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On interval pseudo-homogeneous uninorms

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Abstract

In this paper, we introduce the concept of interval pseudo-homegeneous uninorms. We extend the concept of pseudo-homogeneity of specific functions for interval pseudo-homogeneous functions. It is studied two cases of interval pseudo-homogeneous uninorms, that is, interval pseudo-homogeneous t-norms and interval pseudo-homogeneous t-conorms. It is proved a form of interval pseudo-homogeneous t-norms, that is, $\mathbb{T}_M$ and we also prove that only interval t-conorm which is pseudo-homogeneous is $\mathbb{S}_M$ and that there are no interval pseudo-homogeneous proper uninorms.

Keywords: T-norm, t-conorm, uninorm, pseudo-homogeneous

1. Introduction

Uninorms are a specific kind of aggregation operators that have proved to be useful in many fields like expert systems, aggregation, neural networks, and fuzzy system modeling. It is well known that a uninorm U can be a triangular norm or a triangular conorm whenever $U(1,0) = 0$ or $U(1,0) = 1$, respectively. They are interesting because of their structure as a specific combination of a t-norm and a t-conorm [11], [13].

T-norms and t-conorms have been introduced by Menger [18] and Schweizer and Sklar [22] in the context of the theory of probabilistic metric spaces and in this sense, have found applications in other areas such as the theory of fuzzy sets.

In Mathematics a homogeneous function is a function with conduct scalar multiplicative, i.e., if the arguments are multiplied by a factor, then the result is multiplied by a power of this factor. Generalized homogeneous t-norms (or t-conorms) should reflect the multiplicative constant λ as well as the original value $T(x, y)$ or $S(x, y)$, and thus it should be expressed in the form $T(\lambda x, \lambda y) = F(\lambda, T(x, y))$ or $S(\lambda x, \lambda y) = F(\lambda, S(x, y))$.

Ebanks [8] generalized the concept of homogeneous t-norms, which is called quasi-homogeneous t-norms that are defined by a particular function $G(x, y) = \varphi^{-1}(f(x)\varphi(y))$, namely, $T(\lambda x, \lambda y) = \varphi^{-1}(f(\lambda)\varphi(T(x, y)))$ for all $x, y, \lambda \in [0, 1]$, where $f : [0, 1] \to [0, 1]$ is an arbitrary function and φ is a strictly monotone and continuous function.

Considering function G in the definition of quasi-homogeneous t-norms, Xie [23] generalized it to more general functions and then introduced the concept of pseudo-homogeneous t-norms, t-conorms and proper uninorms.

Based on what was mentioned above, we naturally want to extend the concept of pseudo-homogeneity of specific functions for interval pseudo-homogeneous functions, more precisely, interval pseudo-homogeneous uninorms. This is the motivation of the paper. It is showed two cases of interval pseudo-homogeneous uninorms, i. e., interval pseudo-homogeneous t-norms and interval pseudo-homogeneous t-conorms. Besides we prove that any interval proper uninorm is not interval pseudo-homogeneous.

2. Preliminaries

In this section, we recall the concepts of t-norms, t-conorms, uninorms, pseudo-homogeneity and some results which will be used in the text.

Let $\mathbb{U} = \{[x, y]/0 \leq x \leq y \leq 1\}$ be the set of closed subintervals of [0,1]. $\mathbb{U}$ is associated with two projections: $\Pi_1 : \mathbb{U} \to [0, 1]$ and $\Pi_2 : \mathbb{U} \to [0, 1]$ defined by

$$\Pi_1([\underline{x}, \overline{x}]) = \underline{x} \textit{ and } \Pi_2([\underline{x}, \overline{x}]) = \overline{x}$$

By convention, for any interval variable $X \in \mathbb{U}$, $\Pi_1(X)$ and $\Pi_2(X)$ will be denoted by $\underline{x}$ and $\overline{x}$, respectively.

Definition 2.1 *An interval $X \in \mathbb{U}$ is strictly positive if and only if, $\underline{x} > 0$. The set of strictly positive intervals in $\mathbb{U}$ will be denoted by $\mathbb{U}^+$*

In [21], correctness was formalized through the notion of interval representation, where an interval function $F : \mathbb{U}^n \to \mathbb{U}$ represents a function $f : [0, 1]^n \to [0, 1]$ if for each $X \in \mathbb{U}^n$, $f(x) \in F(X)$ whenever $x \in X$(the interval X represents a x).

