The (s,t)-Relaxed $L(2,1)$-Labeling of Some Balanced Hypercubes

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***Abstract*—For two vertices u and v in a graph G, we denote by $d_G(u,v)$ the distance between u and v. If $d_G(u,v)=i$, we say the vertex v is an i-neighbor of u. Let s, t and k be nonnegative integers. An (s,t)-relaxed $k-L(2,1)$-labeling f of G is an assignment of labels from $\{0,1,\cdots,k\}$ to the vertices of G if each of the following three conditions is met: (1) $f(u)\neq f(v)$ if $d_G(u,v)=1$; (2) for any vertex u of G, there are at most s 1-neighbors of u receiving labels from $\{f(u)-1, f(u)+1\}$; (3) for any vertex u of G, the number of 2-neighbors of u assigned the label $f(u)$ is at most t. The (s,t)-relaxed $L(2,1)$-labeling number $\lambda_{2,1}^{s,t}(G)$ of G is the minimum k such that G admits an (s,t)-relaxed k - $L(2,1)$-labeling. Huang and Wu in [IEEE Transactions on Computers 46 (1997) 484--490] introduced the balanced hypercube BH_n as an interconnection network topology for computing systems. In this paper, the values of the (s,t)-relaxed $L(2,1)$-labeling numbers of balanced hypercubes BH_2 and BH_3 with different pairs (s,t) are given.**

Keywords-relaxed L(2,1)-labeling problem; balanced hypercube; channel assignment problem; graph labeling

I. INTRODUCTION

For two vertices u and v in a graph G, we denote by $d_G(u,v)$ the distance between u and v. If $d_G(u,v)=i$, we say the vertex v is an i-neighbor of u. We denote by $\Delta(G)$ the maximum degree of a graph G and $\Delta_2(G)$ the maximum number of 2-neighbors of a vertex of G. Suppose (s,t) and (s',t') are two pairs of nonnegative integers. If $s\leq s'$ and $t\leq t'$, then we say (s,t) is less than or equal to (s',t'), and we write $(s,t)\preceq(s',t')$.

A kind of Channel Assignment Problem (CAP) asks for assigning frequencies to transmitters in a network with the aim of avoiding undesired interference. Suppose there are many radio transmitters in an area, transmitters that are close must receive frequencies that are sufficiently apart, for otherwise, they may interfere with each other so that they can not work normally. On the other hand, the spectrum of frequencies is a very important resource on which there are increasing demands, and they may be very limited. Therefore, we need an efficient management of the spectrum.

As a theoretical model of the Channel Assignment Problem, the $L(2,1)$-labeling problem was proposed and studied. It has been attracted considerable attention in the literature [1], [3], [4], [5], [6], [7], and there are more than 200 papers to studied CAP as well as its related problems. A k-$L(2,1)$-labeling f of a graph G is an assignment of labels from $\{0,1,\cdots,k\}$ to the vertices of G such that vertices at distance two get different labels and adjacent vertices get labels that are at least two apart. We say the value k to be the span of f. The λ-number $\lambda(G)$ of G is the minimum span k such that G admits a k-$L(2,1)$-labeling.

With the increasing demands of frequencies, the spectrum of frequencies may be a very limited. In such a case, there may be no optimal solution of the $L(2,1)$-labeling of a graph G, i.e., it is impossible to obtain an $L(2,1)$-labeling of G with too smaller span λ. This leads to the proposal of the concept of the (s,t)-relaxed $L(2,1)$-labeling which models CAP in this case [14]. Let s, t and k be nonnegative integers. An (s,t)-relaxed $k-L(2,1)$-labeling f of a graph G is an assignment of labels from $\{0,1,\cdots,k\}$ to the vertices of G if the following three conditions are met: (1) $f(u)\neq f(v)$ if $d_G(u,v)=1$; (2) for any vertex u of G, there are at most s 1-neighbors of u receiving labels from $\{f(u)-1, f(u)+1\}$; (3) for any vertex u of G, the number of 2-neighbors of u assigned the label $f(u)$ is at most t. The above conditions are called the (s,t)-relaxed $L(2,1)$ conditions.

