Symbolic Computation of Exact Solutions of Two Nonlinear Lattice Equations

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Abstract. In this paper, a modified discrete G'/G-expansion method is used to construct exact solutions of Toda lattice equation and Ablowitz-Ladik lattice equations. With the aid of computer symbolic computation, we obtained in a uniform way hyperbolic function solutions, trigonometric function solutions and rational solutions of these two nonlinear lattice equations. When the parameters are taken as special values, some known solutions are recovered. It is shown that the modified method with symbolic computation provides a more effective mathematical tool for solving nonlinear lattice equations in science and enginnering.

Introduction

Solving nonlinear lattice equations plays an important role in many fields of science and enginnering. In the past several decades, many effective methods for constructing exact solutions of nonlinear partial differential equations (PDEs) have been proposed, such as those in [1-15]. Usually, it is hard to generalize one method for nonlinear PDEs to solve nonlinear lattice equations. In 2008, Wang, Li and Zhang [16] proposed a new method called the G'/G-expansion method to find travelling wave solutions of nonlinear PDEs. Some researchers, such as Wang et al. [17] and Ebadi and Biswas [18-20], have done significant work using the method to construct hyperbolic function solutions, trigonometric function solutions and rational solutions of some important equations. This method was generalized by Zhang, Tong and Wang [21] for nonlinear PDEs with variable coefficients. More recently, Zhang et al. [22] found the iterative relations between the lattice indices by careful analysis and devised an effective discrete algorithm for using the G'/G-expansion method [16] to construct hyperbolic function solutions and trigonometric function solutions of nonlinear differential- difference equations (DDEs). Later, Zhang et al. [23] employed an embedded parameter to modify the algorithm in [22] for not only hyperbolic function solutions, trigonometric function but also rational solutions of nonlinear DDEs.

In order to show the validity and advantages of the improved method, we shall use the modified discrete method [23] to solve the Toda lattice equation and the Ablowitz-Ladik lattice equations in [24].

Exact solutions of Toda lattice equation

Let us first consider the famous Toda lattice equation [36]:

$$\frac{\mathrm{d}^2u_n(t)}{\mathrm{d}t^2}-\left(\frac{\mathrm{d}u_n(t)}{\mathrm{d}t}+1\right)[u_{n+1}(t)+u_{n-1}(t)-2u_n(t)]=0. \tag{1}$$

We use the wave transformation $u_n(t)=U_n(\xi_n)$, $\xi_n=d_1n+c_1t+\zeta$, then Eq. (1) becomes

$$c_1^2U_n''-(c_1U_n'+1)(U_{n+1}+U_{n-1}-2U_n)=0. \tag{2}$$

We suppose Eq. (2) has the solution in the form

$$U_n = \alpha_1\left(\frac{G'(\xi_n)}{G(\xi_n)}\right) + \alpha_0, \quad \alpha_1 \neq 0, \tag{3}$$

$$U_{n+1} = \alpha_1\left[\frac{\sqrt{\delta+\varepsilon(\lambda^2-4\mu)}}{2}\left(\frac{\frac{2}{\sqrt{\delta+\varepsilon(\lambda^2-4\mu)}}\left(\frac{G'(\xi_n)}{G(\xi_n)}+\frac{\lambda}{2}\right)+\varepsilon f(\frac{\sqrt{\delta+\varepsilon(\lambda^2-4\mu)}}{2}d_1)}{1+\left(\frac{2}{\sqrt{\delta+\varepsilon(\lambda^2-4\mu)}}\left(\frac{G'(\xi_n)}{G(\xi_n)}+\frac{\lambda}{2}\right)\right)f(\frac{\sqrt{\delta+\varepsilon(\lambda^2-4\mu)}}{2}d_1)}\right)-\frac{\lambda}{2}\right]+\alpha_0, \tag{4}$$