On the other hand, if the functions $f, g : [0, 1]^n \to [0, 1]$ are not asymptotic[1] then the function $\widehat{fg} : \mathbb{U}^n \to \mathbb{U}$ with $f \leq g$ defined by

[1]For us, a real function f is asymptotic if for some interval $[a_1, b_1], \cdots, [a_n, b_n]$, the set $\{f(x_1, \cdots, x_m)/a_j \leq x_j \leq b_j \ for \ all \ j = 1, \cdots, m\}$ either does have not supremum or does have not infimum.

$$\widehat{fg}(X_1,\cdots,X_n) = [inf\{f(x_i,\cdots,x_n)/ x_i \in X_i \ for \ i=1,\cdots,n\}, sup\{g(x_1,\cdots,x_n)/ x_i \in X_i \ for \ i=1,\cdots,n\}]$$

is well defined and it is an interval representation of every function $h : \mathbb{U}^n \to \mathbb{U}$ such that $f \leq h \leq g$[21]. When f and g are increasing we have $\widehat{fg}(X) = [f(\underline{x}_1,\cdots,\underline{x}_n), g(\overline{x}_1,\cdots,\overline{x}_n)]$. It is clear that, if F is also an interval representation of $f : \mathbb{U}^n \to \mathbb{U}$, then for each $X \in \mathbb{U}^n$, $\widehat{f}(X) \subseteq F(X)$. When $f = g$ we will denote $\widehat{fg}$ by $\widehat{f}$. Clearly, $\widehat{f}$ returns a narrower interval than any other interval representation of f and $\widehat{f}$ is therefore its best interval representation.

We define on $\mathbb{U}$ some partial orders:

Product order or Kulisch Miranker order: $X \leq Y \iff \underline{x} \leq \underline{y}$ *and* $\overline{x} \leq \overline{y}$;

Inclusion order: $X \subseteq Y \iff \underline{y} \leq \underline{x}$ *and* $\overline{x} \leq \overline{y}$;

Next, we define other operations that will be useful in this paper.

Definition 2.2 *(Interval Product) Let X and Y be intervals, then the product of those intervals is defined by $X \cdot Y = [\underline{x}\,\underline{y}, \overline{x}\,\overline{y}]$, when $X \geq [0,0]$ and $Y \geq [0,0]$.*

The interval product has the following algebraic properties: associativity, commutativity, the neutral element is the $\mathbf{1} = [1,1]$, subdistributivity with respect to the sum and $X \cdot [0,0] = [0,0]$.

Definition 2.3 *(Interval Power) Let X and $\mathbb{K}$ be strictly positive interval. The interval power of X is given by $X^{\mathbb{K}} = \{x^k / x \in X \, and \, k \in \mathbb{K}\} = [\underline{x}^{\overline{k}}, \overline{x}^{\underline{k}}]$, $0 \leq \underline{k} \leq \overline{k} \leq 1$.*

Observation 2.1 *Observe that when $X, Y \in \mathbb{U}$, $X^{K_1} \cdot X^{K_2} = X^{K_1+K_2}$ and $(XY)^{\mathbb{K}} = X^{\mathbb{K}} Y^{\mathbb{K}}$.*

Proposition 2.1 *[2, Theorem 4.2] Let $f, g : [0,1]^n \to [0,1]$ such that $f \leq g$. $\widehat{f,g}$ is Moore continuous if and only if f and g are continuous.*

Definition 2.4 *A t-norm is a function $T : [0,1]^2 \to [0,1]$ which satisfies the conditions of symmetry, associativity, monotonicity and has 1 as neutral element.*

Example 2.1 *Typical examples of t-norms are:*

i) $T_M(x,y) = min(x,y)$;

ii) $T_P(x,y) = xy$;

iii) $T_W(x,y) = min(x,y)$ if $max(x,y) = 1$ and $T_W(x,y) = 0$ otherwise.

Let T_1 and T_2 be t-norms, we have $T_1 \leq T_2$ if for every $x, y \in [0,1]$, $T_1(x,y) \leq T_2(x,y)$.

