Intuitionistic Fuzzy Implications and the Axioms of Intuitionistic Logic

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Abstract

During years of research, there have been defined 153 intuitionistic fuzzy implications. In the paper, it is checked which of these implications satisfy which axioms of the intuitionistic logic, whether as (classical) tautologies or as intuitionistic fuzzy tautologies. Some open problems, related to the operations of intuitionistic fuzzy propositional calculus, are formulated.

Keywords: Implication, Intuitionistic fuzzy logic, Intuitionistic logic.

1. Introduction

In a series of publications, (see, e.g. [1, 2]), 153 different operations "implication" in intuitionistic fuzzy propositional calculus, have been introduced. They have been found to generate 45 operations "negation" over intuitionistic fuzzy sets.

In the present paper, we study which of these implications satisfy which axioms of the intuitionistic logic, and whether, doing so, they behave tautologies or as intuitionistic fuzzy tautologies.

In intuitionistic fuzzy propositional calculus, if x is a variable, then its truth-value is represented by the ordered couple

$$V(x) = \langle a, b \rangle,$$

so that $a, b, a + b \in [0, 1]$, where a and b are degrees of validity and of non-validity of x.

For the needs of the discussion below, we shall define the notion of Intuitionistic Fuzzy Tautology (IFT, see, e.g. [1, 2]) by:

x is an IFT if and only if
for $V(x) = \langle a, b \rangle$ it holds that $a \geq b$,

while

x will be a (classical) tautology if and only if
$a = 1$ and $b = 0$.

As in the case of ordinary logics, x is a tautology, if $V(x) = \langle 1, 0 \rangle$. Obviously, each tautology is intuitionistic fuzzy tautology, but the opposite is not true.

2. On the operations "implication" and "negation" in intuitionistic fuzzy propositional calculus

Below, we shall assume that for the two variables x and y the equalities: $V(x) = \langle a, b \rangle, V(y) = \langle c, d \rangle$ $(a, b, c, d, a + b, c + d \in [0, 1])$ hold.

First, for the two variables x and y, we define the operations "conjunction" (&) and "disjunction" ($\vee$) (see, e.g. [1, 2]) by:

$$V(x \& y) = \langle \min(a, c), \max(b, d) \rangle,$$

$$V(x \vee y) = \langle \max(a, c), \min(b, d) \rangle.$$

In some definitions, we need to use the auxiliary functions sg and $\overline{sg}$ defined by,

$$\text{sg}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases},$$

$$\overline{sg}(x) = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x \leq 0 \end{cases}$$

The list of all currently existing intuitionistic fuzzy implications and negations are given in Tables 1 and 2, respectively. Another JavaScript powered visualization of these operations with dynamically sortable membership and non-membership parts is given online in [7, 8].

Table 1

$\rightarrow_1$	$\langle \max(b, \min(a, c)), \min(a, d) \rangle$
$\rightarrow_2$	$\langle \overline{sg}(a - c), d.\text{sg}(a - c) \rangle$
$\rightarrow_3$	$\langle 1 - (1 - c).\text{sg}(a - c)), d.\text{sg}(a - c) \rangle$
$\rightarrow_4$	$\langle \max(b, c), \min(a, d) \rangle$
$\rightarrow_5$	$\langle \min(1, b + c), \max(0, a + d - 1) \rangle$
$\rightarrow_6$	$\langle b + ac, ad \rangle$
$\rightarrow_7$	$\langle \min(\max(b, c), \max(a, b), \max(c, d)), \max(\min(a, d), \min(a, b), \min(c, d)) \rangle$
$\rightarrow_8$	$\langle 1 - (1 - \min(b, c)).\text{sg}(a - c), \max(a, d).\text{sg}(a - c), \text{sg}(d - b) \rangle$
$\rightarrow_9$	$\langle b + a^2 c, ab + a^2 d \rangle$
$\rightarrow_{10}$	$\langle c.\overline{sg}(1 - a) + \text{sg}(1 - a).(\overline{sg}(1 - c) + b.\text{sg}(1 - c)), d.\overline{sg}(1 - a) + a.\text{sg}(1 - a).\text{sg}(1 - c) \rangle$