The (s,t)-relaxed $L(2,1)$-labeling number $\lambda_{2,1}^{s,t}(G)$ of G is the minimum k such that G admits an (s,t)-relaxed $k-L(2,1)$-labeling. If $(s,t)=(0,0)$, the (s,t)-relaxed $L(2,1)$-labeling is the standard $L(2,1)$-labeling, and we simply write $\lambda_{2,1}^{0,0}(G)$ as $\lambda_{2,1}(G)$.

The (s,t)-relaxed $L(2,1)$-labeling problem has been studied in the literature for many classes of graphs, including the hexagonal lattice[11], the triangular lattice[12], and the square lattice[13]. It is of interest to investigate other classes of graphs. In this paper, we investigate the values of the (s,t)-relaxed $L(2,1)$-labeling numbers of balanced hypercubes BH_2 and BH_3 with different pairs (s,t).

II. COMPUTER SEARCH

We developed a backtracking procedure **labeling** which is implemented in C++ language. The function **check_labeling**, called by **labeling**, will check if the current labeling satisfies the (s,t)-relaxed $L(2,1)$ conditions.

```
void labeling(int ip, char **g, int **dist, int nv,
        int tot_c, int crs[], char **gc,  int s, int t)
{
   int i;
   if(ip == nv){
g_cn++;
for(i=0; i<nv; i++){
   printf("%d ", crs[i]);
}
printf("\n");

int *rs = new int[nv];
memcpy(rs, crs, nv*sizeof(int));
graph *p_graph = new graph[ng_sz];
memcpy(p_graph, ng_iso, ng_sz);
kg.p_graph = p_graph;
kg.g_sz = ng_sz;
g_crs_map[kg] = rs;
return ;
}
for(i=0; i<tot_c; i++) {
    if(check_labeling(ip,i,crs,tot_c, dist,s,t)) {
        crs[ip] = i;
        labeling(ip+1,g,dist,nv,tot_c, crs,gc,s,t);
    }
}
}
```

The backtracking algorithm corresponds to a search tree. The meaning of the parameters are as follows.

ip: controls the level of this search tree;

g: the tested graph;

dist: distance matrix of the graph g;

nv: vertex number of g;

tot_c: number of colors, i.e., tot_c=k+1 when testing (s,t) relaxed k-L(2,1)-labeling;

crs: an array to store colors of vertices of g;

gc: another graph needed to be compared with g;

By using the above approach, we succeed to obtain some optimal (s,t)-relaxed $L(2,1)$-labelings of balanced hypercubes BH_2 and BH_3.

III. RESULTS

Huang and Wu in [15] introduced the balanced hypercube BH_n as an interconnection network topology for computing systems as follows.

Definition 1: For $n \geq 1$, BH_n has $4n$ vertices, and each vertex has a unique n-component vector on $\{0,1,2,3\}$ for an address, also called an n-bit string. A vertex $(a_0, a_1, \cdots, a_{n-1})$ connects to the following $2n$ vertices:

$$\begin{cases} ((a_0+1) \mod 4, & a_1, \cdots, a_{n-1}) \\ ((a_0-1) \mod 4, & a_1, \cdots, a_{n-1}) \end{cases}$$

$$\begin{cases} ((a_0+1)\text{mod}4, a_1, \cdots, a_{i-1}, a_i + (-1)^{a_0} \text{mod}4, \\ a_{i+1}, \cdots, a_{n-1}) \\ ((a_0-1)\text{mod}4, a_1, \cdots, a_{i-1}, a_i + (-1)^{a_0} \text{mod}4, \\ a_{i+1}, \cdots, a_{n-1}) \end{cases}$$

It can be seen that $BH_2 \cong C_4$ and $BH_3 \cong C_8[2K_1]$, where $C_8[2K_1]$ is the lexicographic product of C_8 and $2K_1$. The graphs BH_2 and BH_3 are presented in Fig. 2 and Fig. 3, respectively.

By using procedure 1, we are able to compute the $\lambda_{2,1}^{s,t}$-numbers of BH_2 and BH_3. Before calling procedure 1, we set $g\_cn = 0$. After the procedure terminates, the tested graph has an (s,t)-relaxed $k-L(2,1)$-labeling if $g\_cn > 0$.

We show an example to obtain the value of $\lambda_{2,1}^{0,0}(BH_2) = 7$ as follows.