Definition 2.5 *[5] A function $\mathbb{T} : \mathbb{U}^2 \to \mathbb{U}$ is an interval t-norm if $\mathbb{T}$ is symmetric, associative, monotonic with respect to the order of Kulisch -Miranker, and [1,1] is the neutral element.*

Definition 2.6 *[6] An interval t-norm $\mathbb{T}$ is t-representable if there exist t-norms T_1 and T_2 such that $T_1 \leq T_2$ and $\mathbb{T} = \widehat{T_1 T_2}$.*

Definition 2.7 *[5] An interval t-norm $\mathbb{T}$ is inclusion monotonic if $\forall X, Y, Z \in \mathbb{U}$, $\mathbb{T}(X,Y) \subseteq \mathbb{T}(X,Z)$ when $Y \subseteq Z$.*

Theorem 2.1 ([9], Corollary 33) *An interval t-norm $\mathbb{T} : \mathbb{U}^2 \to \mathbb{U}$ is t-representable if and only if it is inclusion monotonic.*

Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be interval t-norms. We have $\mathbb{T}_1 \leq \mathbb{T}_2$ if for every $X, Y \in \mathbb{U}$, $\mathbb{T}_1(X,Y) \leq \mathbb{T}_2(X,Y)$.

Proposition 2.2 *Let T_1 and T_2 be t-norms. $T_1 \leq T_2$ if and only if $\widehat{T_1} \leq \widehat{T_2}$.*

Proof: $(\Rightarrow)$ See [3], Proposition 5.1.

$(\Leftarrow)$ Let $x, y \in [0,1]$. Then

$$\widehat{T_1}([x,x],[y,y]) \leq \widehat{T_2}([x,x],[y,y])$$

or in other words,

$$[T_1(x,y), T_1(x,y)] \leq [T_2(x,y), T_2(x,y)]$$

Thus,

$$T_1(x,y) \leq T_2(x,y).$$

□

Similarly to the case of t-norms, many classes of interval t-norms can be defined [3]. We examined only some of them, for example, interval t-norms that have zero divisors, interval Archimedean t-norms and interval idempotent t-norms.

An interval t-norm $\mathbb{T}$ has zero divisors if there is at least one pair of elements $X \neq [0,0]$ and $Y \neq [0,0]$, such that $\mathbb{T}(X,Y) = [0,0]$. For example, $\widehat{T_W}([0.4,0.9],[0.6,0.7]) = [0,0]$. If an interval t-norm has no zero divisor then $\mathbb{T}(X,Y) = \mathbf{0}$ if and only if $X = [0,0]$ or $Y = [0,0]$.

Let $\mathbb{T}$ be an interval t-norm. $\mathbb{T}$ is Archimedean if for each $X, Y \in \mathbb{U} - \{[0,0],[1,1]\}$, there exists a positive integer n such that $X^{(n)} < Y$ where $X^{(1)} = X$ and $X^{(k+1)} = \mathbb{T}(X, X^{(k)})$.

An interval t-norm is idempotent if $\mathbb{T}(X,X) = X$ for all $X \in \mathbb{U}$, for example $\mathbb{T}_M$, where $\mathbb{T}_M(X,Y) = [min(\underline{x},\underline{y}), min(\overline{x},\overline{y})]$.

Proposition 2.3 *[7] The only interval t-norm which is idempotent is $\mathbb{T}_M$.*

Definition 2.8 *A triangular conorm is a function $S : [0,1]^2 \to [0,1]$ that is symmetric, associative, monotonic and has 0 as neutral element.*

A t-norm T and a t-conorm S are dual with respect to $N(x) = 1 - x$ when $T(x, y) = 1 - S(1 - x, 1 - y)$ for all $x, y \in [0, 1]$.

Example 2.2 *An example of a basic t-conorm is:* $S_M(x, y) = max(x, y)$;

Definition 2.9 *[5] A function $\mathbb{S} : \mathbb{U}^2 \to \mathbb{U}$ is an interval t-conorm if $\mathbb{S}$ is symmetric, associative, monotonic and [0,0] is the neutral element.*

Definition 2.10 *An interval t-conorm $\mathbb{S}$ is s-representable if there exist t-conorms S_1 and S_2 such that $S_1 \leq S_2$ and $\mathbb{S} = \widehat{S_1 S_2}$.*

In [23] Xie at al. defined pseudo-homogeneous t-norms and t-conorms and constructed the tuple (T, F) which satisfies the pseudo-homogeneous equation. Next, some of these results are showed.

