Analysis of Stability of Bars with Continuously Varying Cross Section by Means of Differential Equation of Matrix

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Abstract: A general differential equation of matrix with variable coefficient is derived for analysis of stability of the various bars with continuously varying cross section and different constraints at ends, which beneficial for engineering design.

Introduction

In existent methods of analysis of stability of bars with continuously varying cross section, only the stability of several kinds of bars with particular shapes can be analyzed. Euler discussed columns of shapes, including a truncated cone and pyramid. The stability of bars bounded by a surface of revolution of the second degree was discussed by Lagrange.[1]

In this paper a general differential equation of matrix with variable coefficient is derived for analysis of stability of various bars with continuously varying cross section. By means of this equation the stability of the various bars with continuously varying cross section and different constraints at ends can be analyzed.

Methods and example

Consider a bar with continuously varying cross section in the critical state. The critical load is P_{cr}. The coordinate axes and positive directions of the bending moment and the shearing force are shown in Fig. 1.

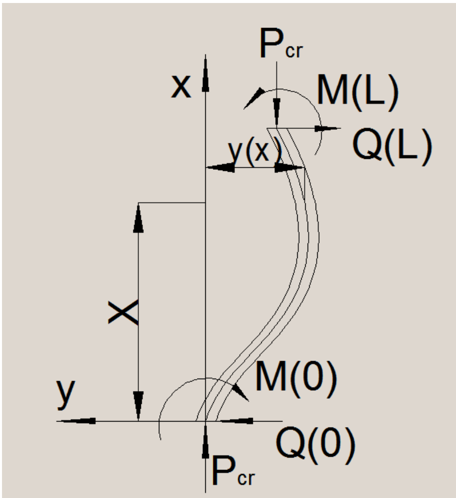

FIG.1

The shearing force Q(x) and bending moment M(x) at x cross section are, respectively,

$$\begin{cases} Q(x) = Q(0) \\ M(x) = M(0) + Q(0)x - P_{cr}y(x) \end{cases} \qquad (1)$$

The deflection y(x) , the rotation angle θ(x) , the bending moment M(x) and the shearing force Q(x) at x cross section exists the following differential relations:

$$\begin{cases} \dfrac{dy(x)}{dx} = \theta(x) \\ \dfrac{d\theta(x)}{dx} = \dfrac{M(x)}{EI(x)} \\ \dfrac{dM(x)}{dx} = Q(0) - P_{cr}\theta(x) = Q(x) - P_{cr}\theta(x) \\ \dfrac{dQ(x)}{dx} = 0 \end{cases} \tag{2}$$

in which I(x) is the moment inertia and EI(x) is the flexural rigidity at x cross section.

Eq. (2) can be written into the differential equation of matrix

$$\frac{d}{dx}\begin{bmatrix} y(x) \\ \theta(x) \\ M(x) \\ Q(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{1}{EI(x)} & 0 \\ 0 & -P_{cr} & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y(x) \\ \theta(x) \\ M(x) \\ Q(x) \end{bmatrix} \tag{3}$$

The following notations are used:

$$A(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{1}{EI(x)} & 0 \\ 0 & -P_{cr} & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and $X(x) = [y(x) \quad \theta(x) \quad M(x) \quad Q(x)]^T$.

X(x) is called state variable which expresses the state of the internal force and the displacements at x cross section. Eq. (3) can be written into the differential equation which is expressed by state variable.

$$\dot{X}(x) = A(x)X(x) \tag{4}$$

It is the differential equation of matrix with variable coefficient. In modern mathematical theory of matrix it has been shown that the solution of Eq. (4) is

$$X(x) = T(x)X(0) \tag{5}$$

when $X(0) = C$, where T(x) is the transfer matrix which is calculated by the method of iteration[2]:

$$T_0(x) = I$$

$$T_{n+1}(x) = I + \int_0^x A(x)T_n(x)dx \quad \text{(n=0,1,2,...)} \tag{6}$$

where I is the unit matrix.