$$U_{n-1} = \alpha_1\left[\frac{\sqrt{\delta+\varepsilon(\lambda^2-4\mu)}}{2}\left(\frac{\frac{2}{\sqrt{\delta+\varepsilon(\lambda^2-4\mu)}}\left(\frac{G'(\xi_n)}{G(\xi_n)}+\frac{\lambda}{2}\right)-\varepsilon f(\frac{\sqrt{\delta+\varepsilon(\lambda^2-4\mu)}}{2}d_1)}{1-\left(\frac{2}{\sqrt{\delta+\varepsilon(\lambda^2-4\mu)}}\left(\frac{G'(\xi_n)}{G(\xi_n)}+\frac{\lambda}{2}\right)\right)f(\frac{\sqrt{\delta+\varepsilon(\lambda^2-4\mu)}}{2}d_1)}\right)-\frac{\lambda}{2}\right]+\alpha_0, \tag{5}$$

where $G(\xi_n)$ satisfies

$$\frac{\mathrm{d}^2G(\xi_n)}{\mathrm{d}\xi_n^2}+\lambda\frac{\mathrm{d}G(\xi_n)}{\mathrm{d}\xi_n}+\mu G(\xi_n)=0. \tag{6}$$

Substituting Eqs. (3)-(5) along with Eq. (6) into Eq. (2) and using *Mathematica*, we obtain a set of algebraic equations for α_0, α_1, d_1 and c_1. Solving the set of algebraic equations, we have three cases

$$\alpha_1 = \pm\frac{2\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}d_1)}{\sqrt{\lambda^2-4\mu}}, \quad \alpha_0=\alpha_0, \quad c_1=\pm\frac{2\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}d_1)}{\sqrt{\lambda^2-4\mu}}, \quad d_1=d_1, \quad \varepsilon=1, \quad \delta=0, \tag{7}$$

$$\alpha_1 = \pm\frac{2\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}d_1)}{\sqrt{4\mu-\lambda^2}}, \quad \alpha_0=\alpha_0, \quad c_1=\pm\frac{2\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}d_1)}{\sqrt{4\mu-\lambda^2}}, \quad d_1=d_1, \quad \varepsilon=-1, \quad \delta=0, \tag{8}$$

$$\alpha_1=\pm d_1, \quad \alpha_0=\alpha_0, \quad c_1=\pm d_1, \quad d_1=d_1, \quad \varepsilon=0, \quad \delta=4, \quad \mu=\frac{\lambda^2}{4}. \tag{9}$$

When $\lambda^2-4\mu>0$, we obtain hyperbolic function solution of Eq. (1):

$$u_n=\pm\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}d_1)\left(\frac{C_1\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n)+C_2\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n)}{C_1\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n)+C_2\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n)}\right)\mp\frac{\lambda\sinh(\frac{\sqrt{4\mu-\lambda^2}}{2}d_1)}{\sqrt{\lambda^2-4\mu}}+\alpha_0, \tag{10}$$

where $\xi_n=d_1n\pm\frac{2\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}d_1)}{\sqrt{\lambda^2-4\mu}}t+\zeta$. Setting $\mu=\frac{\lambda^2-4}{4}$, $\alpha_0=a_0\pm\frac{\lambda\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}d_1)}{\sqrt{\lambda^2-4\mu}}$, $d_1=k$, $\zeta=c$ and $C_2=0$, from solution (10) we obtain

$$u_n=a_0\pm\sinh(k)\tanh(kn\pm\sinh(k)t+c), \tag{11}$$

which is the known kink-type solitary wave solution in [24].

If set $\mu=\frac{\lambda^2+4}{4}$, $\alpha_0=a_0\pm\frac{\lambda\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}d_1)}{\sqrt{\lambda^2-4\mu}}$, $d_1=k$, $\zeta=c$ and $C_1=0$, from solution (10) we obtain

$$u_n = a_0 \pm \sinh(k)\coth(kn \pm \sinh(k)t + c), \tag{12}$$

which is the known singular travelling wave solution in [24].

When $\lambda^2 - 4\mu < 0$, we obtain hyperbolic function solution of Eq. (1):

$$u_n = \pm\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}d_1)\left(\frac{-C_1\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n) + C_2\cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n)}{C_1\cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n) + C_2\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n)}\right) \mp \frac{\lambda\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}d_1)}{\sqrt{4\mu-\lambda^2}} + \alpha_0, \tag{13}$$

where $\xi_n = d_1 n \pm \frac{2\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}d_1)}{\sqrt{4\mu-\lambda^2}}t + \zeta$. Setting $\mu = \frac{\lambda^2+4}{4}$, $\alpha_0 = a_0 \pm \frac{\lambda\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}d_1)}{\sqrt{\lambda^2-4\mu}}$, $d_1 = k$, $\zeta = c$ and $C_2 = 0$, from solution (13) we obtain

$$u_n = a_0 \pm \sin(k)\tan(kn \pm \sin(k)t + c), \tag{14}$$

which is the known periodic wave solution in [24].