Definition 2.11 *[23] A t-norm T is said to be pseudo-homogeneous if it satisfies $T(\lambda x, \lambda y) = F(\lambda, T(x, y))$ for all $x, y, \lambda \in [0, 1]$, where $F : [0, 1]^2 \to [0, 1]$ is a continuous and increasing function with $F(x, 1) = 0 \Leftrightarrow x = 0$.*

Lemma 2.2 *Let T be a pseudo-homogeneous t-norm. Then T is positive.*

Proof
Suppose that there exist $x, y \neq 0$ such that $T(x, y) = 0$. Let $z = min(x, y)$, then $z \neq 0$ and because T is increasing, $T(z, z) = 0$. Thus, $T(z, z) = F(z, T(1, 1)) = F(z, 1) \neq 0$ which is a contradiction.

□

Lemma 2.3 *[23] If T is a pseudo-homogeneous t-norm, then it must be continuous.*

Lemma 2.4 *[23] Let T be a t-norm. Then $T(x, xy) = T(y, xy)$ for any $x, y \in [0, 1]$ if and only if $T = T_M$.*

Lemma 2.5 *[23] Let T be a pseudo-homogeneous t-norm and F be the same as in Definition 2.11. Then F is commutative if and only if $T(x, x) = x$ for any $x \in (0, 1)$, i.e., $T = T_M$.*

Definition 2.12 *A t-conorm S is called pseudo-homogeneous if it satisfies $S(\lambda x, \lambda y) = F(\lambda, S(x, y))$ for all $x, y, \lambda \in [0, 1]$, where $F : [0, 1]^2 \to [0, 1]$ is an increasing function.*

Theorem 2.6 *[23] A t-conorm S is pseudo-homogeneous if and only if $S = S_M$ and $F(x, y) = xy$.*

Definition 2.13 *A function $U : [0, 1]^2 \to [0, 1]$ is called a uninorm if it is commutative, associative and increasing and has a neutral element $e \in [0, 1]$.*

There are two cases: if $e = 1$, it leads back to t-norms. If $e = 0$, it leads back to t-conorms. Any uninorm with neutral element in (0,1) is called **proper uninorm** [11].

Definition 2.14 *[23] A proper uninorm U is called pseudo-homogeneous if it satisfies $U(\lambda x, \lambda y) = F(\lambda, U(x, y))$ for all $x, y, \lambda \in [0, 1]$, where $F : [0, 1]^2 \to [0, 1]$ is an increasing function.*

Proposition 2.4 *[23] Let U be a proper uninorm. Then U is never pseudo-homogeneous.*

3. Interval pseudo-homogeneous uninorms

As previously stated any uninorm with neutral element $e \in (0, 1)$ is called proper [11]. In this section, we introduce the concept of interval pseudo-homogeneous uninorms and we show there are no interval pseudo-homogeneous proper uninorms.

Definition 3.1 *[20] A function $\mathcal{U} : \mathbb{U}^2 \to \mathbb{U}$ is called an interval uninorm if it is commutative, associative, increasing and has a neutral element $e \in \mathbb{U}$. When the neutral element e is neither $[0, 0]$ nor $[1, 1]$ the interval uninorm $\mathcal{U}$ is called of proper.*

Definition 3.2 *A interval uninorm $\mathcal{U}$ is called interval pseudo-homogeneous if there exist $\mathbb{F} : \mathbb{U}^2 \to \mathbb{U}$ such that*

$$\mathcal{U}(\lambda X, \lambda Y) = \mathbb{F}(\lambda, \mathcal{U}(X, Y)), \tag{1}$$

for all $X, Y, \lambda \in \mathbb{U}$.

Proposition 3.1 *Let $\mathcal{U}$ be an interval uninorm with neutral element e. If $\mathcal{U}$ is interval pseudo-homogeneous with respect to a function $\mathbb{F} : \mathbb{U}^2 \to \mathbb{U}$, then $\mathbb{F}$ satisfies the following properties:*

1. $\mathbb{F}([0, 0], X) = \mathbb{F}(X, [0, 0]) = [0, 0]$;
2. $\mathbb{F}$ *is increasing;*
3. $\mathbb{F}(e, Y) \leq eY$;
4. $\mathbb{F}(X, e) \leq eX$.