$\to_{11}$	$\langle 1-(1-c).\mathrm{sg}(a-c), d.\mathrm{sg}(a-c).\mathrm{sg}(d-b)\rangle$
$\to_{12}$	$\langle \max(b,c), 1-\max(b,c)\rangle$
$\to_{13}$	$\langle b+c-b.c, a.d\rangle$
$\to_{14}$	$\langle 1-(1-c).\mathrm{sg}(a-c)-d.\overline{\mathrm{sg}}(a-c).\mathrm{sg}(d-b),$ $d.\mathrm{sg}(d-b)\rangle$
$\to_{15}$	$\langle 1-(1-\min(b,c)).\mathrm{sg}(a-c).\mathrm{sg}(d-b)$ $-\min(b,c).\mathrm{sg}(a-c).\mathrm{sg}(d-b),$ $1-(1-\max(a,d)).\mathrm{sg}(\overline{\mathrm{sg}}(a-c)+\overline{\mathrm{sg}}(d-b))$ $-\max(a,d).\overline{\mathrm{sg}}(a-c).\overline{\mathrm{sg}}(d-b)\rangle$
$\to_{16}$	$\langle \max(\overline{\mathrm{sg}}(a),c), \min(\mathrm{sg}(a),d)\rangle$
$\to_{17}$	$\langle \max(b,c), \min(a.b+a^2,d)\rangle$
$\to_{18}$	$\langle \max(b,c), \min(1-b,d)\rangle$
$\to_{19}$	$\langle \max(1-\mathrm{sg}(\mathrm{sg}(a)+\mathrm{sg}(1-b)),c),$ $\min(\mathrm{sg}(1-b),d)\rangle$
$\to_{20}$	$\langle \max(\overline{\mathrm{sg}}(a),\mathrm{sg}(c)), \min(\mathrm{sg}(a),\overline{\mathrm{sg}}(c))\rangle$
$\to_{21}$	$\langle \max(b,c.(c+d)),$ $\min(a.(a+b),d.(c^2+d+c.d))\rangle$
$\to_{22}$	$\langle \max(b,1-d), 1-\max(b,1-d)\rangle$
$\to_{23}$	$\langle 1-\min(\mathrm{sg}(1-b),\overline{\mathrm{sg}}(1-d)),$ $\min(\mathrm{sg}(1-b),\overline{\mathrm{sg}}(1-d))\rangle$
$\to_{24}$	$\langle \overline{\mathrm{sg}}(a-c).\overline{\mathrm{sg}}(d-b), \mathrm{sg}(a-c).\mathrm{sg}(d-b)\rangle$
$\to_{25}$	$\langle \max(b,\overline{\mathrm{sg}}(a).\overline{\mathrm{sg}}(1-b),$ $c.\overline{\mathrm{sg}}(d).\overline{\mathrm{sg}}(1-c)), \min(a,d)\rangle$
$\to_{26}$	$\langle \max(\overline{\mathrm{sg}}(1-b),c), \min(\mathrm{sg}(a),d)\rangle$
$\to_{27}$	$\langle \max(\overline{\mathrm{sg}}(1-b),\mathrm{sg}(c)),$ $\min(\mathrm{sg}(a),\overline{\mathrm{sg}}(1-d))\rangle$
$\to_{28}$	$\langle \max(\overline{\mathrm{sg}}(1-b),c), \min(a,d)\rangle$
$\to_{29}$	$\langle \max(\overline{\mathrm{sg}}(1-b),\overline{\mathrm{sg}}(1-c)),$ $\min(a,\overline{\mathrm{sg}}(1-d))\rangle$
$\to_{30}$	$\langle \max(1-a,\min(a,1-d)), \min(a,d)\rangle$
$\to_{31}$	$\langle \overline{\mathrm{sg}}(a+d-1), d.\mathrm{sg}(a+d-1)\rangle$
$\to_{32}$	$\langle 1-d.\mathrm{sg}(a+d-1), d.\mathrm{sg}(a+d-1)\rangle$
$\to_{33}$	$\langle 1-\min(a,d), \min(a,d)\rangle$
$\to_{34}$	$\langle \min(1,2-a-d), \max(0,a+d-1)\rangle$
$\to_{35}$	$\langle 1-a.d, a.d\rangle$
$\to_{36}$	$\langle \min(1-\min(a,d),\max(a,1-a),$ $\max(1-d,d)), \max(\min(a,d),$ $\min(a,1-a),\min(1-d,d))\rangle$
$\to_{37}$	$\langle 1-\max(a,d).\mathrm{sg}(a+d-1),$ $\max(a,d).\mathrm{sg}(a+d-1)\rangle$
$\to_{38}$	$\langle 1-a+(a^2.(1-d)), a.(1-a)+a^2.d\rangle$
$\to_{39}$	$\langle (1-d).\overline{\mathrm{sg}}(1-a)+\mathrm{sg}(1-a)$ $.(\overline{\mathrm{sg}}(d)+(1-a).\mathrm{sg}(d)),$ $d.\overline{\mathrm{sg}}(1-a)+a.\mathrm{sg}(1-a).\mathrm{sg}(d)\rangle$
$\to_{40}$	$\langle 1-\mathrm{sg}(a+d-1), 1-\overline{\mathrm{sg}}(a+d-1)\rangle$
$\to_{41}$	$\langle \max(\overline{\mathrm{sg}}(a),1-d), \min(\mathrm{sg}(a),d)\rangle$
$\to_{42}$	$\langle \max(\overline{\mathrm{sg}}(a),\mathrm{sg}(1-d)),$ $\min(\mathrm{sg}(a),\overline{\mathrm{sg}}(1-d))\rangle$
$\to_{43}$	$\langle \max(\overline{\mathrm{sg}}(a),1-d), \min(\mathrm{sg}(a),d)\rangle$
$\to_{44}$	$\langle \max(\overline{\mathrm{sg}}(a),1-d), \min(a,d)\rangle$
$\to_{45}$	$\langle \max(\overline{\mathrm{sg}}(a),\overline{\mathrm{sg}}(d)), \min(a,\overline{\mathrm{sg}}(1-d))\rangle$
$\to_{46}$	$\langle \max(b,\min(1-b,c)), 1-\max(b,c)\rangle$
$\to_{47}$	$\langle \overline{\mathrm{sg}}(1-b-c), (1-c).\mathrm{sg}(1-b-c)\rangle$
$\to_{48}$	$\langle 1-(1-c).\mathrm{sg}(1-b-c),$ $(1-c).\mathrm{sg}(1-b-c)\rangle$
$\to_{49}$	$\langle \min(1,b+c), \max(0,1-b-c)\rangle$
$\to_{50}$	$\langle b+c-b.c, 1-b-c+b.c\rangle$
$\to_{51}$	$\langle \min(\max(b,c),\max(1-b,b),$ $\max(c,1-c)), \max(1-\max(b,c),$ $\min(1-b,b),\min(c,1-c))\rangle$