If set $\mu = \frac{\lambda^2+4}{4}$, $\alpha_0 = a_0 \pm \frac{\lambda sin(\frac{\sqrt{4\mu-\lambda^2}}{2}d_1)}{\sqrt{4\mu-\lambda^2}}$, $d_1 = k$, $\zeta = c$ and $C_1 = 0$, from solution (13) we obtain

$$u_n = a_0 \pm \sin(k)\cot(kn \pm \sin(k)t + c), \tag{15}$$

which is the known periodic wave solution in [24].

When $\lambda^2 - 4\mu = 0$, we obtain rational solution of Eq. (1):

$$u_n = \alpha_0 \mp \frac{1}{2}d_1\lambda \pm \frac{d_1 C_2}{C_1 + C_2\xi_n}, \tag{16}$$

where $\xi_n = d_1 n \pm d_1 t + \zeta$.

Setting $\alpha_0 = a_0 \pm \frac{1}{2}d_1\lambda$, $d_1 = k$, $\zeta = c$ and $C_1 = 0$, from solution (16) we obtain

$$u_n = a_0 \pm \frac{k}{kn \pm kt + c}, \tag{17}$$

which is the known rational solution in [24].

Exact solutions of Ablowitz-Ladik lattice equations

We next consider the Ablowitz-Ladik lattice equations [24]:

$$\frac{\mathrm{d}u_n(t)}{\mathrm{d}t} - [\alpha + u_n(t)v_n(t)][u_{n+1}(t) + u_{n-1}(t)] + 2\alpha u_n(t) = 0, \tag{18}$$

$$\frac{\mathrm{d}v_n(t)}{\mathrm{d}t} + [\alpha + u_n(t)v_n(t)][v_{n+1}(t) + v_{n-1}(t)] - 2\alpha v_n(t) = 0. \tag{19}$$

We use the wave transformation $u_n = U_n(\xi_n)$, $v_n = V_n(\xi_n)$, $\xi_n = d_1 n + c_1 t + \zeta$, then Eqs. (18) and (19) become

$$c_1 U_n' - (\alpha + U_n V_n)(U_{n+1} + U_{n-1}) + 2\alpha U_n = 0, \tag{20}$$

$$c_1 V_n' + (\alpha + U_n V_n)(V_{n+1} + V_{n-1}) - 2\alpha V_n = 0. \tag{21}$$

According to the homogeneous balance procedure, we suppose that Eqs. (20) and (21) have the following formal solutions:

$$U_n = \alpha_1\left(\frac{G'(\xi_n)}{G(\xi_n)}\right)+\alpha_0, \quad \alpha_1 \neq 0, \tag{22}$$

$$U_{n+1} = \alpha_1\left[\frac{\sqrt{\delta+\varepsilon(\lambda^2-4\mu)}}{2}\left(\frac{\frac{2}{\sqrt{\delta+\varepsilon(\lambda^2-4\mu)}}\left(\frac{G'(\xi_n)}{G(\xi_n)}+\frac{\lambda}{2}\right)+\varepsilon f(\frac{\sqrt{\delta+\varepsilon(\lambda^2-4\mu)}}{2}d_1)}{1+\left(\frac{2}{\sqrt{\delta+\varepsilon(\lambda^2-4\mu)}}\left(\frac{G'(\xi_n)}{G(\xi_n)}+\frac{\lambda}{2}\right)\right)f(\frac{\sqrt{\delta+\varepsilon(\lambda^2-4\mu)}}{2}d_1)}\right)-\frac{\lambda}{2}\right]+\alpha_0, \tag{23}$$