Proof

1. $\mathbb{F}([0, 0], X) = \mathbb{F}([0, 0], \mathcal{U}(X, e)) = \mathcal{U}([0, 0]X, [0, 0]e) = \mathcal{U}([0, 0], [0, 0]) = [0, 0]$ and $\mathbb{F}(X, [0, 0]) = \mathbb{F}(X, \mathcal{U}([0, 0], [0, 0])) = \mathcal{U}(X[0, 0], X[0, 0]) = \mathcal{U}([0, 0], [0, 0]) = [0, 0]$.
2. If $Y \leq Z$ then $\mathbb{F}(X, Y) = \mathbb{F}(X, \mathcal{U}(Y, e)) = \mathcal{U}(XY, Xe) \leq \mathcal{U}(XZ, Xe) = \mathbb{F}(X, \mathcal{U}(Z, e)) = \mathbb{F}(X, Z)$ and $\mathbb{F}(Y, X) = \mathbb{F}(Y, \mathcal{U}(X, e)) = \mathcal{U}(YX, Ye) \leq \mathcal{U}(ZX, Ze) = \mathbb{F}(Z, \mathcal{U}(X, e)) = \mathbb{F}(Z, X)$.
3. $\mathbb{F}(e, Y) = \mathbb{F}(e, \mathcal{U}(Y, e)) = \mathcal{U}(eY, e^2) \leq \mathcal{U}(eY, e) = eY$;
4. $\mathbb{F}(X, e) = \mathbb{F}(X, \mathcal{U}(e, e)) = \mathcal{U}(Xe, Xe) \leq \mathcal{U}(Xe, e) = Xe$.

Theorem 3.1 *Let $\mathbb{T}$ be an interval t-norm. If $\mathbb{T}$ is interval pseudo-homogeneous with respect a function $\mathbb{F} : \mathbb{U}^2 \to \mathbb{U}$ then $\mathbb{F}$ is an interval conjunctive aggregation function.*

Proof From Proposition 3.1, $\mathbb{F}$ is increasing and $\mathbb{F}([0,0],[0,0]) = [0,0]$. Since, $\mathbb{F}([1,1],[1,1]) = \mathbb{F}([1,1],\mathbb{T}([1,1],[1,1])) = \mathbb{T}([1,1],[1,1]) = [1,1]$. Therefore, $\mathbb{F}$ is an aggregation function. In addition, because $\mathbb{F}$ is increasing and from Proposition 3.1, $\mathbb{F}(X,Y) \leq \mathbb{F}([1,1],Y) \leq [1,1]Y = Y$ and $\mathbb{F}(X,Y) \leq \mathbb{F}(X,[1,1]) \leq X[1,1] = X$. So, $\mathbb{F}(X,Y) \leq \inf(X,Y)$.

□

Proposition 3.2 *Let $\mathcal{U}$ be an interval proper uninorm. Then $\mathcal{U}$ is never interval pseudo-homogeneous.*

Proof
Suppose that $\mathcal{U}$ is an interval pseudo-homogeneous proper uninorm with neutral element $e \in \mathbb{U} - \{[0,0],[1,1]\}$. According to (1), we have

$$\begin{aligned}[\underline{e},\overline{e}]^2 &= \mathcal{U}([\underline{e},\overline{e}][\underline{e},\overline{e}],[\underline{e},\overline{e}][1,1]) \\ &= \mathbb{F}([\underline{e},\overline{e}],\mathcal{U}([\underline{e},\overline{e}],[1,1])) \\ &= \mathbb{F}([\underline{e},\overline{e}],[1,1]).\end{aligned}$$

Therefore, $\mathbb{F}([\underline{e},\overline{e}],[1,1]) = [\underline{e},\overline{e}]^2$.

$$\begin{aligned}[\underline{e},\overline{e}][1,1] &= \mathcal{U}([\underline{e},\overline{e}][1,1],[\underline{e},\overline{e}][1,1]) \\ &= \mathbb{F}([\underline{e},\overline{e}],\mathcal{U}([1,1],[1,1])) \\ &= \mathbb{F}([\underline{e},\overline{e}],[1,1]) = [\underline{e},\overline{e}]^2,\end{aligned}$$

which is a contradiction. Thus, there is no interval proper uninorm which is interval pseudo-homogeneous.