Example 1: $\lambda_{2,1}^{0,0}(BH_2) = 7$

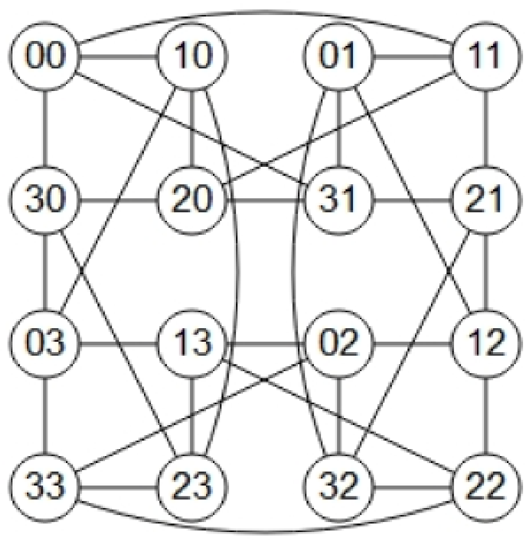

FIGURE I. THE GRAPH BH_2

TABLE I. $\lambda_{2,1}^{s,t}$ -NUMBER OF BH_2

t / s	0	1	2	3	4	≥ 5
0	7	4	4	4	4	4
1	7	4	4	4	3	2
2	7	3	3	3	3	2
3	7	3	3	3	3	2
≥ 4	7	3	3	2	2	1

Proof: By setting the parameters in the procedure **labeling** that $s=0, t=0, tot\_c=7$ and the global variable $gc\_n=0$, when the procedure terminates, we have that the variable $gc\_n=0$. Therefore, BH_2 does not admit a $(0,0)$-relaxed 6-$L(2,1)$-labeling, and so $\lambda_{2,1}^{0,0}(BH_2)\geq 7$. The labeling depicted in Fig. 3 is a $(0,0)$-relaxed 7-$L(2,1)$-labeling of BH_2, and so $\lambda_{2,1}^{0,0}(BH_2)\leq 7$.

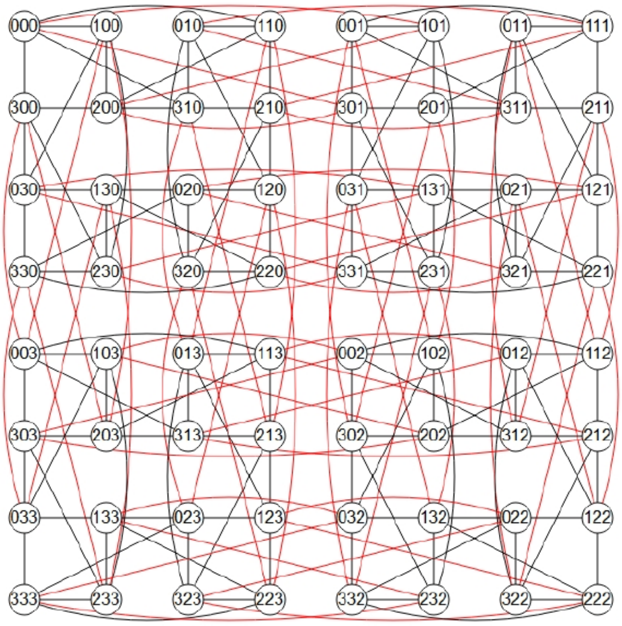

FIGURE II. THE GRAPH BH_3

In the following, we will use patterns to represent the labelings of BH_2 and BH_3. The pattern $\begin{bmatrix}5&2&3&0\\1&4&7&2\\6&4&0&6\\3&7&5&1\end{bmatrix}$ is the labeling corresponding to Fig. 3.

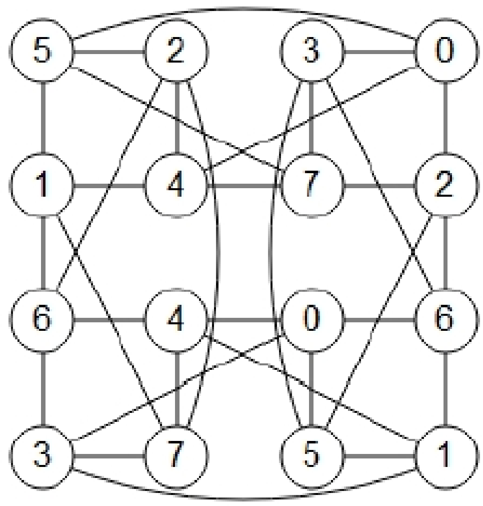

FIGURE III. THE GRAPH BH_2

In [14], the following five lemmas were established.