$$U_{n-1} = \alpha_1\left[\frac{\sqrt{\delta+\varepsilon(\lambda^2-4\mu)}}{2}\left(\frac{\frac{2}{\sqrt{\delta+\varepsilon(\lambda^2-4\mu)}}\left(\frac{G'(\xi_n)}{G(\xi_n)}+\frac{\lambda}{2}\right)-\varepsilon f(\frac{\sqrt{\delta+\varepsilon(\lambda^2-4\mu)}}{2}d_1)}{1-\left(\frac{2}{\sqrt{\delta+\varepsilon(\lambda^2-4\mu)}}\left(\frac{G'(\xi_n)}{G(\xi_n)}+\frac{\lambda}{2}\right)\right)f(\frac{\sqrt{\delta+\varepsilon(\lambda^2-4\mu)}}{2}d_1)}\right)-\frac{\lambda}{2}\right]+\alpha_0, \tag{24}$$

$$V_n = \beta_1\left(\frac{G'(\xi_n)}{G(\xi_n)}\right)+\beta_0, \quad \beta_1 \neq 0, \tag{25}$$

$$V_{n+1} = \beta_1\left[\frac{\sqrt{\delta+\varepsilon(\lambda^2-4\mu)}}{2}\left(\frac{\frac{2}{\sqrt{\delta+\varepsilon(\lambda^2-4\mu)}}\left(\frac{G'(\xi_n)}{G(\xi_n)}+\frac{\lambda}{2}\right)+\varepsilon f(\frac{\sqrt{\delta+\varepsilon(\lambda^2-4\mu)}}{2}d_1)}{1+\left(\frac{2}{\sqrt{\delta+\varepsilon(\lambda^2-4\mu)}}\left(\frac{G'(\xi_n)}{G(\xi_n)}+\frac{\lambda}{2}\right)\right)f(\frac{\sqrt{\delta+\varepsilon(\lambda^2-4\mu)}}{2}d_1)}\right)-\frac{\lambda}{2}\right]+\beta_0, \tag{26}$$

$$V_{n-1} = \beta_1\left[\frac{\sqrt{\delta+\varepsilon(\lambda^2-4\mu)}}{2}\left(\frac{\frac{2}{\sqrt{\delta+\varepsilon(\lambda^2-4\mu)}}\left(\frac{G'(\xi_n)}{G(\xi_n)}+\frac{\lambda}{2}\right)-\varepsilon f(\frac{\sqrt{\delta+\varepsilon(\lambda^2-4\mu)}}{2}d_1)}{1-\left(\frac{2}{\sqrt{\delta+\varepsilon(\lambda^2-4\mu)}}\left(\frac{G'(\xi_n)}{G(\xi_n)}+\frac{\lambda}{2}\right)\right)f(\frac{\sqrt{\delta+\varepsilon(\lambda^2-4\mu)}}{2}d_1)}\right)-\frac{\lambda}{2}\right]+\beta_0, \tag{27}$$

where $G(\xi_n)$ satisfies Eq. (6).

Substituting Eqs. (22)-(27) along with Eq. (6) into Eqs. (20) and (21) and using *Mathematica*, we obtain a set of algebraic equations for α_0, α_1, β_0, β_1, d_1 and c_1. Solving the set of algebraic equations, we have three cases

$$\alpha_1 = \alpha_1, \quad \alpha_0 = \frac{1}{2}\alpha_1(\lambda \pm \sqrt{\lambda^2-4\mu}), \quad \beta_0 = \frac{2\alpha(-\lambda \pm \sqrt{\lambda^2-4\mu})\sinh^2(\frac{\sqrt{\lambda^2-4\mu}}{2}d_1)}{\alpha_1(\lambda^2-4\mu)}, \tag{28}$$

$$\beta_1 = -\frac{4\alpha\sinh^2(\frac{\sqrt{\lambda^2-4\mu}}{2}d_1)}{\alpha_1(\lambda^2-4\mu)}, \quad c_1 = \pm\frac{4\alpha\sinh^2(\frac{\sqrt{\lambda^2-4\mu}}{2}d_1)}{\sqrt{\lambda^2-4\mu}}, \quad d_1 = d_1, \quad \varepsilon = 1, \quad \delta = 0, \tag{29}$$

$$\alpha_1 = \alpha_1, \quad \alpha_0 = \frac{1}{2}\alpha_1(\lambda \pm \mathrm{i}\sqrt{4\mu-\lambda^2}), \quad \beta_0 = \frac{2\alpha(-\lambda \pm \sqrt{4\mu-\lambda^2})\sin^2(\frac{\sqrt{4\mu-\lambda^2}}{2}d_1)}{\alpha_1(4\mu-\lambda^2)}, \tag{30}$$

