

Novel Stability Criterion for Neutral Systems with Mixed Time-Delay

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Abstract—The aim of this paper is to improve stability condition for neutral systems with mixed time-delay. By introducing a novel integral inequality based on the optimization theory, and constructing the novel Lyapunov functionals, an improved delay-dependent stability criterion is established. It is shown that the integral inequality obtained in this paper can be utilized to get less conservative delay-dependent stability condition for neutral systems. A numerical example is provided to demonstrate the theoretical result.

Keywords—new integral inequality; neutral systems with mixed time-delays; delay-dependent stability

I. INTRODUCTION

It is well known that neutral systems are frequently encountered in various engineering systems, including population, ecology, distributed networks containing lossless transmission lines, heat exchangers, and repetitive control. In the past two decades, a great deal of attention have been drawn to the delay-dependent stability problem of neutral dynamical systems (see, e.g., [1]-[31]). Various important methods have been introduced, such as model transformation approach[4], delay partitioning technique[11], discretized Lyapunov functional method[12], free-weighting matrix approach[5], and descriptor system approach[17]. Among those techniques, the Jensen's like inequality has played an important role to obtain delay-dependent stability conditions [21].

However, Jensen's like inequality inevitably introduces an undesirable conservatism, and improving it is always an open problem [25]. Fortunately, it has been improved in [24]-[26] which dealt with single integral terms. Recently, a new multiple integral inequality was introduced following a similar line as in proof of Jensen inequality in [20], and a novel delay-dependent stability criterion was established, which has been unfortunately observed that the computational burden is slightly heavy. Actually, [19] observed that the upper bounds of double integral terms should also be estimated if triple integral terms are introduced in the Lyapunov-Krasovskii functional to obtain less conservative conditions. Based on the Wirtinger inequality, [23] improved the double Jensen's like double integral, and a double integral form of the Wirtinger-based integral was introduced. However, they didn't consider the derivative of the integrand directly. We may get another double integral inequality if we consider this case of derivative. Therefore, if one can introduce a less conservative double integral Jensen's like inequality which contains its derivative,

the delay-dependent stability results would be improved greatly, which may reduce the complexity of calculation.

In this paper, we will introduce a new integral inequality based on the optimization theory and construction techniques, and then investigate the conventional neutral systems to illustrate the advantages of applying this inequality to obtain delay-dependent stability criterion. The rest of the paper is organized as follows. In Section II, we formulate the problem and the novel inequality is introduced, which is proved to be less conservative than the double integral Jensen's like inequality. In Section III, as an application of the introduced inequality, the stability condition for neutral systems with mixed delay is presented. The sufficient condition is formed in terms of LMIs, which can be easily calculated by Matlab control toolbox [32]. Many cases are compared in the tables based on the conventional numerical example, which is used to show the validity and less conservativeness of our approach in Section IV. The paper is concluded by Section V. The acknowledge is presented in Section VI

II. PROBLEM STATEMENT AND PRELIMINARIES

In this paper, we will improve the stability condition for the following neutral system with mixed time-delays:

$$\begin{cases} \dot{x}(t) - Cx(t - \tau) = Ax(t) + Bx(t - h) \\ x(t_0 + \theta) = \varphi(\theta), \forall \theta \in [-\max(\tau, h), 0] \end{cases} \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, τ, h are time-delays, $\varphi(\theta)$ is the initial condition function. A, B, C are known matrices. In this paper, we always assume that the spectral radius of C less than 1.

As is well known, inequalities and Lyapunov functionals are both important to improve the delay-dependent stability condition for neutral systems with mixed time-delays. Accordingly, we will first introduce a novel double integral inequality to deal with the $\int_{-h}^0 \int_{t+\theta}^t w^T(s) R w(s) ds d\theta$ for the derivative of constructed Lyapunov functionals.

