An Exact Analytical Solution to the Static Cracks in Finite-width Single-edge Cracked Strips

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Abstract. Using the method of complex analysis and through constructing new conformal mapping functions, we analyzed the plane elasticity problem of static cracks in finite-width single-edge cracked strips, and provided an exact analytical solution to the crack-tip stress intensity factor.

Introduction

As of present, there have been various studies regarding finite-height crack strips of different materials. Article [1], for instance, provides an analytical solution to two semi-infinite collinear crack strips; article [2] provides an analytical solution to Type II cracks in piezoelectric ceramic strips. Article [3] provides a conformal mapping function and conformal mapped the finite-width single-edged crack strip to the upper half plane, from which the stress intensity factors (SIFs) $K_{\Box}$, $K_{\Box}$ of static cracks are obtained. In this paper we have further analyzed and improved the results of article [3] by constructing new conformal mapping functions, and have obtained the exact analytical solution to crack tip SIFs.

The Static Crack Problem in Finite-width Single-edge Cracked Tips

Let there be a single crack in a finite-with strip, under plane stress or plane strain state as shown in figure 1. The governing equation for this problem is[4]

$$\nabla^2\,\nabla^2\,U=0 \tag{1}$$

in which ∇^2 is the two-dimension Laplace operator.

$$U(x,y)=\mathrm{Re}\left[\bar{z}f_1(z)+\int y_1(z)\,dz\right] \tag{2}$$

In which $f_1(z)$ and $y_1(z)$ represent two analytic functions of the complex variable $z=x+\mathrm{i}y$, $\bar{z}=x-\mathrm{i}y$ is the complex conjugate of z, Re represents real part of the complex number. From the fundamentals of electrostatics and equation (2), we have the following expression of stress and displacement

$$\begin{cases} s_{xx}+s_{yy}=4\,\mathrm{Re}\,f_1'(z) \\ s_{yy}-s_{xx}+2\mathrm{i}s_{xy}=2\left[\bar{z}f_1''(z)+y_1'(z)\right] \end{cases} \tag{3}$$

$$2m\left(u_x+\mathrm{i}u_y\right)=kf_1(z)-z\overline{f_1'(z)}-\overline{y_1(z)} \tag{4}$$

in which $k=\begin{cases}(3-n)/(1+n), & \text{plane stress}\\ 3-4n, & \text{plane strain}\end{cases}$

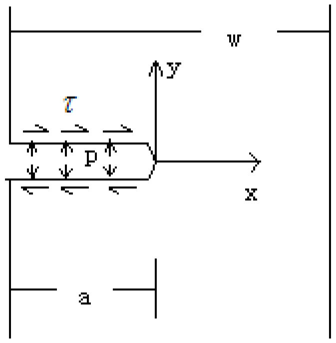

Fig. 1 Finite-width strip with static crack

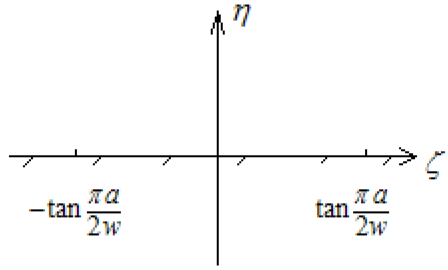

Fig. 2 Region and boundary on the z plane

In this study we construct a new conformal mapping function

$$z = w(z) = \left(\frac{2w}{p}\right) arc\tan\left[\sqrt{\tan^2\frac{pa}{2w} - z^2}\right] - a \tag{5}$$

The conformal mapping function equation (5) maps the upper half plane (mathematical plane) of the $z\,(= x + ih)$ plane, shown in figure 2, to the z plane (mathematical plane) of the strip region of strip cracks, shown in figure 1. So that crack tip $z = 0$ corresponds to $z = 0$, and $z = -a^+$ (upper part of the crack) corresponds to $z = -\tan\frac{pa}{2w}$, $z = -a^-$ (lower part of the crack) corresponds to $z = \tan\frac{pa}{2w}$.