$\to_{52}$	$\langle 1-(1-\min(b,c)).\mathrm{sg}(1-b-c),$ $1-\min(b,c).\mathrm{sg}(1-b-c)\rangle$
$\to_{53}$	$\langle b+(1-b)^2.c,$ $(1-b).b+(1-b)^2.(1-c)\rangle$
$\to_{54}$	$\langle c.\overline{\mathrm{sg}}(b)+\mathrm{sg}(b).(\overline{\mathrm{sg}}(1-c)+b.\mathrm{sg}(1-c)),$ $(1-c).\overline{\mathrm{sg}}(b)+(1-b).\mathrm{sg}(b).\mathrm{sg}(1-c)\rangle$
$\to_{55}$	$\langle 1-\mathrm{sg}(1-b-c), 1-\overline{\mathrm{sg}}(1-b-c)\rangle$
$\to_{56}$	$\langle \max(\overline{\mathrm{sg}}(1-b),c),$ $\min(\mathrm{sg}(1-b),(1-c))\rangle$
$\to_{57}$	$\langle \max(\overline{\mathrm{sg}}(1-b),$ $\mathrm{sg}(c)), \min(\mathrm{sg}(1-b),\overline{\mathrm{sg}}(c))\rangle$
$\to_{58}$	$\langle \max(\overline{\mathrm{sg}}(1-b),\overline{\mathrm{sg}}(1-c)), 1-\max(b,c)\rangle$
$\to_{59}$	$\langle \max(\overline{\mathrm{sg}}(1-b),c), (1-\max(b,c))\rangle$
$\to_{60}$	$\langle \max(\overline{\mathrm{sg}}(1-b),\overline{\mathrm{sg}}(1-c)),$ $\min((1-b),\overline{\mathrm{sg}}(c))\rangle$
$\to_{61}$	$\langle \max(c,\min(b,d)), \min(a,d)\rangle$
$\to_{62}$	$\langle \overline{\mathrm{sg}}(d-b), a.\mathrm{sg}(d-b)\rangle$
$\to_{63}$	$\langle 1-(1-b).\mathrm{sg}(d-b), a.\mathrm{sg}(d-b)\rangle$
$\to_{64}$	$\langle c+b.d, a.d\rangle$
$\to_{65}$	$\langle 1-(1-\min(b,c)).\mathrm{sg}(d-b),$ $\max(a,d).\mathrm{sg}(d-b).\mathrm{sg}(a-c)\rangle$
$\to_{66}$	$\langle c+d^2.b, b.d+d^2.a\rangle$
$\to_{67}$	$\langle b.\overline{\mathrm{sg}}(1-d)+\mathrm{sg}(1-d).(\overline{\mathrm{sg}}(1-b)$ $+c.\mathrm{sg}(1-b)),$ $a.\overline{\mathrm{sg}}(1-d)+d.\mathrm{sg}(1-d).\mathrm{sg}(1-b)\rangle$
$\to_{68}$	$\langle 1-(1-b).\mathrm{sg}(d-b), a.\mathrm{sg}(d-b).\mathrm{sg}(a-c)\rangle$
$\to_{69}$	$\langle 1-(1-b).\mathrm{sg}(d-b)-a.\overline{\mathrm{sg}}(d-b).\mathrm{sg}(a-c),$ $a.\mathrm{sg}(a-c)\rangle$
$\to_{70}$	$\langle \max(\overline{\mathrm{sg}}(d),b), \min(\mathrm{sg}(d),a)\rangle$
$\to_{71}$	$\langle \max(b,c), \min(c.d+d^2,a)\rangle$
$\to_{72}$	$\langle \max(b,c), \min(1-c,a)\rangle$
$\to_{73}$	$\langle \max(1-\max(\mathrm{sg}(d),\mathrm{sg}(1-c)),b),$ $\min(\mathrm{sg}(1-c),a)\rangle$
$\to_{74}$	$\langle \max(\mathrm{sg}(b),\overline{\mathrm{sg}}(d)), \min(\overline{\mathrm{sg}}(b),\mathrm{sg}(d))\rangle$
$\to_{75}$	$\langle \max(c,b.(a+b)), \min(d.(c+d),$ $a.(b^2+a)+a.b)\rangle$
$\to_{76}$	$\langle \max(c,1-a), \min(1-c,a)\rangle$
$\to_{77}$	$\langle (1-\min(\overline{\mathrm{sg}}(1-a),\mathrm{sg}(1-c))),$ $\min(\overline{\mathrm{sg}}(1-a),\mathrm{sg}(1-c))\rangle$
$\to_{78}$	$\langle \max(\overline{\mathrm{sg}}(1-c),b), \min(\mathrm{sg}(d),a)\rangle$
$\to_{79}$	$\langle \max(\overline{\mathrm{sg}}(1-c),\mathrm{sg}(b)),$ $\min(\mathrm{sg}(d),\overline{\mathrm{sg}}(1-a))\rangle$
$\to_{80}$	$\langle \max(\overline{\mathrm{sg}}(1-c),b), \min(d,a)\rangle$
$\to_{81}$	$\langle \max(\overline{\mathrm{sg}}(1-b),\overline{\mathrm{sg}}(1-c)),$ $\min(d,\overline{\mathrm{sg}}(1-a))\rangle$
$\to_{82}$	$\langle \max(1-d,\min(d,1-a)), \min(d,a)\rangle$
$\to_{83}$	$\langle \overline{\mathrm{sg}}(a+d-1), a.\mathrm{sg}(a+d-1)\rangle$
$\to_{84}$	$\langle 1-a.\mathrm{sg}(a+d+1), a.\mathrm{sg}(a+d+1)\rangle$
$\to_{85}$	$\langle 1-d+d^2.(1-a), d.(1-d)+d^2.\rangle$
$\to_{86}$	$\langle (1-a).\overline{\mathrm{sg}}(1-d)+\mathrm{sg}(1-d)$ $.\overline{\mathrm{sg}}(a+\min(1-d,\mathrm{sg}(a))),$ $a.\overline{\mathrm{sg}}(1-d)+d.\mathrm{sg}(1-d).\mathrm{sg}(a)\rangle$
$\to_{87}$	$\langle \max(\overline{\mathrm{sg}}(d),1-a), \min(\mathrm{sg}(d),a)\rangle$
$\to_{88}$	$\langle \max(\overline{\mathrm{sg}}(d),\mathrm{sg}(1-a)),$ $\min(\mathrm{sg}(d),\overline{\mathrm{sg}}(1-a))\rangle$
$\to_{89}$	$\langle \max(\overline{\mathrm{sg}}(d),1-a), \min(d,a)\rangle$
$\to_{90}$	$\langle \max(\overline{\mathrm{sg}}(a),\overline{\mathrm{sg}}(d)), \min(d,\overline{\mathrm{sg}}(1-a))\rangle$
$\to_{91}$	$\langle \max(c,\min(1-c,b)), 1-\max(b,c)\rangle$
$\to_{92}$	$\langle \overline{\mathrm{sg}}(1-b-c), \min(1-b,\mathrm{sg}(1-b-c))\rangle$