Substituting $x = l$ into Eq. (5), the relation of state variables of both ends of bar is obtained:

$$X(l) = T(l)X(0) \tag{7}$$

i.e.

$$\begin{bmatrix} y(l) \\ \theta(l) \\ M(l) \\ Q(l) \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{41} & t_{42} & t_{43} & t_{44} \end{bmatrix} \begin{bmatrix} y(0) \\ \theta(0) \\ M(0) \\ Q(0) \end{bmatrix} \tag{8}$$

Substituting the conditions of constraints at both ends of bar into Eq. (8) ,the critical compressive load of bars can be found out.

Example: A bar with continuously varying circular cross-section is shown in Fig.2. Its bottom diameter is D_0 and every section diameters accord with $D_x = D_0 e^{-m\frac{x}{l}}$.

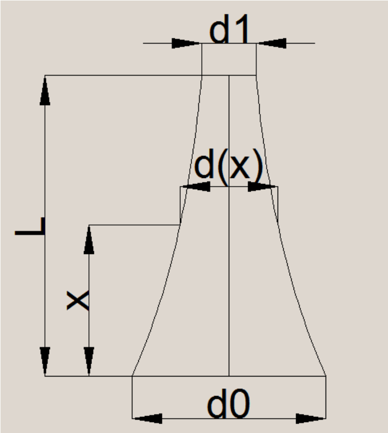

Fig.2

Let $K = \dfrac{Pl^2}{EI_0}$ in which I_0 is the moment of inertia at the bottom.

Then $P_{cr} = K\dfrac{EI_0}{l^2}$.

While $m = 0.5$, its top diameter is $D_1 = D_0 e^{-0.5} = 0.6065 D_0$.

Calculation the K values of two situations:

1) the top end fixed and the bottom end hinged;

2) the top end hinged and the bottom end fixed.

The boundary conditions of the top end fixed and the bottom end hinged are

$$\begin{cases} x = 0 & y^b = 0 \quad \theta^b = 0 \\ x = l & y^u = 0 \quad M^u = 0 \end{cases}$$

Using Eq. (8) it can be obtained that

$$\begin{cases} y(l) = t_{12}\theta(0) + t_{14}Q(0) = 0 \\ Q(l) = t_{22}\theta(0) + t_{24}Q(0) = 0 \end{cases}$$

Cause $M(0)$ and $Q(0)$ are not zero, so

$$\Delta(K) = \begin{vmatrix} t_{12} & t_{14} \\ t_{22} & t_{24} \end{vmatrix} = 0.$$

Likewise, the boundary conditions of the top end hinged and the bottom end fixed are

$$\begin{cases} x = 0 & y^b = 0 \quad M^b = 0 \\ x = l & y^u = 0 \quad \theta^u = 0 \end{cases}$$

Using Eq. (8) it can be obtained that

$$\begin{cases} y(l) = t_{13}M(0) + t_{14}Q(0) = 0 \\ Q(l) = t_{33}M(0) + t_{34}Q(0) = 0 \end{cases}$$

Cause $M(0)$ and $Q(0)$ are not zero, so

$$\Delta(K)=\begin{vmatrix} t_{13} & t_{14} \\ t_{33} & t_{34} \end{vmatrix}=0.$$

The calculated K values are shown in Table 1

Iteration times n	The top end fixed and the bottom end hinged	The top end hinged and the bottom end fixed
10	7.372681388387784544412543	7.472739910044923593494 03
12	6.74931824141975773406656	6.79913891555028959009334
14	6.79123476880484794557665	6.84310972346347084211993
16	6.78755586815925818722193	6.83921651904333802960563
18	6.78779748421229039392066	6.83947399301546075537347
20	6.78778466588288617394692	6.83946023396047644590350

Table 1. The calculated K values while m=0.5

Different from the constant section bars, as shown in Table 1, the critical compressive loads of bars with continuously varying cross sections have the same boundary conditions but at the different ends are different.

Conclusion

Using the methods in this paper, we can calculate the compressive loads of bars with continuously varying cross section easily. And using the methods in my early paper[3] at the same time, we can calculate the compressive loads of more complex bars.

References

[1] S. P. Timoshenko, J. F. Gere, Theory of elastic stability, second ed., Dover, New York, 2009, Chap.1;

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[3] Y. Zhang, The Matrix transfer method to analyze the stability of bars, Journal of Guizhou University of technology (Natural science edition), Vol.29, 5(2000), 7-12.