Under transformation equation (5), functions $f_1(z)$ and $y_1(z)$ as well as their derivatives are transformed to

$$\begin{cases} f_1(z) = f_1[w(z)] = f(z), & y_1(z) = y_1[w(z)] = y(z) \\ f_1'(z) = f'(z)/w'(z), & y_1'(z) = y'(z)/w'(z) \end{cases} \tag{6}$$

When the crack plane is under uniform internal pressure (Type I problem), we have the following boundary conditions

$$\begin{cases} s_{yy} = s_{xy} = 0, & y = \pm\infty, & -a < x < w - a & \\ s_{xx} = s_{xy} = 0, & x = -a, & x = w - a, & -\infty < y < \infty \\ s_{yy} = f_1(x), & s_{xy} = 0, & y = \pm 0, & -a < x < 0 \end{cases} \tag{7}$$

in which $f_1(x)$ is a known arbitrary function, here $f_1(x) = -p$ = constant. Therefore, the solution to the problem comes down to solving the following equation set [5]

$$f(z) + \frac{1}{2pi}\int_g \frac{w(s)}{\overline{w'(s)}}\frac{\overline{f'(s)}}{s - z}ds + \overline{y(0)} = \frac{1}{2pi}\int_g \frac{f_0}{s - z}ds \tag{8}$$

$$y(z) + \frac{1}{2pi}\int_g \frac{\overline{w(s)}}{w'(s)}\frac{f'(s)}{s - z}ds + \overline{f(0)} = \frac{1}{2pi}\int_g \frac{\overline{f_0}}{s - z}ds \tag{9}$$

in which

$$f_0 = i\int(\overline{X} + i\overline{Y})ds = \begin{cases} -p(z + a), & z \in \overset{\frown}{AOB} \\ 0, & z \notin \overset{\frown}{AOB} \end{cases} \tag{10}$$

Note

$$F(s) = \frac{w(s)}{w'(s)}\overline{f'(s)} \tag{11}$$

then

$$F(z) = \frac{w(z)}{w'\left(\overline{z}\right)}\overline{f'\left(\overline{z}\right)}$$

$$= \frac{\left[\frac{pa}{2w} - arc\tan\left(\sqrt{\tan^2\frac{pa}{2w} - z^2}\right)\right]\left[\left(1+\tan^2\frac{pa}{2w} - z^2\right)\sqrt{\tan^2\frac{pa}{2w} - z^2}\right]}{z}\overline{f'\left(\overline{z}\right)} \tag{12}$$

Therefore $F(z)$ is the analytic function on the lower half plane.

From the stress-free condition at infinity we get [6]

$$\lim_{z\to\infty} z\overline{f'(z)} = 0 \tag{13}$$

Then from equation (5) and equation (13) we get

$$\lim_{z\to\infty} F(z) = \lim_{z\to\infty}\frac{w(z)}{w'\left(\overline{z}\right)}\overline{f'\left(\overline{z}\right)} = \lim_{z\to\infty} z\overline{f'(z)} = 0 \tag{14}$$

Using the Canchy intergral formula on the straight line we get

$$\frac{1}{2p\mathrm{i}}\int_r \frac{F(s)}{s-z}\mathrm{d}s = F(\infty) = 0 \tag{15}$$

Then from equations(5), (10), (15), the equation (8) can become

$$f(z) + \overline{y(0)} = \frac{1}{2p\mathrm{i}}\int_{-\tan\frac{pa}{2w}}^{\tan\frac{pa}{2w}} \frac{-p[w(s)+a]}{s-z}\mathrm{d}s \tag{16}$$

Similarly, $y(z)$ can also be determined from solving equation (9). Below we will calculate the vital physical quantity—the stress intensity factor[5]

$$K_{\square} - \mathrm{i}K_{\square} = 2\sqrt{p}\lim_{z\to 0}\frac{f'(z)}{\sqrt{w''(z)}} = 2\sqrt{p}\,\frac{f'(0)}{\sqrt{w''(0)}} \tag{17}$$

Using integration by parts we obtain from equation (16)

$$f'(z) = \frac{-p}{2p\mathrm{i}}\int_{-\tan\frac{pa}{2w}}^{\tan\frac{pa}{2w}} \frac{w'(s)}{s-z}\mathrm{d}s \tag{18}$$

From equation (5) we get

$$w'(z) = \left(\frac{2w}{p}\right)\frac{-z}{\left(1+\tan^2\frac{pa}{2w} - z^2\right)\sqrt{\tan^2\frac{pa}{2w} - z^2}} \tag{19}$$

