Synchronization Problems on Small-world Networks with time delay

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Abstract. In this paper, we study the synchronization problem on small-world network with time delay. In particular, we give rigorous mathematical analysis and arrive with sufficient condition of the synchronization problem when the synchronous states are global stable. The general closed-form analytical results build the relationship between network topological characteristic and the network dynamical characteristic of time delay.

Introduction

Synchronization problem for a coupled dynamics system origins from the synchronization phenomenon in physics and biology. The study can be traced back to the Kuramoto system [1], which is now a research area on whether coupled agents with continuous and discrete time chaos, circuit as well as bifurcation systems are stable or not [2] [3] [4] [5] [6] [7] [8] [9] [10] [11]. The stability of coupled system is related to both network topology and the dynamic system of individual agents. In [4] [12] [13] [14] [15], different sufficient conditions are investigated for the synchronization problem.

Synchronization problem in small-world networks has been carefully studied [16] [17] [18] [19] [20]. It has been observed that the ability to archive synchronization is greatly enhanced by making the network become a small-world network. Small-world networks are those network that exhibit two important characterizations: i.e., large clustering coefficient, and the small world phenomenon (small diameter, or short average distance) in the real complex networks. Watts and Strogatz [21] in 1998 proposed a famous small-world network model which possesses both small average distance and large clustering coefficient by simulation. Newman and Watts proposed another model [22] satisfying the two properties which is the union of an E-R random graph[23] and a 2k regular cycle. Gu, Huang and Zhang further introduced a generalized small-world model [24] to cover both Watts-Strogatz model and Newman-Watts model.

In this paper, we study the synchronization problem on Newman-Watts type small-world network with time delay. In particular, we give the sufficient condition of the synchronization problem based on linear feedback controllers which can also be applied in the situation without feedback as the special case. Our result provides a general closed-form analytical result that builds the relationship between network topological characteristic parameters and the network dynamical characteristic parameter of time delay.

Network Dynamics and Spectral Analysis of Newman-Watts Network

Suppose there are N nodes (oscillators) and $\mathbf{x}^i$ be the m-dimensional vector of dynamical variables of the i_{th} node. The dynamics of the i_{th} node are

$$\dot{x}^i(t) = F[x^i(t)] + \sigma \sum_j L_{ij} H x^j(t-\tau) \qquad (1)$$

where σ is a coupling strength and L is a Laplacian matrix. Synchronization on such a network means when time is long enough, $\lim_{t\to\infty} x_1(t) = ... = \lim_{t\to\infty} x_n(t)$. This defines the synchronization manifold. Pecora and Carroll shows that it is a invariant manifold with some additional assumptions and therefore it can apply linear approximation near synchronization state. And therefore one get

the corresponding variational equation, here *DF* and *DH* are correspondent Jacobian matrix of *F* and *H*.

$$\dot{\xi}^i(t) = DF[\xi^i(t)] + \sigma\sum_j L_{ij} DH\xi^j(t-\tau) \tag{2}$$

Supposing the small-world network model is an undirected graph. With matrix transformation we can diagonalize *L*, the matrix is symmetric and hence the eigenvalues are all real, denoted by $\lambda_k, k=1,...n$. Then the stability of Equation (2) is equivalent to the stability of the stability of n collection of following equations.

$$\dot{\xi}^i(t) = DF[\xi^i(t)] + \sigma\lambda_k DH\xi^j(t-\tau) \tag{3}$$

Therefore arrive with the following result[13]:

Suppose the n-1 system are all asymptotically stable, that

$$\dot{\xi}^i(t) = DF[\xi^i(t)] + \sigma\lambda_k DH\xi^j(t-\tau) \tag{4}$$

for all $\lambda_k, k=2,...n.$ Where $0 = \lambda_1 \le \lambda_2 \le ... \le \lambda_n$ are the eigenvalues of Laplacian matrix, then the coupled system is also asymptotically stable. Here the stability means local stable near the synchronous states.

Moreover, Wu and Jiao [15] proposed a time-delay coupled system with linear feedback controllers

$$\dot{X}(t) = F[X(t)] + \sigma L \times A(t-\tau) + \kappa I_n \times A(X(t) - S(t)) \tag{5}$$

where S(t) is the synchronous state of the system. Moreover it is assuming that there exists $\bar{x}(t)$ so that $f(x(t)) - f(y(t)) = Df(\bar{x}(t))$, where *Df* could be the Jacobian *DF* or other related matrix [15].

Wu and Jiao [15] give the sufficient condition of global stablity, exists $\varepsilon > 0$

$$Df[\bar{x}(t)]^T + Df[\bar{x}(t)] - \kappa(A^T + A) + \lambda_n^2 AA^T \prec -(\varepsilon+1)/\sqrt{N}I_n \tag{6}$$

Here matrix $A \prec B$ means that matrix B – A is nonnegative definite matrix.

Equation (6) implies that the sufficient condition is determined by the largest eigenvalue of the Laplacian, especially for Newman-Watts small-world network we have the following result.

Lemma 1: Let NW(n,c,k) be the Newman-Watts network of n nodes which is the union of 2k-regular lattice and random graph $\mathbb{G}_\delta(n.c/n)$, then the largest eigenvalue of correspondent Laplacian is $(2+o(1))\log n/\log\log n$.

Proof:

In [18] Equation (8) shows that the maximum degree of NW(n,c,k) is $(1+o(1))\log n/\log\log n$. Moreover Lemma 2.4 in [18] shows that the largest eigenvalue is smaller than two times maximum degree, and the result follows directly.

Now we are able to arrive with our main result as follows

Theorem 1: Consider the time-delayed coupled system $\dot{X}(t) = F[X(t)] + \sigma L \times A(t-\tau) + \kappa I_n \times A(X(t) - S(t))$, where L is the Laplacian matrix of Newman-Watts network NW(n,c,k). Letting $B(\varepsilon) = [\kappa(A^T + A) - (\varepsilon+1)/\sqrt{N}I_n - Df[\bar{x}(t)]^T - Df[\bar{x}(t)]](A^T)^{-1}A^{-1}$, then the synchronous state of dynamic system is global stable if $\lambda_{\min}(B(\varepsilon)) \ge (4+o(1))\log^2 n/\log\log^2 n$.

Proof:

Rewrite equation (6) becomes $\lambda_n^2 AA^T \prec \kappa(A^T + A) - (\varepsilon+1)/\sqrt{N}I_n - Df[\bar{x}(t)]^T - Df[\bar{x}(t)]$. Multiply both side by $(A^T)^{-1}A^{-1}$, the equation becomes $\lambda_n^2 I_N \prec B(\varepsilon)$. Remember that by Lemma 1, $\lambda_n \le (2+o(1))\log n/\log\log n$ and B is symmetric matrix, the result follows directly.

Theorem 1 provides a general closed-form analytical result that builds the relationship between network topological characteristic parameters and the network dynamical characteristic parameter of time delay. Moreover, setting $\kappa=0$, equation (5) degenerates into equation (1) so that the

situation without feedback play as a special case.

Conclusion

In this paper, we study the synchronization problem on small-world network with time delay. In particular, we give rigorous mathematical analysis and arrive with sufficient condition of the synchronization problem when the synchronous states are global stable. The general closed-form analytical results build the relationship between network topological characteristic and the network dynamical characteristic of time delay.

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