□

3.1. Two cases of interval pseudo-homogeneous uninorms

Uninorms are a generalization of both t-norms and t-conorms [24]. Here we show that when $e = [1,1]$ we have an interval pseudo-homogeneous t-norm and when $e = [0,0]$ we have an interval pseudo-homogeneous t-conorm. In this sense, there are only two cases of interval pseudo-homogeneous uninorms.

Definition 3.3 *[17] A interval t-norm $\mathbb{T}$ is said to be interval pseudo-homogeneous if it satisfies*

$$\mathbb{T}(\lambda X, \lambda Y) = \mathbb{F}(\lambda, \mathbb{T}(X,Y)) \ for\ all\ X,Y,\lambda \in \mathbb{U}, \quad (2)$$

where $\mathbb{F} : \mathbb{U}^2 \to \mathbb{U}$ is a Moore continuous and increasing function with $\mathbb{F}(X,[1,1]) = [0,0] \Leftrightarrow X = [0,0]$.

Observation 3.1 *The above definition is a specific case of Definition 3.2, since all interval t-norm are interval uninorms and any function $\mathbb{F}$ with the above condition, also satisfies Definition 3.2.*

Lemma 3.2 *Let $\mathbb{T}$ be an interval pseudo-homogeneous t-norm. Then $\mathbb{T}$ has no zero divisors.*

Proof
Suppose that there exist $X,Y \neq [0,0]$ such that $\mathbb{T}(X,Y) = [0,0]$. Let $Z = \inf(X,Y)$. Then, $Z \neq [0,0]$ and, because $\mathbb{T}$ is increasing, $\mathbb{T}(Z,Z) = [0,0]$. On the other hand, $\mathbb{T}(Z,Z) = \mathbb{F}(Z,\mathbb{T}([1,1],[1,1])) = \mathbb{F}(Z,[1,1]) \neq [0,0]$ which is a contradiction.

□

Lemma 3.3 *[17] Let $\mathbb{T}$ be an interval pseudo-homogeneous t-norm with respect to a function $\mathbb{F}$. If $\mathbb{T}$ is t-representable then there exist continuous and increasing functions $F_1, F_2 : [0,1]^2 \to [0,1]$ satisfying the condition $F_i(x,1) = 0 \Leftrightarrow x = 0$ and such that $\mathbb{F} = \widehat{F_1F_2}$.*

Theorem 3.4 *[17] Let $\mathbb{T}$ be a t-representable interval t-norm. $\mathbb{T}$ is interval pseudo-homogeneous if and only if its represents are pseudo homogeneous.*

Lemma 3.5 *Let $\mathbb{T}$ be a t-norm. Then $\mathbb{T}(X,XY) = \mathbb{T}(Y,XY)$ for any $X,Y \in \mathbb{U} \Leftrightarrow \mathbb{T} = \mathbb{T}_M$.*

Proof
($\Rightarrow$) Suppose that $\mathbb{T}(X,XY) = \mathbb{T}(XY,Y)$. By fixing $X = [1,1]$, we get that $\mathbb{T}(\mathbf{1},Y) = \mathbb{T}(Y,Y)$ for any $Y \in \mathbb{U}$. Thus $\mathbb{T}(Y,Y) = Y$.
($\Leftarrow$) On the other hand, if $\mathbb{T} = \mathbb{T}_M$, then $\mathbb{T}(X,XY) = XY = \mathbb{T}(XY,Y)$.