$\rightarrow_{93}$	$\langle(1-\min(1-b,\mathrm{sg}(1-b-c)),$ $\min(1-b,\mathrm{sg}(1-b-c))\rangle$
$\rightarrow_{94}$	$\langle c+(1-c)^2.b,(1-c).c+(1-c)^2.(1-b)\rangle$
$\rightarrow_{95}$	$\langle\min(b,\overline{\mathrm{sg}}(c))+\mathrm{sg}(c).(\overline{\mathrm{sg}}(1-b)$ $+\min(c,\mathrm{sg}(1-b))),\ (\min(1-b,\overline{\mathrm{sg}}(c))$ $+\min(1-c,\mathrm{sg}(c),\mathrm{sg}(1-b))\rangle$
$\rightarrow_{96}$	$\langle\max(\overline{\mathrm{sg}}(1-c),b),\min(\mathrm{sg}(1-b),1-c)\rangle$
$\rightarrow_{97}$	$\langle\max(\overline{\mathrm{sg}}(1-c),\mathrm{sg}(b)),$ $\min(\mathrm{sg}(1-c),\overline{\mathrm{sg}}(b))\rangle$
$\rightarrow_{98}$	$\langle\max(\overline{\mathrm{sg}}(1-c),b),1-\max(b,c)\rangle$
$\rightarrow_{99}$	$\langle\max(\overline{\mathrm{sg}}(1-c),\overline{\mathrm{sg}}(1-b)),$ $\min(1-c,\overline{\mathrm{sg}}(b))\rangle$
$\rightarrow_{100}$	$\langle\max(\min(b,\mathrm{sg}(a)),c),$ $\min(a,\mathrm{sg}(b),d)\rangle$
$\rightarrow_{101}$	$\langle\max(\min(b,\mathrm{sg}(a)),\min(c,\mathrm{sg}(d))),$ $\min(a,\mathrm{sg}(b),\mathrm{sg}(c),d)\rangle$
$\rightarrow_{102}$	$\langle\max(b,\min(c,\mathrm{sg}(d))),\min(a,\mathrm{sg}(c),d)\rangle$
$\rightarrow_{103}$	$\langle\max(\min(1-a,\mathrm{sg}(a)),1-d),$ $\min(a,\mathrm{sg}(1-a),d)\rangle$
$\rightarrow_{104}$	$\langle\max(\min(1-a,\mathrm{sg}(a)),$ $\min(1-d,\mathrm{sg}(d))),$ $\min(a,\mathrm{sg}(1-a),d,\mathrm{sg}(1-d))\rangle$
$\rightarrow_{105}$	$\langle\max(1-a,\min(1-d,\mathrm{sg}(d))),$ $\min(a,d,\mathrm{sg}(1-d))\rangle$
$\rightarrow_{106}$	$\langle\max(\min(b,\mathrm{sg}(1-b)),c),$ $\min(1-b,\mathrm{sg}(b),1-c)\rangle$
$\rightarrow_{107}$	$\langle\max(\min(b,\mathrm{sg}(1-b)),$ $\min(c,\mathrm{sg}(1-c))),$ $\min(1-b,\mathrm{sg}(b),1-c,\mathrm{sg}(c))\rangle$
$\rightarrow_{108}$	$\langle\max(b,\min(c,\mathrm{sg}(1-c))),$ $\min(1-b,1-c,\mathrm{sg}(c))\rangle$
$\rightarrow_{109}$	$\langle b+\min(\overline{\mathrm{sg}}(1-a),c),$ $a.b+\min(\overline{\mathrm{sg}}(1-a),d))\rangle$
$\rightarrow_{110}$	$\langle\max(b,c),\min(a.b+\overline{\mathrm{sg}}(1-a),d)\rangle$
$\rightarrow_{111}$	$\langle\max(b,c.d+\overline{\mathrm{sg}}(1-c)),$ $\min(a.b+\overline{\mathrm{sg}}(1-a),d.(c.d+\overline{\mathrm{sg}}(1-c))$ $+\overline{\mathrm{sg}}(1-d))\rangle$
$\rightarrow_{112}$	$\langle b+c-b.c,a.b+\overline{\mathrm{sg}}(1-a).d\rangle$
$\rightarrow_{113}$	$\langle b+c.d-b.(c.d+\overline{\mathrm{sg}}(1-c)),$ $(a.b+\overline{\mathrm{sg}}(1-a)).(d.(c.d+\overline{\mathrm{sg}}(1-c))$ $+\overline{\mathrm{sg}}(1-d))\rangle$
$\rightarrow_{114}$	$\langle1-a+\min(\overline{\mathrm{sg}}(1-a),1-d),$ $a.(1-a)+\min(\overline{\mathrm{sg}}(1-a),d)\rangle$
$\rightarrow_{115}$	$\langle1-\min(a,d),$ $\min(a.(1-a)+\overline{\mathrm{sg}}(1-a),d)\rangle$
$\rightarrow_{116}$	$\langle\max(1-a,(1-d).d+\overline{\mathrm{sg}}(d)),$ $\min(a.(1-a)+\overline{\mathrm{sg}}(1-a),d.((1-d).d$ $+\overline{\mathrm{sg}}(d))+\overline{\mathrm{sg}}(1-d))\rangle$
$\rightarrow_{117}$	$\langle1-a-d+a.d,$ $(a.(1-a)+\overline{\mathrm{sg}}(1-a)).d\rangle$
$\rightarrow_{118}$	$\langle1-a+(1-d).d-(1-a).((1-d)$ $.d+\overline{\mathrm{sg}}(d)),\ (a.(1-a)+\overline{\mathrm{sg}}(1-a))$ $.d.((1-d).d+\overline{\mathrm{sg}}(d))+\overline{\mathrm{sg}}(1-d))\rangle$
$\rightarrow_{119}$	$\langle b+\min(\overline{\mathrm{sg}}(b),c),$ $(1-b).b+\min(\overline{\mathrm{sg}}(b),1-c)\rangle$
$\rightarrow_{120}$	$\langle\max(b,c),\min((1-b).b+\overline{\mathrm{sg}}(b),1-c)\rangle$
$\rightarrow_{121}$	$\langle\max(b,c.(1-c)+\overline{\mathrm{sg}}(1-c)),$ $\min((1-b).b+\overline{\mathrm{sg}}(b),(1-c).(c.(1-c)$ $+\overline{\mathrm{sg}}(1-c)))+\overline{\mathrm{sg}}(c))\rangle$
$\rightarrow_{122}$	$\langle b+c-b.c,((1-c).b+\overline{\mathrm{sg}}(b)).(1-c)\rangle$