Lemma 2.1 For a give matrix $R > 0$, the following inequality holds for all continuously differentiable function $w \in [t - h, t] \rightarrow \mathbb{R}^n$:

$$\int_{-h}^0 \int_{t+\theta}^t w^\top(s) R w(s) ds d\theta \geq 6\Omega^\top \mathfrak{R} \Omega \quad (2)$$

where

$$\mathfrak{R} = \begin{pmatrix} R & R & -4R \\ * & 3R & -8R \\ * & * & 24R \end{pmatrix},$$

$$\Omega^\top = \left(w^\top(t), \frac{1}{h} \int_{t-h}^t w^\top(s) ds, \frac{1}{h^2} \int_{-h}^0 \int_{t+\theta}^t w^\top(s) ds d\theta \right).$$

Proof. For any continuous function $w(t) \in [t-h, t]$ and which admits a continuous derivative, define the function z by

$$z(s) = w(s) - \frac{2}{h^2} A + \frac{4g(s)}{h^3} B, \text{ for } s \in [t-h, t],$$

where

$$A = hw(t) - \int_{t-h}^t w(s) ds, \quad g(s) = 3t - h - 3s,$$

$$B = 2 \int_{t-h}^t w(s) ds + hw(t) - \frac{6}{h} \int_{-h}^0 \int_{t+\theta}^t w(s) ds d\theta.$$

The computation of $\int_{-h}^0 \int_{t+\theta}^t z^\top(s) R z(s) ds d\theta$ leads to

$$\begin{aligned} & \int_{-h}^0 \int_{t+\theta}^t z^\top(s) R z(s) ds d\theta = \\ & \int_{-h}^0 \int_{t+\theta}^t w^\top(s) R w(s) ds d\theta - \frac{4}{h^2} \int_{-h}^0 \int_{t+\theta}^t w^\top(s) ds d\theta R A \\ & + \frac{4}{h^4} \int_{-h}^0 \int_{t+\theta}^t ds d\theta A^\top R A - \frac{16}{h^5} \int_{-h}^0 \int_{t+\theta}^t g(s) ds d\theta A^\top R B \\ & + \frac{16}{h^6} \int_{-h}^0 \int_{t+\theta}^t g^2(s) ds d\theta B^\top R B \\ & + \frac{8}{h^3} \int_{-h}^0 \int_{t+\theta}^t g(s) w^\top(s) ds d\theta R B \end{aligned} \quad (3)$$

Simple calculation ensure that

$$\int_{-h}^0 \int_{t+\theta}^t w(s) ds d\theta = A, \quad \int_{-h}^0 \int_{t+\theta}^t ds d\theta = \frac{1}{2} h^2,$$

$$\int_{-h}^0 \int_{t+\theta}^t g(s) ds d\theta = 0, \quad \int_{-h}^0 \int_{t+\theta}^t g^2(s) ds d\theta = \frac{1}{4} h^4,$$

and integration by parts ensures that

$$\int_{-h}^0 \int_{t+\theta}^t g(s) w(s) ds d\theta = -hB. \quad (4)$$

It thus follows from $R > 0$ that

$$\begin{aligned} \int_{-h}^0 \int_{t+\theta}^t z^\top(s) R z(s) ds d\theta &= \int_{-h}^0 \int_{t+\theta}^t w^\top(s) R w(s) ds d\theta \\ &\quad - \frac{2}{h^2} A^\top R A - \frac{4}{h^2} B^\top R B \geq 0 \end{aligned}$$

More specifically, one can see

$$\int_{-h}^0 \int_{t+\theta}^t w^\top(s) R w(s) ds d\theta \geq \frac{2}{h^2} (A^\top R A + 2B^\top R B) \quad (5)$$

or equivalently

$$\int_{-h}^0 \int_{t+\theta}^t w^\top(s) R w(s) ds d\theta \geq 6\Omega^\top \mathfrak{R} \Omega \quad (6)$$

with \mathfrak{R} and Ω are defined in Lemma 2.1. This completes the proof.