Substituting equation (19) into equation (18), we get

$$f'(0)=\frac{pw}{p^{2}\mathrm{i}}\lim_{z\to 0}\int_{-\tan\frac{pa}{2w}}^{\tan\frac{pa}{2w}}\frac{s}{\sqrt{\tan^{2}\frac{pa}{2w}-s^{2}}\,(s-z)\left(1+\tan^{2}\frac{pa}{2w}-s^{2}\right)}\mathrm{d}s$$

$$=\frac{-\mathrm{i}pw}{p^{2}}\lim_{x\to\tan\frac{pa}{2w}}\frac{1}{\sqrt{1+x^{2}}}\left(arc\tan\frac{\tan\frac{pa}{2w}\sqrt{1+x^{2}}-x^{2}}{\sqrt{x^{2}-\tan^{2}\frac{pa}{2w}}}+arc\tan\frac{\tan\frac{pa}{2w}\sqrt{1+x^{2}}+x^{2}}{\sqrt{x^{2}-\tan^{2}\frac{pa}{2w}}}\right)$$

$$=\frac{pw}{p\mathrm{i}}\cos\frac{pa}{2w} \tag{20}$$

In the above equation we have already taken $arc\tan(+\infty)=\frac{p}{2}$.

From equation (5) we can get

$$w''(0)=-\frac{2w}{p}c\tan\frac{pa}{2w}\cdot\cos^{2}\frac{pa}{2w} \tag{21}$$

Substituting equation (20), (21) into equation (17) and simplify to get $K_{\Box}$- i$K_{\Box}$=$\sqrt{2}p\sqrt{w\tan\frac{pa}{2w}}$

Then

$$K_{\Box}=\sqrt{2}p\sqrt{w\tan\frac{pa}{2w}} \tag{22}$$

The changing curve of $K_{\Box}$ following parameters p , a , w is shown in figure 3.

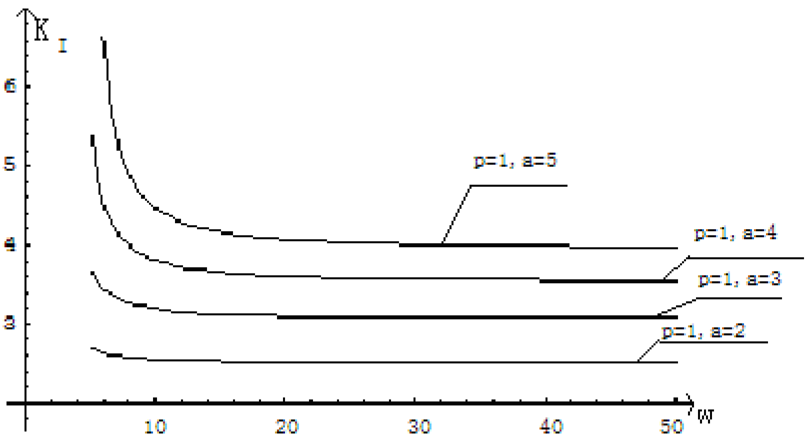

Figure 3 Changing curve of the crack tip SIF following parameters p, a, w

Similarly, when crack surface is under uniform shear and the shearing strength is $-t$, we can obtain

$$K_{\Box}=\sqrt{2}t\sqrt{w\tan\frac{pa}{2w}} \tag{23}$$

Through constructing new conformal mapping, our study re-derived the static stress intensity factor of finite-width single-edged crack strips. It is slightly different from the results of article [3]. After careful reading we have found errors within some of the steps in article [3], such as those that would have influenced the calculation of results: the original equation, $f'(0)=\frac{pa}{p\mathrm{i}}\sin\frac{pa}{2w}$, should have been $f'(0)=\frac{pw}{p\mathrm{i}}\sin\frac{pa}{2w}$ (equation(22)). Calculating with the corrected results, the stress intensity factor expression is identical to the conclusion of our study.

Conclusions

Regarding the static crack problem in finite-width single-edge cracked strips, the conformal mapping equation (5) provided by this article is a transcendental function, using conformal mapping to simplify the complicated crack problem in order to obtain a solution, and the calculation method is relatively simple. The method in this article is an extension to the Muskhelishvili[4] complex potential method, expanding the scope of application and enriching the content of the latter. An exact analytical solution to the static crack in finite-width single-edge cracked strips also provides great significance in solving many practical problems in engineering fracture.

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