$\rightarrow_{123}$	$\langle b+c.(1-c)-(b.(c.(1-c)+\overline{\mathrm{sg}}(1-c))),$ $((1-b).b+\overline{\mathrm{sg}}(b)).(((1-c)$ $.(c.(1-c)+\overline{\mathrm{sg}}(1-c)))+\overline{\mathrm{sg}}(c))\rangle$
$\rightarrow_{124}$	$\langle c+\min(\overline{\mathrm{sg}}(1-d),b),$ $c.d+\min(\overline{\mathrm{sg}}(1-d),a)\rangle$
$\rightarrow_{125}$	$\langle\max(b,c),\min(c.d+\overline{\mathrm{sg}}(1-d),a)\rangle$
$\rightarrow_{126}$	$\langle\max(c,a.b+\overline{\mathrm{sg}}(1-b)),$ $\min(c.d+\overline{\mathrm{sg}}(1-d),a.(a.b$ $+\overline{\mathrm{sg}}(1-b))+\overline{\mathrm{sg}}(1-a))\rangle$
$\rightarrow_{127}$	$\langle b+c-b.c,(c.d+\overline{\mathrm{sg}}(1-d)).a\rangle$
$\rightarrow_{128}$	$\langle c+a.b-c.(a.b+\overline{\mathrm{sg}}(1-b)),$ $(c.d+\overline{\mathrm{sg}}(1-d)).(a.(a.b+\overline{\mathrm{sg}}(1-b))$ $+\overline{\mathrm{sg}}(1-a))\rangle$
$\rightarrow_{129}$	$\langle1-d+\min(\overline{\mathrm{sg}}(1-d),1-a),$ $d.(1-d)+\min(\overline{\mathrm{sg}}(1-d),a)\rangle$
$\rightarrow_{130}$	$\langle1-\min(d,a),$ $\min(d.(1-d)+\overline{\mathrm{sg}}(1-d),a)\rangle$
$\rightarrow_{131}$	$\langle\max(1-d,(1-a).a+\overline{\mathrm{sg}}(a)),$ $\min(d.(1-d)+\overline{\mathrm{sg}}(1-d),$ $a.((1-a).a+\overline{\mathrm{sg}}(a))+\overline{\mathrm{sg}}(1-a))\rangle$
$\rightarrow_{132}$	$\langle1-a.d,(d.(1-d)+\overline{\mathrm{sg}}(1-d)).a\rangle$
$\rightarrow_{133}$	$\langle1-d+(1-a).a-(1-d)$ $.((1-a).a+\overline{\mathrm{sg}}(a)),$ $(d.(1-d)+\overline{\mathrm{sg}}(1-d))$ $.(a.((1-a).a+\overline{\mathrm{sg}}(a))+\overline{\mathrm{sg}}(1-a))\rangle$
$\rightarrow_{134}$	$\langle c+\min(\overline{\mathrm{sg}}(c),b),$ $(1-c).c+\min(\overline{\mathrm{sg}}(c),(1-b))\rangle$
$\rightarrow_{135}$	$\langle\max(b,c),$ $\min((1-c).c+\overline{\mathrm{sg}}(c),1-b)\rangle$
$\rightarrow_{136}$	$\langle\max(c,(b.(1-b)+\overline{\mathrm{sg}}(1-b))),$ $\min((1-c).c+\overline{\mathrm{sg}}(c),$ $(1-b).(b.(1-b)+\overline{\mathrm{sg}}(1-b))$ $+\overline{\mathrm{sg}}(b)))\rangle$
$\rightarrow_{137}$	$\langle b+c-b.c,$ $((1-c).c+\overline{\mathrm{sg}}(c)).(1-b)\rangle$
$\rightarrow_{138}$	$\langle c+b.(1-b)-c.(b.(1-b)+\overline{\mathrm{sg}}(1-b)),$ $((1-c).c+\overline{\mathrm{sg}}(c)).((1-b)$ $.(b.(1-b)+\overline{\mathrm{sg}}(1-b))+\overline{\mathrm{sg}}(b))\rangle$
$\rightarrow_{139}$	$\langle\frac{b+c}{2},\frac{a+d}{2}\rangle$
$\rightarrow_{140}$	$\langle\frac{b+c+\min(b,c)}{3},\frac{a+d+\max(a,d)}{3}\rangle$
$\rightarrow_{141}$	$\langle\frac{b+c+\max(b,c)}{3},\frac{a+d+\min(a,d)}{3}\rangle$
$\rightarrow_{142}$	$\langle\frac{3-a-d-\max(a,d)}{3},\frac{a+d+\max(a,d)}{3}\rangle$
$\rightarrow_{143}$	$\langle\frac{1-a+c+\min(1-a,c)}{3},\frac{2+a-c-\min(1-a,c)}{3}\rangle$
$\rightarrow_{144}$	$\langle\frac{1+b-d+\min(b,1-d)}{3},\frac{2-b+d+\min(b,1-d)}{3}\rangle$
$\rightarrow_{145}$	$\langle\frac{b+c+\min(b,c)}{3},\frac{3-b-c-\min(b,c)}{3}\rangle$
$\rightarrow_{146}$	$\langle\frac{3-a-d-\min(a,d)}{3},\frac{a+d+\min(a,d)}{3}\rangle$
$\rightarrow_{147}$	$\langle\frac{1-a+c+\max(1-a,c)}{3},\frac{2+a-c-\max(1-a,c)}{3}\rangle$
$\rightarrow_{148}$	$\langle\frac{1+b-d+\max(b,1-d)}{3},\frac{2-b+d-\max(b,1-d)}{3}\rangle$
$\rightarrow_{149}$	$\langle\frac{b+c+\max(b,c)}{3},\frac{3-b-c-\max(b,c)}{3}$
$\rightarrow_{150,\lambda}$	$\langle\frac{b+c+\lambda-1}{2\lambda},\frac{a+d+\lambda-1}{2\lambda}$, where $\lambda\geq1$
$\rightarrow_{151,\gamma}$	$\langle\frac{b+c+\gamma}{2\gamma+1},\frac{a+d+\gamma-1}{2\gamma+1}$, where $\gamma\geq1$
$\rightarrow_{152,\alpha,\beta}$	$\langle\frac{b+c+\alpha-1}{\alpha+\beta},\frac{a+d+\beta-1}{\alpha+\beta},$ where $\alpha\geq1,\beta\in[0,\alpha]$
$\rightarrow_{153,\varepsilon,\eta}$	$\langle\min(1,\max(c,b+\varepsilon)),$ $\max(0,\min(d,a-\eta))\rangle$ where $\varepsilon,\eta\in[0,1]$ and $\varepsilon\leq\eta$

Table 2

$\neg_1$	$\langle b, a \rangle$
$\neg_2$	$\langle \overline{\mathrm{sg}}(a), \mathrm{sg}(a) \rangle$
$\neg_3$	$\langle b, a.b + a^2 \rangle$
$\neg_4$	$\langle b, 1 - b \rangle$
$\neg_5$	$\langle \overline{\mathrm{sg}}(1 - b), \mathrm{sg}(1 - b) \rangle$
$\neg_6$	$\langle \overline{\mathrm{sg}}(1 - b), \mathrm{sg}(a) \rangle$
$\neg_7$	$\langle \overline{\mathrm{sg}}(1 - b), a \rangle$
$\neg_8$	$\langle 1 - a, a \rangle$
$\neg_9$	$\langle \overline{\mathrm{sg}}(a), a \rangle$
$\neg_{10}$	$\langle \overline{\mathrm{sg}}(1 - b), 1 - b \rangle$
$\neg_{11}$	$\langle \mathrm{sg}(b), \overline{\mathrm{sg}}(b) \rangle$
$\neg_{12}$	$\langle b.(b + a), \min(1, a.(b^2 + a + b.a)) \rangle$
$\neg_{13}$	$\langle \mathrm{sg}(1 - a), \overline{\mathrm{sg}}(1 - a) \rangle$
$\neg_{14}$	$\langle \mathrm{sg}(b), \overline{\mathrm{sg}}(1 - a) \rangle$
$\neg_{15}$	$\langle \overline{\mathrm{sg}}(1 - b), \overline{\mathrm{sg}}(1 - a) \rangle$
$\neg_{16}$	$\langle \overline{\mathrm{sg}}(a), \overline{\mathrm{sg}}(1 - a) \rangle$
$\neg_{17}$	$\langle \overline{\mathrm{sg}}(1 - b), \overline{\mathrm{sg}}(b) \rangle$
$\neg_{18}$	$\langle b.\mathrm{sg}(a), a.\mathrm{sg}(b) \rangle$
$\neg_{19}$	$\langle b.\mathrm{sg}(a), 0 \rangle$
$\neg_{20}$	$\langle b, 0 \rangle$
$\neg_{21}$	$\langle \min(1 - a, \mathrm{sg}(a)), \min(a, \mathrm{sg}(1 - a)) \rangle$
$\neg_{22}$	$\langle \min(1 - a, \mathrm{sg}(a)), 0 \rangle$
$\neg_{23}$	$\langle 1 - a, 0 \rangle$
$\neg_{24}$	$\langle \min(b, \mathrm{sg}(1 - b)), \min(1 - b, \mathrm{sg}(b)) \rangle$
$\neg_{25}$	$\langle \min(b, \mathrm{sg}(1 - b)), 0 \rangle$
$\neg_{26}$	$\langle b, a.b + \overline{\mathrm{sg}}(1 - a) \rangle$
$\neg_{27}$	$\langle 1 - a, a.(1 - a) + \overline{\mathrm{sg}}(1 - a) \rangle$
$\neg_{28}$	$\langle b, (1 - b).b + \overline{\mathrm{sg}}(b) \rangle$
$\neg_{29}$	$\langle \max(0, b.a + \overline{\mathrm{sg}}(1 - b)), \\ \min(1, a.(b.a + \overline{\mathrm{sg}}(1 - b)) + \overline{\mathrm{sg}}(1 - a)) \rangle$
$\neg_{30}$	$\langle a.b, \; a.(a.b + \overline{\mathrm{sg}}(1 - b)) \\ + \overline{\mathrm{sg}}(1 - a) \rangle$
$\neg_{31}$	$\langle \max(0, (1 - a).a + \overline{\mathrm{sg}}(a)), \\ \min(1, a.((1 - a).a + \overline{\mathrm{sg}}(a)) \\ + \overline{\mathrm{sg}}(1 - a)) \rangle$
$\neg_{32}$	$\langle (1 - a).a, \; a.((1 - a).a \\ + \overline{\mathrm{sg}}(a)) + \overline{\mathrm{sg}}(1 - a) \rangle$
$\neg_{33}$	$\langle b.(1 - b) + \overline{\mathrm{sg}}(1 - b), \\ (1 - b).(b.(1 - b) + \overline{\mathrm{sg}}(1 - b)) + \overline{\mathrm{sg}}(b) \rangle$
$\neg_{34}$	$\langle b.(1 - b), \; (1 - b).(b.(1 - b) \\ + \overline{\mathrm{sg}}(1 - b)) + \overline{\mathrm{sg}}(b) \rangle$
$\neg_{35}$	$\langle \frac{b}{2}, \frac{1+a}{2} \rangle$
$\neg_{36}$	$\langle \frac{b}{3}, \frac{2+a}{3} \rangle$
$\neg_{37}$	$\langle \frac{2b}{3}, \frac{2a+1}{3} \rangle$
$\neg_{38}$	$\langle \frac{1-a}{3}, \frac{2+a}{3} \rangle$
$\neg_{39}$	$\langle \frac{b}{3}, \frac{3-b}{3} \rangle$
$\neg_{40}$	$\langle \frac{2-2a}{3}, \frac{1+2a}{3} \rangle$
$\neg_{41}$	$\langle \frac{2b}{3}, \frac{3-2b}{3} \rangle$
$\neg_{42,\lambda}$	$\langle \frac{b+\lambda-1}{2\lambda}, \frac{a+\lambda}{2\lambda}, \text{ where } \lambda \geq 1$
$\neg_{43,\gamma}$	$\langle \frac{b+\gamma}{2\gamma+1}, \frac{a+\gamma}{2\gamma+1}, \text{ where } \gamma \geq 1$
$\neg_{44,\alpha,\beta}$	$\langle \frac{b+\alpha-1}{\alpha+\beta}, \frac{a+\beta}{\alpha+\beta}, \text{ where } \alpha \geq 1, \beta \in [0, \alpha]$
$\neg_{45,\varepsilon,\eta}$	$\langle \min(1, b + \varepsilon), \max(0, a - \eta) \rangle \\ \text{where } \varepsilon, \eta \in [0, 1] \text{ and } \varepsilon \leq \eta$