Remark 2.1 It is worth mentioning that the choice of function $z(s)$ is essential in the proof of Lemma 2.1. On the one hand, the function $g(s)$ should be constructed to mark $\int_{-h}^0 \int_{t+\theta}^t g(s) ds d\theta \times A^\top R B = 0$, and lead to the right-hand of (5) in a standard quadratic form. Therefore, according to the computation for the integrations, we choose $g(s) = 3t - h - 3s$. On the other hand, to unify the coefficient $\frac{1}{h^2}$ of the expression (5), $z(s)$ should contain the coefficient $\frac{1}{h^3}$.

Remark 2.2 The inequality in (2) not only contains the information of $w^\top(t)$ and $\int_{t-h}^t w^\top(s) ds$, but also includes the information of $\int_{-h}^0 \int_{t+\theta}^t w^\top(s) ds d\theta$. Clearly, inequality (2) of Lemma 2.1 is less conservative than the conventional Jensen's inequality in [28].

III. MAIN RESULT

In this section, by constructing a novel Lyapunov functional, and using many inequalities which contain the above introduced inequality, we will give much better delay-dependent stability condition.

Theorem 3.1 The system (1) is asymptotically stable if there exist positive definite matrices $P, W_1, W_2, Q_1, Q_2, R_1, R_2$, any symmetric matrices Y_{ij}, Z_{ij} ($i, j = 1, 2, 3, 4$), $\mathfrak{R}_{2 \times 2}$ with appropriate dimensions satisfying the following linear matrix inequalities

$$\Psi_{6 \times 6} > 0, \quad (7)$$

$$\varphi_{8 \times 8} < 0, \quad (8)$$

where

$$\begin{aligned}
\Psi_{11} &= P + Y_{11} + Z_{11} + \frac{4}{\tau}R_{22} + 6hW_2 + 6\tau W_1 + \frac{3h}{2}R_2 + \frac{3\tau}{2}R_1, \\
\Psi_{12} &= Y_{12} + Z_{12} + \frac{2}{\tau}R_{22}, \quad \Psi_{13} = \tau Y_{13} - 2R_{12}^T - \frac{6}{\tau}R_{22} + 6\tau W_1, \\
\Psi_{14} &= 6hW_2 + \tau^2 Z_{13}, \quad \Psi_{15} = 6R_{12}^T - 24\tau W_1 - 3\tau R_1 + hY_{14}, \\
\Psi_{16} &= -24hW_2 - 3hR_2 + h^2 Z_{14}, \quad \Psi_{22} = Y_{22} + Z_{22} + \frac{4}{\tau}R_{22}, \\
\Psi_{23} &= \tau Y_{23} - 4R_{12}^T - \frac{6}{\tau}R_{22}, \quad \Psi_{24} = \tau^2 Z_{23}, \quad \Psi_{25} = 6R_{12}^T + hY_{24}, \\
\Psi_{26} &= h^2 Z_{24}, \\
\Psi_{33} &= 4\tau Q_1 + 4\tau R_{11} + \frac{12}{\tau}R_{22} + 6R_{12} + 6R_{12}^T + 18\tau W_1 + \tau^2 Y_{33}, \\
\Psi_{35} &= -6\tau Q_1 - 2\tau \left(3R_{11} + \frac{6}{\tau}R_{12}^T \right) - 48\tau W_1 + h\tau Y_{34}, \\
\Psi_{44} &= 4hQ_2 + 18hW_2 + \tau^4 Z_{33}, \\
\Psi_{46} &= -6hQ_2 - 48hW_2 + h^2 \tau^2 Z_{34}, \\
\Psi_{55} &= 12\tau Q_1 + 12\tau R_{11} + 144\tau W_1 + 6\tau R_1 + h^2 Y_{44}, \\
\Psi_{66} &= 12hQ_2 + 144hW_2 + 6hR_2 + h^4 Z_{44}, \\
\varphi_{11} &= Q_1 + Q_2 - 6R_1 + R_{11} - 6R_2 - 9W_1 - 9W_2 \\
&\quad + Y_{13} + Y_{13}^T + Y_{14} + Y_{14}^T + hZ_{14} + hZ_{14}^T + \tau Z_{13} \\
&\quad + \tau Z_{13}^T + PA + A^T P + Y_{11}A + A^T Y_{11}^T + Z_{11}A \\
&\quad + A^T Z_{11}^T + R_{12}A + A^T R_{12}^T + A^T R_{22}A + \frac{h^2}{2}A^T R_2 A \\
&\quad + \frac{\tau^2}{2}A^T R_1 A + h^2 A^T W_2 A + \tau^2 A^T W_1 A \\
\varphi_{12} &= Y_{23}^T - Y_{13} + 3W_1 + Y_{24}^T + A^T Y_{12} + A^T Z_{12} + hZ_{24}^T + \tau Z_{23}^T, \\
\varphi_{13} &= 3W_2 - Y_{14} + R_{12}B + A^T R_{22}B + PB + Y_{11}B + Z_{11}B \\
&\quad + \frac{h^2}{2}A^T R_2 B + \frac{\tau^2}{2}A^T R_1 B + h^2 A^T W_2 B + \tau^2 A^T W_1 B, \\
\varphi_{22} &= -Q_1 - R_{11} - 9W_1 - Y_{23} - Y_{23}^T, \quad \varphi_{23} = Y_{12}^T B - Y_{24} + Z_{12}^T B, \\
\varphi_{33} &= B^T R_{22}B - 9W_2 - Q_2 + \frac{h^2}{2}B^T R_2 B + \frac{\tau^2}{2}B^T R_1 B \\
&\quad + h^2 B^T W_2 B + \tau^2 B^T W_1 B, \\
\varphi_{14} &= Y_{12} + Z_{12} + R_{12}C + A^T R_{22}C + PC + Y_{11}C + Z_{11}C \\
&\quad + \frac{h^2}{2}A^T R_2 C + \frac{\tau^2}{2}A^T R_1 C + h^2 A^T W_2 C + \tau^2 A^T W_1 C,
\end{aligned}$$