Lemma 1: Let G be a graph and H a subgraph of G. Then $\lambda_{2,1}^{s,t}(H)\leq\lambda_{2,1}^{s,t}(G)$ for any two nonnegative integers s and t.

Lemma 2: Let (s,t) and (s',t') be two pairs of nonnegative integers. If $(s,t)\preceq(s',t')$, then $\lambda_{2,1}^{s,t}(G)\geq\lambda_{2,1}^{s',t'}(G)$

Lemma 3: If $s\geq\Delta(G)$ and $t\geq\Delta_2(G)$, then $\lambda_{2,1}^{s,t}(G)=\chi(G)-1$.

It can be seen that $\Delta(BH_2)=4$, $\Delta_2(BH_2)=5$, $\Delta(BH_3)=6$, and $\Delta_2(BH_3)=13$. With the above procedure, we are able to determine the $\lambda_{2,1}^{s,t}$-numbers of BH_2 for $s\leq 4, t\leq 5$, and of BH_3 for $s\leq 6, t\leq 13$. Together with Lemmas 1-3, the results of $\lambda_{2,1}^{s,t}$-numbers of BH_2 and BH_3 for all pairs of (s,t) can be determined, and they are shown in Table I and II, respectively. Let $P_{s,t}$ and $Q_{s,t}$ be the patterns of an (s,t)-relaxed $L(2,1)$-labeling of BH_2 and BH_3, respectively. We provide some patterns (which give upper bounds of $\lambda_{2,1}^{s,t}$-numbers) as follows.

$$\boldsymbol{P}_{3,0}=\begin{bmatrix}5&3&2&7\\4&6&1&3\\2&7&6&4\\5&0&0&1\end{bmatrix},\ \boldsymbol{P}_{1,3}=\begin{bmatrix}4&2&3&0\\2&4&1&2\\0&3&2&4\\4&0&0&1\end{bmatrix},$$

$$\boldsymbol{P}_{0,5}=\begin{bmatrix}2&4&2&0\\4&2&4&2\\0&3&0&4\\4&0&4&1\end{bmatrix},\ \boldsymbol{P}_{3,4}=\begin{bmatrix}3&1&3&2\\1&0&1&0\\3&2&0&1\\1&0&1&0\end{bmatrix},$$

$$\boldsymbol{P}_{3,5}=\begin{bmatrix}2&0&2&0\\0&2&0&2\\2&0&2&0\\0&2&0&2\end{bmatrix},\ \boldsymbol{P}_{4,2}=\begin{bmatrix}2&3&2&1\\3&0&1&3\\2&3&1&0\\0&1&0&1\end{bmatrix},$$

$$\boldsymbol{P}_{4,4}=\begin{bmatrix}1&2&2&0\\2&1&0&2\\0&1&0&1\\1&0&1&0\end{bmatrix},\ \boldsymbol{P}_{4,5}=\begin{bmatrix}1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&1\end{bmatrix}.$$

TABLE II. $\lambda_{2,1}^{s,t}$ -NUMBER OF BH_3

s \ t	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**≥ 13**
0	10	6	6	5	5	4	4	4	4	3	3	3	3	2
1	7	6	6	5	5	4	4	4	4	3	3	3	3	2
2	7	4	4	4	4	3	3	3	3	3	3	3	3	2
3	7	4	4	4	4	3	3	3	3	3	3	3	3	2
4	7	3	3	3	3	3	3	3	3	3	3	3	3	2
5	7	3	3	3	3	3	3	3	3	3	3	3	3	2
≥ 6	7	3	3	3	3	3	3	2	2	2	2	2	2	1

$$Q_{0,0}=\begin{bmatrix}10&7&3&1&1&4&6&10\\6&9&0&4&3&0&9&7\\3&9&1&7&7&0&9&4\\10&4&6&0&1&6&3&10\\0&3&6&10&9&7&3&0\\4&1&9&7&6&10&1&4\\6&1&9&3&4&10&1&7\\0&7&4&10&9&3&6&0\end{bmatrix},$$

