.e. six for each node, this stiffness matrix is corresponding to fourteen DOFs, i.e. seven for each node. This matrix could be used for the finite element analysis of open thin-walled rod, with section warping considered.

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