$$\begin{aligned}
\varphi_{24} &= Y_{22} - R_{12} + Z_{22} + Y_{12}^T C + Z_{12}^T C, \\
\varphi_{34} &= B^T R_{22}C + \frac{h^2}{2}B^T R_2 C + \frac{\tau^2}{2}B^T R_1 C + h^2 B^T W_2 C + \tau^2 B^T W_1 C, \\
\varphi_{44} &= C^T R_{22}C - R_{22} + \frac{h^2}{2}C^T R_2 C + \frac{\tau^2}{2}C^T R_1 C \\
&\quad + h^2 C^T W_2 C + \tau^2 C^T W_1 C, \\
\varphi_{15} &= \tau Y_{33} + \tau Y_{34}^T + \tau A^T Y_{13} - 24W_1 - \tau Z_{13} - 6R_1, \\
\varphi_{25} &= 36W_1 - \tau Y_{33} - \tau Z_{23}, \quad \varphi_{35} = -\tau (Y_{34}^T - B^T Y_{13}), \\
\varphi_{45} &= \tau (Y_{23} + C^T Y_{13}), \quad \varphi_{55} = -18R_1 - 192W_1, \\
\varphi_{16} &= h (Y_{34} + Y_{44} + A^T Y_{14}) - 24W_2 - hZ_{14} - 6R_2, \\
\varphi_{26} &= -hY_{34} - hZ_{24}, \quad \varphi_{36} = 36W_2 - h (Y_{44} - B^T Y_{14}), \\
\varphi_{46} &= h (Y_{24} + C^T Y_{14}), \quad \varphi_{66} = -18R_2 - 192W_2, \\
\varphi_{17} &= 24R_1 + 60W_1 + \tau^2 (A^T Z_{13} + \tau Z_{33} + hZ_{34}^T), \\
\varphi_{27} &= -60W_1, \quad \varphi_{37} = \tau^2 B^T Z_{13}, \quad \varphi_{47} = \tau^2 (Z_{23} + C^T Z_{13}), \\
\varphi_{57} &= 48R_1 + 360W_1 - \tau^3 Z_{33}, \\
\varphi_{67} &= -h\tau^2 Z_{34}^T, \quad \varphi_{77} = -144R_1 - 720W_1, \quad \varphi_{88} = -144R_2 - 720W_2, \\
\varphi_{18} &= 24R_2 + 60W_2 + h^2 (A^T Z_{14} + hZ_{44} + \tau Z_{34}), \\
\varphi_{38} &= h^2 B^T Z_{14} - 60W_2, \quad \varphi_{48} = h^2 (Z_{24} + C^T Z_{14}), \\
\varphi_{58} &= -\tau h^2 Z_{34}, \quad \varphi_{68} = 48R_2 + 360W_2 - h^3 Z_{44}, \\
\varphi_{88} &= -144R_2 - 720W_2,
\end{aligned}$$