$$Q_{1,2}=\begin{bmatrix}1&4&1&4&2&6&6&4\\0&2&3&1&5&0&5&0\\3&6&3&6&3&0&6&1\\1&2&6&4&0&4&2&6\\6&2&6&1&1&4&5&3\\3&6&1&5&5&0&3&5\\4&0&3&0&3&6&4&2\\0&5&1&2&6&2&2&4\end{bmatrix},$$

$$\boldsymbol{Q}_{1,4}=\begin{bmatrix}2&4&3&5&5&0&4&0\\4&2&5&3&0&5&1&4\\1&5&3&1&3&5&3&1\\5&2&5&3&0&2&1&4\\0&3&0&5&5&3&3&0\\3&1&4&0&2&5&0&2\\0&2&0&2&5&1&4&1\\2&0&2&4&0&4&0&4\end{bmatrix},$$

$$\boldsymbol{Q}_{1,8}=\begin{bmatrix}2&0&2&0&4&1&4&0\\0&3&4&2&0&3&0&4\\4&2&0&4&4&2&4&2\\2&4&4&0&1&4&2&0\\3&0&2&0&3&1&4&0\\1&4&0&2&1&3&0&4\\4&2&0&4&4&2&0&2\\2&4&4&0&2&4&2&0\end{bmatrix},$$

$$\boldsymbol{Q}_{3,4}=\begin{bmatrix}3&2&4&0&4&0&0&3\\2&4&0&4&0&1&3&1\\0&2&4&1&3&4&2&3\\3&1&1&0&4&3&3&2\\4&1&1&3&0&2&1&2\\3&0&3&1&3&0&2&1\\4&2&0&3&0&1&2&4\\2&4&4&0&1&4&0&2\end{bmatrix},$$

$$\boldsymbol{Q}_{5,12}=\begin{bmatrix}3&1&0&2&3&1&0&1\\1&0&1&0&1&0&1&0\\0&2&3&1&0&2&3&1\\1&3&1&0&1&0&1&0\\3&2&3&1&3&2&0&2\\1&3&1&0&1&0&1&0\\3&1&0&2&3&1&0&1\\1&0&1&0&1&0&1&0\end{bmatrix},$$

$$\boldsymbol{Q}_{6,6}=\begin{bmatrix}3&1&0&2&3&2&2&1\\2&0&2&3&1&0&1&0\\0&3&0&2&3&2&3&1\\3&0&2&0&1&3&1&3\\0&3&3&2&0&2&3&1\\3&0&1&3&2&0&1&3\\2&3&1&0&1&0&1&0\\0&1&0&1&0&1&0&1\end{bmatrix},$$

$$\boldsymbol{Q}_{6,12}=\begin{bmatrix}1&2&1&2&2&0&2&0\\2&1&2&1&0&2&0&2\\0&2&1&2&2&1&2&0\\2&0&2&1&1&2&0&2\\0&1&0&1&0&1&0&1\\1&0&1&0&1&0&1&0\\0&1&0&1&0&1&0&1\\1&0&1&0&1&0&1&0\end{bmatrix},$$

$$\boldsymbol{Q}_{6,13}=\begin{bmatrix}1&0&1&0&1&0&1&0\\0&1&0&1&0&1&0&1\\1&0&1&0&1&0&1&0\\0&1&0&1&0&1&0&1\\1&0&1&0&1&0&1&0\\0&1&0&1&0&1&0&1\\1&0&1&0&1&0&1&0\\0&1&0&1&0&1&0&1\end{bmatrix}.$$

IV. CONCLUSION

The proposal of the (s,t) -relaxed $L(2,1)$ -labeling problem is motivated with Channel Assignment Problem (CAP) with limited spectrum frequencies. In this paper, we applied a backtracking algorithm to study the (s,t) -relaxed $L(2,1)$ -labeling for an interconnection network topology for computing systems: the balanced hypercubes BH_2 and BH_3. We succeed to determine all the (s,t) -relaxed $L(2,1)$ -labeling numbers of BH_2 and BH_3 for all cases (s,t). Since the (s,t) -relaxed $L(2,1)$ -labeling numbers of few classes of graphs are known (e.g., square lattice, triangular lattice, hexagonal lattice), it is still of interest to investigate the (s,t) -relaxed $L(2,1)$ -labeling numbers of other classes of graphs.

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