The relations between the implications and negations are shown in Table 3.

Table 3

$\neg_1$	$\to_1, \to_4, \to_5, \to_6, \to_7, \to_{10}, \to_{13}, \to_{61}, \\ \to_{63}, \to_{64}, \to_{66}, \to_{67}, \to_{68}, \to_{69}, \to_{70}, \to_{71}, \\ \to_{72}, \to_{73}, \to_{78}, \to_{80}, \to_{124}, \to_{125}, \to_{127}$
$\neg_2$	$\to_2, \to_3, \to_8, \to_{11}, \to_{16}, \to_{20}, \to_{31}, \to_{32}, \\ \to_{37}, \to_{40}, \to_{41}, \to_{42}$
$\neg_3$	$\to_9, \to_{17}, \to_{21}$
$\neg_4$	$\to_{12}, \to_{18}, \to_{22}, \to_{46}, \to_{49}, \to_{50}, \to_{51}, \to_{53}, \\ \to_{54}, \to_{91}, \to_{93}, \to_{94}, \to_{95}, \to_{96}, \to_{98}, \to_{134}, \\ \to_{135}, \to_{137}$
$\neg_5$	$\to_{14}, \to_{15}, \to_{19}, \to_{23}, \to_{47}, \to_{48}, \to_{52}, \to_{55}, \\ \to_{56}, \to_{57}$
$\neg_6$	$\to_{24}, \to_{26}, \to_{27}, \to_{65}$
$\neg_7$	$\to_{25}, \to_{28}, \to_{29}, \to_{62}$
$\neg_8$	$\to_{30}, \to_{33}, \to_{34}, \to_{35}, \to_{36}, \to_{38}, \to_{39}, \to_{76}, \\ \to_{82}, \to_{84}, \to_{85}, \to_{86}, \to_{87}, \to_{89}, \to_{129}, \\ \to_{130}, \to_{132}$
$\neg_9$	$\to_{43}, \to_{44}, \to_{45}, \to_{83}$
$\neg_{10}$	$\to_{58}, \to_{59}, \to_{60}, \to_{92}$
$\neg_{11}$	$\to_{74}, \to_{97}$
$\neg_{12}$	$\to_{75}$
$\neg_{13}$	$\to_{77}, \to_{88}$
$\neg_{14}$	$\to_{79}$
$\neg_{15}$	$\to_{81}$
$\neg_{16}$	$\to_{90}$
$\neg_{17}$	$\to_{99}$
$\neg_{18}$	$\to_{100}$
$\neg_{19}$	$\to_{101}$
$\neg_{20}$	$\to_{102}, \to_{108}$
$\neg_{21}$	$\to_{103}$
$\neg_{22}$	$\to_{104}$
$\neg_{23}$	$\to_{105}$
$\neg_{24}$	$\to_{106}$
$\neg_{25}$	$\to_{107}$
$\neg_{26}$	$\to_{109}, \to_{110}, \to_{111}, \to_{112}, \to_{113}$
$\neg_{27}$	$\to_{114}, \to_{115}, \to_{116}, \to_{117}, \to_{118}$
$\neg_{28}$	$\to_{119}, \to_{120}, \to_{121}, \to_{122}, \to_{123}$
$\neg_{29}$	$\to_{126}$
$\neg_{30}$	$\to_{128}$
$\neg_{31}$	$\to_{131}$
$\neg_{32}$	$\to_{133}$
$\neg_{33}$	$\to_{136}$
$\neg_{34}$	$\to_{138}$
$\neg_{35}$	$\to_{139}$
$\neg_{36}$	$\to_{140}$
$\neg_{37}$	$\to_{141}$
$\neg_{38}$	$\to_{142}, \to_{143}$
$\neg_{39}$	$\to_{144}, \to_{145}$
$\neg_{40}$	$\to_{146}, \to_{147}$
$\neg_{41}$	$\to_{148}, \to_{149}$
$\neg_{42}$	$\to_{150}$
$\neg_{43}$	$\to_{151}$
$\neg_{44}$	$\to_{152}$
$\neg_{45}$	$\to_{153}$

3. Intuitionistic fuzzy implications and the axioms of intuitionistic logic

First, we mention that the last four from the above intutionistic fuzzy implications ($\rightarrow_{150,\lambda}$, $\rightarrow_{151,\gamma}$, $\rightarrow_{152,\alpha,\beta}$, $\rightarrow_{153,\varepsilon,\eta}$) contain parameters and by this reason the method used here is not applicable to them. Their properties will be discussed in a next research.