Proof. Construct a Lyapunov functional candidate as

$$V(x_t, t) = \sum_{i=1}^5 V_i(x_t, t), \quad (9)$$

$$V_1 = x^T(t) P x(t),$$

$$V_2 = \int_{t-\tau}^t x^T(s) Q_1 x(s) ds + \int_{t-h}^t x^T(s) Q_2 x(s) ds$$

$$+ \int_{t-\tau}^t \begin{pmatrix} x(s) \\ x(s) \end{pmatrix}^T \Re \begin{pmatrix} x(s) \\ x(s) \end{pmatrix} ds,$$

$$V_3 = \int_{-\tau}^0 \int_{t+\theta}^t x^T(s) (\tau W_1) x(s) ds d\theta$$

$$+ \int_{-h}^0 \int_{t+\theta}^t x^T(s) (h W_2) x(s) ds d\theta,$$

$$V_4 = \int_{-\tau}^0 \int_{\theta}^0 \int_{t+\lambda}^t x^T(s) R_1 x(s) ds d\lambda d\theta \\ + \int_{-h}^0 \int_{\theta}^0 \int_{t+\lambda}^t x^T(s) R_2 x(s) ds d\lambda d\theta$$

$$V_5 = G_1^T Z G_1 + G_2^T Y G_2,$$

where, Y and Z are 4-order matrices with its elements $Y_{ij}, Z_{ij} (i, j = 1, 2, 3, 4)$ expressed in theorem 3.1,

$$G_1 = \begin{pmatrix} x(t) \\ x(t-\tau) \\ \int_{-\tau}^0 \int_{t+\theta}^t x(s) ds d\theta \\ \int_{-h}^0 \int_{t+\theta}^t x(s) ds d\theta \end{pmatrix}, \quad G_2 = \begin{pmatrix} x(t) \\ x(t-\tau) \\ \int_{t-\tau}^t x(s) ds \\ \int_{t-h}^t x(s) ds \end{pmatrix}$$

For the sake of convenience, we set $e_i \in \mathbb{R}^{8 \times n} (i = 1, \dots, 8)$ are elementary matrices, for example

$$e_2^T = [0 \quad I \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0],$$

and let

$$e_0 = A e_1 + B e_3 + C e_4,$$

$$\xi^T = \begin{pmatrix} x^T(t), x^T(t-\tau), x^T(t-h), x^T(t-\tau), \\ \frac{1}{\tau} \int_{t-\tau}^t x^T(s) ds, \frac{1}{h} \int_{t-h}^t x^T(s) ds, \\ \frac{1}{\tau^2} \int_{-\tau}^0 \int_{t+\theta}^t x^T(s) ds d\theta, \frac{1}{h^2} \int_{-h}^0 \int_{t+\theta}^t x^T(s) ds d\theta \end{pmatrix},$$