$$\beta_1 = -\frac{4\alpha\sin^2(\frac{\sqrt{4\mu-\lambda^2}}{2}d_1)}{\alpha_1(4\mu-\lambda^2)}, \quad c_1 = \pm\frac{4\mathrm{i}\alpha\sin^2(\frac{\sqrt{4\mu-\lambda^2}}{2}d_1)}{\sqrt{4\mu-\lambda^2}}, \quad d_1 = d_1, \quad \varepsilon = -1, \quad \delta = 0, \tag{31}$$

$$\alpha_1 = \alpha_1, \quad \alpha_0 = \frac{1}{2}\alpha_1\lambda, \quad \beta_0 = -\frac{\alpha\lambda d_1^2}{2\alpha_1}, \quad \beta_1 = -\frac{\alpha d_1^2}{\alpha_1}, \quad c_1 = 0, \quad d_1 = d_1, \quad \varepsilon = 0, \quad \delta = 4, \quad \mu = \frac{\lambda^2}{4}, \tag{32}$$

When $\lambda^2 - 4\mu > 0$, we obtain hyperbolic function solutions of Eqs. (18) and (19):

$$u_n = \frac{1}{2}\alpha_1\sqrt{\lambda^2-4\mu}\left(\frac{C_1\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n) + C_2\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n)}{C_1\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n) + C_2\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n)}\right) \pm \frac{1}{2}\alpha_1\sqrt{\lambda^2-4\mu}, \tag{33}$$

$$v_n = -\frac{2\alpha\sinh^2(\frac{\sqrt{\lambda^2-4\mu}}{2}d_1)}{\alpha_1\sqrt{\lambda^2-4\mu}}\left(\frac{C_1\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n) + C_2\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n)}{C_1\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n) + C_2\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi_n)}\right) \pm \frac{2\alpha\sinh^2(\frac{\sqrt{\lambda^2-4\mu}}{2}d_1)}{\alpha_1\sqrt{\lambda^2-4\mu}}, \tag{34}$$

where $\xi_n = d_1 n \pm \frac{4\alpha\sinh^2(\frac{\sqrt{\lambda^2-4\mu}}{2}d_1)}{\sqrt{\lambda^2-4\mu}}t + \zeta$. Setting $\mu = \frac{\lambda^2-4}{4}$ and $C_2 = 0$, from solutions (33) and (34) we have

$$u_n = \pm\alpha_1 + \alpha_1\tanh(d_1 n \pm 2\alpha\sinh^2(d_1)t + \zeta), \tag{35}$$

$$v_n = -\frac{\alpha\sinh^2(d_1)}{\alpha_1}\tanh(d_1 n \pm 2\alpha\sinh^2(d_1)t + \zeta) \pm \frac{\alpha\sinh^2(d_1)}{\alpha_1}, \tag{36}$$

which are equivalent to the kink-type solitary wave solutions in [24].

If set $\mu = \frac{\lambda^2-4}{4}$ and $C_1 = 0$, then solutions (33) and (34) become

$$u_n = \pm\alpha_1 + \alpha_1\coth(d_1 n \pm 2\alpha\sinh^2(d_1)t + \zeta), \tag{37}$$

$$v_n = -\frac{\alpha\sinh^2(d_1)}{\alpha_1}\coth(d_1 n \pm 2\alpha\sinh^2(d_1)t + \zeta) \pm \frac{\alpha\sinh^2(d_1)}{\alpha_1}, \tag{38}$$

which are equivalent to the singular travelling wave solutions in [24].