Here, we give the 17 axioms of the intuitionistic logic (see, e.g. [6]) If A, B and C are arbitrary propositional forms, then:

(a) $A \rightarrow A$,
(b) $A \rightarrow (B \rightarrow A)$,
(c) $A \rightarrow (B \rightarrow (A\&B))$,
(d) $(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$,
(e) $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$,
(f) $A \rightarrow \neg\neg A$,
(g) $\neg(A\&\neg A)$,
(h) $(\neg A \vee B) \rightarrow (A \rightarrow B)$,
(i) $\neg(A \vee B) \rightarrow (\neg A\&\neg B)$,
(j) $(\neg A\&\neg B) \rightarrow \neg(A \vee B)$,
(k) $(\neg A \vee \neg B) \rightarrow \neg(A\&B)$,
(l) $(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$,
(m) $(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)$,
(n) $\neg\neg\neg A \rightarrow \neg A$,
(o) $\neg A \rightarrow \neg\neg\neg A$,
(p) $\neg\neg(A \rightarrow B) \rightarrow (A \rightarrow \neg\neg B)$,
(q) $(C \rightarrow A) \rightarrow ((C \rightarrow (A \rightarrow B)) \rightarrow (C \rightarrow B))$.

Now, we formulate the following

Theorem. The intuitionistic fuzzy implications (from Table 1) and negations (from Table 2) related to them (see Table 3) that satisfy the axioms of intuitionistic logic as (classical) tautologies, are marked in Table 4 by "•" and these that satisfy the same axioms (only) as IFTs – by "○".

Table 4

	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q
$\rightarrow_1$	●	○	○	○	○	●	●	●	○	●	●	●	●	●	●	●	●
$\rightarrow_2$	●					●	●	●		●	●	●	●	●	●	●	●
$\rightarrow_3$	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
$\rightarrow_4$	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
$\rightarrow_5$	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
$\rightarrow_6$	○	○	○	○		○	○	○	○	○	○	○	○	○	○	○	○
$\rightarrow_7$	○	○				○	○		○	○	○	○	○	○	○	○	○
$\rightarrow_8$	●					●	●		●	●	●	●	●	●	●	●	●
$\rightarrow_9$	○	○			○	○	○	○	○	○	○	○	○	○	○	○	○
$\rightarrow_{10}$							○										
$\rightarrow_{11}$	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●
$\rightarrow_{12}$				○	○	○	○		○	○	○	○	○	○	○	○	○
$\rightarrow_{13}$	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
$\rightarrow_{14}$	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●
$\rightarrow_{15}$	●					●	●		●	●	●	●	●	●	●	●	●
$\rightarrow_{16}$						●	●		●	●	●	●	●	●	●	●	●
$\rightarrow_{17}$	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
$\rightarrow_{18}$	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
$\rightarrow_{19}$						●	●		●	●	●	●	●	●	●	●	●
$\rightarrow_{20}$	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●

	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q
$\rightarrow_{21}$	○	○	○			○	○	○	○	○	○	○	○		○	○	○
$\rightarrow_{22}$	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
$\rightarrow_{23}$	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●
$\rightarrow_{24}$	●					○	●	○		●	●	●	●	●	●	●	●
$\rightarrow_{25}$				○	○	○	○		○	○	○	○	○	○	○	○	○
$\rightarrow_{26}$						○	○		○	○	○	○	○	○	○	○	○
$\rightarrow_{27}$	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
$\rightarrow_{28}$	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
$\rightarrow_{29}$	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
$\rightarrow_{30}$	○	○	○	○		○	○	○	○	○	○		○	○	○		
$\rightarrow_{31}$	●	●	●	●	●		●	●	●	●	●	●		●	●	●	
$\rightarrow_{32}$	●	●	●	●		●	●	●	●	●	●	●		●	●	●	
$\rightarrow_{33}$	○	○	○	○		○	○	○	○	○	○		○	○	○		
$\rightarrow_{34}$	●	●	●	●		●	●	●	●	●	●	●		●	●	●	
$\rightarrow_{35}$	○	○	○	○		○	○	○	○	○	○		○	○	○		
$\rightarrow_{36}$	○	○	○	○		○	○	○	○	○	○		○	○	○		
$\rightarrow_{37}$	●					●	●		●	●	●		●	●	●		
$\rightarrow_{38}$	○	○	○	○		○	○	○	○	○	○		○	○	○		
$\rightarrow_{39}$							○										
$\rightarrow_{40}$	●					●	●		●	●	●		●	●	●		
$\rightarrow_{41}$						●	●		●	●	●		●	●	●		
$\rightarrow_{42}$	●	●	●	●		●	●	●	●	●	●		●	●	●		
$\rightarrow_{43}$				●		●	●		●	●	●		●	●	●		
$\rightarrow_{44}$	○	○	○	○		●	●	○		●	●		●	●	●		
$\rightarrow_{45}$	●	●	●			●	●	●	●	●	●		●	●	●		
$\rightarrow_{46}$		○	○	○	○		○	○	○	○	○	○	○	○	○	○	○
$\rightarrow_{47}$						●	●		●	●	●	●	●	●	●	●	●
$\rightarrow_{48}$		●	●	●	●		●	●	●	●	●	●	●	●	●	●	○
$\rightarrow_{49}$		●	○	●	○		●	●	●	●	●	●	●	●	●	●	●
$\rightarrow_{50}$		○	○	○	○		○	○	○	○	○	○	○	○	○	○	○
$\rightarrow_{51}$		○	○	○	○		○	○	○	○	○	○	○	○	○	○	○
$\rightarrow_{52}$						●	●		●	●	●	●	●	●	●	●	●
$\rightarrow_{53}$		○	○	○	○		●	○	○	○	○	○	○	○	○	○	○
$\rightarrow_{54}$							○										
$\rightarrow_{55}$						●	●	●		●	●	●	●	●	●	●	●
$\rightarrow_{56}$						●	●		●	●	●	●	●	●	●	●	●
$\rightarrow_{57}$			●	●	●	●		●	●	●	●	●	●	●	●	●	●
$\rightarrow_{58}$																	
$\rightarrow_{59}$																	
$\rightarrow_{60}$																	
$\rightarrow_{61}$	○	○	○	○	○		○	○	○	○	○	○	○	○	○	○	○
$\rightarrow_{62}$	●					●	○		●	●	●	○		○	●	○	
$\rightarrow_{63}$	●					●	○		●	●	●			●	●	●	
$\rightarrow_{64}$	○	○	○	○		○	○	○	○	○	○	○	○	○	○	○	○
$\rightarrow_{65}$	●					●	○		●	●	●	●		●	●	●	
$\rightarrow_{66}$	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
$\rightarrow_{67}$							○										
$\rightarrow_{68}$	●					●	○		●	●	●			●	●	●	
$\rightarrow_{69}$	●					●	○		●	●	●			●	●	●	
$\rightarrow_{70}$							○										
$\rightarrow_{71}$	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
$\rightarrow_{72}$	○	○		○		○	○	○	○	○	○	○	○	○	○		
$\rightarrow_{73}$							○										
$\rightarrow_{74}$	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●
$\rightarrow_{75}$	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
$\rightarrow_{76}$	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
$\rightarrow_{77}$	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●	●