First, with the inequality (8) of Lemma 5.1 in [29] and Lemma 2.1, the Lyapunov functionals V_2 and V_3 can be scaled as

$$V_2 \geq \xi^T \times \{ \tau e_5^T Q_1 e_5 + 3\tau(e_5 - 2e_7)^T Q_1 (e_5 - 2e_7) \\ + h e_6^T Q_2 e_6 + 3h(e_6 - 2e_8)^T Q_2 (e_6 - 2e_8) \\ + \frac{1}{\tau} [\tau e_5^T \quad e_1^T - e_2^T] \Re [\tau e_5^T \quad e_1^T - e_2^T]^T \\ + \frac{3}{\tau} [\tau e_5^T - 2\tau e_7^T \quad -e_1^T - e_2^T + 2e_5^T] \Re \\ \times [\tau e_5^T - 2\tau e_7^T \quad -e_1^T - e_2^T + 2e_5^T]^T \} \times \xi \quad (10)$$

$$V_3 \geq \xi^T \times \{ 2\tau(e_1 - e_5)^T W_1 (e_1 - e_5) \\ + 4\tau(e_1 + 2e_5 - 6e_7)^T W_1 (e_1 + 2e_5 - 6e_7) \\ + 2h(e_1 - e_6)^T W_2 (e_1 - e_6) \\ + 4h(e_1 + 2e_6 - 6e_8)^T W_2 (e_1 + 2e_6 - 6e_8) \} \times \xi \quad (11)$$

Following the same line to prove the Jensen inequality in [30], V_4 can be bounded as

$$V_4 \geq \xi^T \times \left\{ \frac{6}{\tau^3} \left(\frac{\tau^2}{2} e_1 - \tau^2 e_7 \right)^T R_1 \left(\frac{\tau^2}{2} e_1 - \tau^2 e_7 \right) \right. \\ \left. + \frac{6}{h^3} \left(\frac{h^2}{2} e_1 - h^2 e_8 \right)^T R_2 \left(\frac{h^2}{2} e_1 - h^2 e_8 \right) \right\} \times \xi \quad (12)$$

On the other hand, V_1 and V_5 can be presented as

$$V_1 = \xi^T \times \{ e_1^T P e_1 \} \times \xi, \quad (13)$$

$$V_5 = \xi^T \times \left\{ \begin{bmatrix} e_1^T & e_2^T & \tau^2 e_7^T & h^2 e_8^T \end{bmatrix} Z \begin{bmatrix} e_1^T & e_2^T & \tau^2 e_7^T & h^2 e_8^T \end{bmatrix}^T \right. \\ \left. + \begin{bmatrix} e_1^T & e_2^T & \tau e_5^T & h e_6^T \end{bmatrix} Y \begin{bmatrix} e_1^T & e_2^T & \tau e_5^T & h e_6^T \end{bmatrix}^T \right\} \times \xi \quad (14)$$

From (10) to (14), we can get that

$$V \geq \xi_1^T \Psi \xi_1. \quad (15)$$

where

$$\xi_1^T = \begin{pmatrix} x^T(t), x^T(t-\tau), \frac{1}{\tau} \int_{t-\tau}^t x^T(s) ds, \frac{1}{h} \int_{t-h}^t x^T(s) ds, \\ \frac{1}{\tau^2} \int_{-\tau}^0 \int_{t+\theta}^t x^T(s) ds d\theta, \frac{1}{h^2} \int_{-h}^0 \int_{t+\theta}^t x^T(s) ds d\theta \end{pmatrix}$$