When $\lambda^2 - 4\mu < 0$, we obtain hyperbolic function solutions of Eqs. (18) and (19):

$$u_n = \frac{1}{2}\alpha_1\sqrt{4\mu-\lambda^2}\left(\frac{-C_1\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n) + C_2\cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n)}{C_1\cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n) + C_2\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n)}\right) \pm \frac{\mathrm{i}}{2}\alpha_1\sqrt{4\mu-\lambda^2}, \tag{39}$$

$$v_n = -\frac{2\alpha\sin^2(\frac{\sqrt{4\mu-\lambda^2}}{2}d_1)}{\alpha_1\sqrt{4\mu-\lambda^2}}\left(\frac{-C_1\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n) + C_2\cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n)}{C_1\cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n) + C_2\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi_n)}\right) \pm \frac{2\mathrm{i}\alpha\sin^2(\frac{\sqrt{\lambda^2-4\mu}}{2}d_1)}{\alpha_1\sqrt{4\mu-\lambda^2}}, \tag{40}$$

where $\xi_n = d_1 n \pm \frac{4\mathrm{i}\alpha\sin^2(\frac{\sqrt{4\mu-\lambda^2}}{2}d_1)}{\sqrt{4\mu-\lambda^2}}t + \zeta$.

When $\lambda^2 - 4\mu = 0$, we obtain rational solutions of Eqs. (18) and (19):

$$u_n = \frac{\alpha_1 C_2}{C_1 + C_2\xi_n}, \quad v_n = -\frac{\alpha C_2 d_1^2}{\alpha_1(C_1 + C_2\xi_n)}, \tag{41}$$

where $\xi_n = d_1 n + \zeta$.

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References

[1] S.Y. Lou and X.Y. Tang: *Method of Nonlinear Mathematical Physics* (Science Press, Beijing 2006)

[2] C.S. Garder, J.M. Greene M.D. Kruskal and R.M. Miura: Phys. Rev. Lett. Vol. 19 (1965), p. 1095

[3] M.J. Ablowitz and P.A. Clarkson: *Soliton, Nonlinear Evolution Equations and Inverse Scattering*, (Cambridge University Press, Cambridge 1991)

[4] M.R. Miurs: *Bäcklund Transformation* (Springer-Verlag, Berlin 1978)

[5] R. Hirota: Phys. Rev. Lett. Vol. 27 (1971), p. 1192

[6] M.L. Wang: Phys. Lett. A Vol. 213 (1996), p. 279

[7] E.G. Fan: Phys. Lett. A Vol. 300 (2002), p. 243

[8] S. Zhang and T.C. Xia: Commun. Theor. Phys. Vol. 45 (2006), p. 985

[9] S. Zhang, Y.Y. Zhou, IAENG Int. J. Appl. Math. Vol. 44 (2014), p. 177

[10]S. Zhang, B. Xu and H.Q. Zhang, Int. J. Comput. Math. Vol. 91 (2014), p. 1601

[11]S. Zhang and D. Wang: Therm. Sci. Vol. 18 (2014), p. 1555

[12]S. Zhang, J.L. Tong and W. Wang: Comput. Math. Appl. Vol. 58 (2009), p. 2294

[13]S. Zhang and B. Cai: Nonlinear Dyn. Vol. 78 (2014), p. 1593

[14]S. Zhang, B. Xu and A.X. Peng: Appl. Mech. Mater. Vol. 394 (2013), p. 571

[15]S. Zhang, Y.Y. Zhou and B. Cai: Adv. Mater. Res. Vol. 989-994 (2014), p. 1716

[16]M.L. Wang, X.Z. Li and J.L. Zhang: Phys. Lett. A 372 (2008), p. 417

[17]M.L. Wang, J.L. Zhang and X.Z. Li: Appl. Math. Comput. Vol. 206 (2008), p. 321

[18]G. Ebadi and A. Biswas: J. Franklin Inst. Vol. 347 (2010), p. 1391

[19]G. Ebadi and A. Biswas: Commun. Nonlinear Sci. Numer. Simul. Vol. 16 (2011), p. 2377

[20]G. Ebadi and A. Biswas: Math. Comput. Model. Vol. 53 (2011), p. 694

[21]S. Zhang, J.L. Tong and W. Wang: Phys. Lett. A Vol. 372 (2008), p. 2254

[22]S. Zhang, L. Dong, J.M. Ba and Y.N. Sun, Phys. Lett. A Vol. 373 (2009), p. 905

[23]S. Zhang, L. Dong, J.M. Ba and Y.N. Sun: Pramana-J. Phys. Vol. 74 (2010), p. 207

[24]Z. Wang: Comput. Phys. Commu. Vol. 180 (2009), p. 1104