	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q
→78							○										
→79	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
→80	○	○		○			○	○	○	○	○	○	○	○	○	○	○
→81	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
→82	○	○	○	○			○	○	○	○	○	○		○	○	○	
→83	●					●	●		●	●	●	●		●	●	●	
→84	●					●	○		●	●	●			●	●		
→85	○	○	○	○			○	○	○	○	○	○		○	○	○	
→86							○										
→87							○										
→88	●	●	●	●		●	●	●	●	●	●	●		●	●	●	
→89	○	○		○			○	○	○	○	○	○		○	○	○	
→90	○	○	○	●		●	●	○	●	●	●		●	●	●		
→91			○	○	○	○		○	○	○	○	○	○	○	○	○	○
→92																	
→93						●	○		●	●	●			●	●	●	
→94			○	○	○	○		○	○	○	○	○	○	○	○	○	○
→95							○										
→96							○										
→97						●	●	●	●		●	●	●	●	●	●	●
→98							○	○		○	○	○			○	●	
→99							○	○							○		
→100	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
→101	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
→102	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
→103	○	○	○	○			○	○	○	○	○	○		○	○	○	
→104	○	○	○	○			○	○	○	○	○	○	○	○	○	○	
→105	○	○	○	○			○	○	○	○	○	○		○	○	○	
→106			○	○	○	○			○			○	○	○	○		○
→107			○			○	○	○	○	○	○			○	○	○	○
→108			○	○										○			
→109	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
→110	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
→111	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
→112	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
→113	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
→114	○	○	○	○			○	○	○	○	○	○		○	○	○	
→115	○	○	○	○			○	○	○	○	○	○		○	○	○	
→116	○	○	○	○			○	○	○	○	○	○		○	○	○	
→117	○	○	○	○			○	○	○	○	○	○		○	○	○	
→118	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
→119				○	○			○	○	○				○	○	○	
→120			○			○	○		○	○	○	○	○	○	○	○	
→121			○	○	○	○		○	○	○	○	○	○	○	○	○	○
→122			○			○	○		○	○	○	○	○	○	○	○	
→123				○	○			○	○	○				○	○		
→124	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
→125	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
→126	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
→127	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
→128	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○	○
→129	○	○	○	○			○	○	○	○	○	○		○	○	○	
→130	○	○	○	○			○	○	○	○	○	○		○	○	○	
→131	○	○	○	○			○	○	○	○	○	○		○	○	○	
→132	○	○	○	○			○	○	○	○	○	○		○	○	○	
→133	a	b	c	○			g	h	i	j	k	l		○	○	○	
→134			○	○	○	○		○	○	○	○	○	○	○	○	○	○
→135			○	○	○	○		○	○	○	○	○	○	○	○	○	○

	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q
→136		○	○	○	○			○	○	○	○	○	○	○	○	○	○
→137			○	○	○	○			○	○	○	○	○	○	○	○	○
→138							○										
→139	○							○	○	○							
→140																	
→141	○							○	○	○			○				
→142																	
→143																	
→145																	
→146	○							○	○	○			○				
→147	○							○	○	○			○				
→148	○							○	○	○			○				
→149								○	○	○			○				

The check of the validity of the assertion in Theorem 1 was made by the software application IFSTool [5, 9], developed as a tool for automatic check of the properties of intuitionistic fuzzy implications and negations. The software has an option – to either check for intuitionistic fuzzy tautologies or only for fuzzy tautologies. For the needs of the present paper, each axiom was tested with all implications and their corresponding negations, and manual backup checks of some of the properties were made, as well.

On the other hand, each one of the above mentioned checks can be realized directly. For example, for implication $\to_{16}$, Axiom (f) is checked as follows. It has the form (see Table 1):

$$\langle a, b\rangle \to_{16} \langle c, d\rangle = \langle \max(\overline{sg}(a), c), \min(sg(a), d)\rangle$$

and it generates (see Table 3) the negation $\neg_2$ with the form (see Table 2):

$$\neg_2\langle a, b\rangle = \langle \overline{sg}(a), sg(a)\rangle.$$

Therefore, we construct the expression

$$A \to \neg\neg A = \langle a, b\rangle \to_{16} \neg_2\neg_2\langle a, b\rangle$$

$$= \langle a, b\rangle \to_{16} \neg_2\langle \overline{sg}(a), sg(a)\rangle$$

$$= \langle a, b\rangle \to_{16} \langle \overline{sg}(\overline{sg}(a)), sg(\overline{sg}(a))\rangle$$

$$= \langle a, b\rangle \to_{16} \langle sg(a), \overline{sg}(a)\rangle$$

$$= \langle \max(\overline{sg}(a), sg(a)), \min(sg(a), \overline{sg}(a))\rangle$$

$$= \langle 1, 0\rangle.$$

All the rest checks are similar, but, of course, in some cases, they are essentially longer and difficult. For the case of intuitionistic fuzzy sets, the intuitionistic logic axioms were checked for a part of the above implications and negations in [1].

The Theorem gives us information which implications satisfy all, no one or part of the axions of intuitionistic logic. Therefore, we can call the implications that satisfy all (part of; or no one of) axions of the intuitionistic logic as tautologies or as IFTs, respectively, *tautology- (or IFT-) good, partly good or bad implications from intuitionistic logic point of view.*

4. Conclusion: Open problems

We finish with three interesting open problems.

Open problem 1. *In classical logic, the operation "disjunction" is obtained by the formula $x \vee y = \neg x \rightarrow y$. Which will be the form of the new disjunctions, that can be generated by the above implications?*

Open problem 2. *The same question is interesting for the new possible operations "conjunction".*

In both cases, probably, from some (different) implication coinciding conjunctions or disjunctions will be generated. So, one of them must be eliminated in the obtained list of new operations.

Open problem 3. *Will the new disjunctions generate new implications using the classical logical formula $x \rightarrow y = \neg x \vee y$?*

Open problem 4. *Which applications in practice can find the separate and more essentially, the tautology- (or IFT-) good implications from intuitionistic logic point of view?*

In a next research, other properties of the implications will be studied. The present research will be an object of future research from point of view of the InterCriteria Analysis (see [4]).

Acknowledgements

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