According to the inequality (7), the functional (9) is positive definite. Then, the time derivative of $V(t)$ along the trajectories of equation (1) can be firstly computed as follows:

$$V(x, t) = \sum_{i=1}^5 V_i(x, t), \quad (16)$$

with

$$V_1 = 2x^T(t) P [Ax(t) + Bx(t-h) + Cx(t-h)] \\ = 2\xi_1^T e_1^T P e_0 \xi, \quad (17)$$

$$V_2 = x^T(Q_1 + Q_2)x - x^T(t-\tau)Q_1x(t-\tau) \\ - x^T(t-h)Q_2x(t-h) + \begin{pmatrix} x(t) \\ x(t) \end{pmatrix}^T \Re \begin{pmatrix} x(t) \\ x(t) \end{pmatrix} \\ - \begin{pmatrix} x(t-\tau) \\ x(t-\tau) \end{pmatrix}^T \Re \begin{pmatrix} x(t-\tau) \\ x(t-\tau) \end{pmatrix}$$

$$\begin{aligned}
&= \xi^T \left[\begin{pmatrix} e_1 \\ e_0 \end{pmatrix}^T \Re \begin{pmatrix} e_1 \\ e_0 \end{pmatrix} - \begin{pmatrix} e_2 \\ e_4 \end{pmatrix}^T \Re \begin{pmatrix} e_2 \\ e_4 \end{pmatrix} \right] \xi \\
&+ \xi^T \left[e_1^T (Q_1 + Q_2) e_1 - e_2^T Q_1 e_2 - e_3^T Q_2 e_3 \right] \xi \quad (18)
\end{aligned}$$

$$\begin{aligned}
V_3 &= \xi^T [e_0^T (\tau^2 W_1 + h^2 W_2) e_0 - \tau \int_{t-\tau}^t x^T(s) W_1 x(s) ds \\
&\quad - h \int_{t-h}^t x^T(s) W_2 x(s) ds] \xi
\end{aligned}$$

With the inequality (24) of Lemma 5.1 in [29], one can obtain that

$$\begin{aligned}
V_3 &\leq \xi^T [e_0^T (\tau^2 W_1 + h^2 W_2) e_0 - (e_1 - e_2)^T W_1 (e_1 - e_2) \\
&\quad - (e_1 - e_3)^T W_2 (e_1 - e_3) - 3(e_1 + e_2 - 2e_5)^T W_1 (e_1 + e_2 - 2e_5) \\
&\quad \quad - 3(e_1 + e_3 - 2e_6)^T W_2 (e_1 + e_3 - 2e_6) \\
&\quad - 5(e_1 - e_2 + 6e_3 - 12e_7)^T \times W_2 \times (e_1 - e_2 + 6e_3 - 12e_7) \\
&\quad - 5(e_1 - e_3 + 6e_6 - 12e_8)^T \times W_2 \times (e_1 - e_3 + 6e_6 - 12e_8)] \xi \quad (19)
\end{aligned}$$

$$\begin{aligned}
V_4 &= \xi^T \left[\frac{1}{2} e_0^T (\tau^2 R_1 + h^2 R_2) e_0 \right] \xi - \int_{-h}^0 \int_{t+\theta}^t x^T(s) R_1 x(s) ds d\theta \\
&\quad - \int_{-h}^0 \int_{t+\theta}^t x^T(s) R_2 x(s) ds d\theta
\end{aligned}$$

By using the Lemma 2.1, one can see that

$$\begin{aligned}
V_4 &\leq \xi^T \left[\frac{1}{2} e_0^T (\tau^2 R_1 + h^2 R_2) e_0 \right. \\
&\quad - (e_1^T \quad e_5^T \quad e_7^T) K_1 (e_1^T \quad e_5^T \quad e_7^T)^T \\
&\quad \left. - (e_1^T \quad e_6^T \quad e_8^T) K_2 (e_1^T \quad e_6^T \quad e_8^T)^T \right] \xi \quad (20)
\end{aligned}$$

with

$$\begin{aligned}
K_1 &= 6R_1 \begin{pmatrix} 1 & 1 & -4 \\ 1 & 3 & -8 \\ -4 & -8 & 24 \end{pmatrix}, \quad K_2 = 6R_2 \begin{pmatrix} 1 & 1 & -4 \\ 1 & 3 & -8 \\ -4 & -8 & 24 \end{pmatrix} \\
V_5 &= G_1^T Z G_1 + G_1^T Y G_1 + G_2^T Y G_2 + G_2^T Z G_2 \\
&= \xi^T [2(e_1^T \quad e_2^T \quad \tau^2 e_7^T \quad h^2 e_8^T) \\
&\quad \times Z (e_0^T \quad e_4^T \quad \tau e_1^T - \tau e_5^T \quad h e_1^T - h e_6^T)^T + 2(e_1^T \quad e_2^T \quad \tau e_5^T \quad h e_6^T) \\
&\quad \times Y (e_0^T \quad e_4^T \quad e_1^T - e_2^T \quad e_1^T - e_3^T)^T] \xi \quad (21)
\end{aligned}$$

From (16) to (21), one can see that

$$V = \xi^T \varphi \xi < 0. \quad (22)$$

Obviously, the inequality (22) holds if the condition (8) from Theorem 3.1. Following the theory in [30], the system (1) is asymptotically stable.

Remark 3.1 In the proof of this theorem, the functional V is general for considering much more information about the state vector. In particular, V_5 is very important to improve the delay-dependent stability condition. If one attempt to delete $x(t-\tau)$ from G_1 or G_2 , the maximum of time-delay may less than those of original V_5 . It is very important to use a new class of inequalities for improving delay-dependent stability condition. In short, the delay-dependent stability condition would be less conservative, which is shown in next section.

IV. EXAMPLE

In this section, we will show the superiority of our approach in the above section according to the conventional systems.

Example 1. Consider the system (1) with the state matrices are listed as the following:

$$A = \begin{pmatrix} -0.9 & 0.2 \\ 0.1 & -0.9 \end{pmatrix}, \quad B = \begin{pmatrix} -1.1 & -0.2 \\ -0.1 & -1.1 \end{pmatrix}, \quad C = \begin{pmatrix} -0.2 & 0 \\ 0.2 & -0.1 \end{pmatrix},$$

With different cases, we can obtain different maximum time delay h as listed in Table 1 and Table 2, compared with some existing references.

The first case is $\tau \neq h$, the maximum of h for different τ compared to the existing results are listed in Table 1. Clearly, the maximum of h using the Theorem 3.1 is very close to analytical bounds, which shows our approach is less conservative than the existing results. It is also shows that we could also obtain good condition without discrete Lyapunov functional approach.

The second case is $\tau = h$, the maximum of h compared to the existing results are listed in Table 2. Although our result is little less than that in [12], our condition is the best result except the condition obtain by using the approach in [12].

Furthermore, our approach may bring simple computation than that in [12]. Furthermore, using our approach may be more efficient to deal with neutral systems with mixed time-varying delay.

In a word, it is observed that our method presented in this paper is less conservative than most of the existing results.

TABLE I. THE MAXIMUM OF h FOR DIFFERENT τ

References	$\tau = 0.1$	$\tau = 0.5$	$\tau = 1$
[5]	1.7100	1.6718	1.6543
[6]	1.7844	1.7495	1.7201
[7]	1.8307	1.7755	1.7213
[12]	2.1229	2.1229	2.1229
[13]	2.2951	2.3471	2.3752
Theorem 3.1	2.2959	2.3488	2.3769
Analytical bounds	2.2963	2.3491	2.3775

TABLE II. THE MAXIMUM OF h_{AS} $h = \tau$

References	$\tau = 0.1$
[21]	1.3718
[5]	1.6527
[14]	1.7191
[15]	1.7220
[6]	1.7844
[16]	1.7856
[7]	1.8307
[12]	2.2254
Theorem 3.1	2.2250
Analytical bounds	2.2255

V. CONCLUSION

In this paper, an improved integral inequality has been introduced, as applications of this inequality, the delay-dependent stability problems for neutral systems have been investigated. Based on a new constructed Lyapunov Krasovskii functional, and combined with the new integral inequalities, a new stability criterion has been obtained for neutral systems with mixed time-delay. Finally, one example has shown the effectiveness and less conservativeness of the condition presented in this paper.

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