□

Proposition 3.3 *Let $\mathbb{T}$ be an interval pseudo-homogeneous t-norm and $\mathbb{F}$ be the same as in Definition 3.3. Then $\mathbb{F}$ is commutative if and only if $\mathbb{T}(X,X) = X$ for any $X \in \mathbb{U}$. i.e., $\mathbb{T} = \mathbb{T}_M$.*

Proof
By Eq. (2), we have $\mathbb{T}(X,XY) = \mathbb{F}(X,Y)$ for any $X,Y \in \mathbb{U}$. Similary, $\mathbb{T}(XY,Y) = \mathbb{F}(Y,X)$ for any $X,Y \in \mathbb{U}$. Therefore,

$$\begin{aligned}\mathbb{F}\ is\ commutative &\Leftrightarrow \mathbb{F}(X,Y) = \mathbb{F}(Y,X) \forall X,Y \in \mathbb{U} \\ &\Leftrightarrow \mathbb{T}(X,XY) = \mathbb{T}(XY,Y) \forall X,Y \in \mathbb{U} \\ &\Leftrightarrow \mathbb{T} = \mathbb{T}_M\ by\ Lemma\, 3.5 \\ &\Leftrightarrow \mathbb{T}(X,X) = X \forall X \in \mathbb{U}\ by\ Prop.\, 2.3\end{aligned}$$

□

Observe that there are interval pseudo-homogeneous interval t-norms in the sense of Def.3.2 which are not interval pseudo-homogeneous in the sense of Def. 3.3, e.g. the interval t-norm

$$\mathbb{T}(X,Y) = \begin{cases} [0,0] & \text{if } \sup(X,Y) < [1,1] \\ \inf(X,Y) & \text{otherwise} \end{cases}$$

Alsina at al. [1] proved that if S is a homogeneous t-conorm, then k=1 and $S = S_M$. Here, we will extend the concept of pseudo-homogeneous t-conorm and will show that the only interval t-conorm which is pseudo-homogeneous is $\mathbb{S}_M$.

Definition 3.4 *A t-conorm $\mathbb{S}$ is called interval pseudo-homogeneous if it satisfies $\mathbb{S}(\lambda X, \lambda Y) = \mathbb{F}(\lambda, \mathbb{S}(X,Y))$ for all $X, Y, \lambda \in \mathbb{U}$, where $\mathbb{F} : \mathbb{U}^2 \to \mathbb{U}$ is an increasing function.*

Theorem 3.6 *A interval t-conorm $\mathbb{S}$ is pseudo-homogeneous if and only if $\mathbb{S} = \mathbb{S}_M$ and $\mathbb{F}(X,Y) = XY$.*

Proof
($\Rightarrow$) Suppose that for some increasing function $\mathbb{F} : \mathbb{U}^2 \to \mathbb{U}$, $\mathbb{S}$ satisfies

$$\mathbb{S}(\lambda X, \lambda Y) = \mathbb{F}(\lambda, \mathbb{S}(X,Y))$$

for all $X, Y, \lambda \in \mathbb{U}$, then for any $X \in \mathbb{U}$

$$\begin{aligned} \mathbb{S}(X,X) &= \mathbb{S}(X \cdot [1,1], X \cdot [1,1]) \\ &= \mathbb{F}(X, \mathbb{S}([1,1],[1,1])) \\ &= \mathbb{F}(X,[1,1]), \end{aligned}$$

and

$$\begin{aligned} X = \mathbb{S}([0,0],X) &= \mathbb{S}(X \cdot [0,0], X \cdot [1,1]) \\ &= \mathbb{F}(X, \mathbb{S}([0,0],[1,1])) \\ &= \mathbb{F}(X,[1,1]), \end{aligned}$$

Thus, $\mathbb{S}(X,X) = X$ for any $X \in \mathbb{U}$ and $\mathbb{S} = \mathbb{S}_M$. Moroever, since

$$\begin{aligned} XY &= \mathbb{S}(XY, XY) \\ &= \mathbb{F}(X, \mathbb{S}(Y,Y)) \\ &= \mathbb{F}(X,Y) \end{aligned}$$

Thus,

$$\mathbb{F}(X,Y) = XY$$

($\Leftarrow$) Let $\mathbb{S} = \mathbb{S}_M$ and $\mathbb{F}(X,Y) = XY$, then $\mathbb{S}(\lambda X, \lambda Y) = sup(\lambda X, \lambda Y) = \lambda sup(X,Y)$ and

$$\begin{aligned} \mathbb{F}(\lambda, \mathbb{S}(X,Y)) &= \mathbb{F}(\lambda, sup(X,Y)) \\ &= \lambda sup(X,Y) \end{aligned}$$

Thus,

$$\mathbb{S}(\lambda X, \lambda Y) = \mathbb{F}(\lambda, \mathbb{S}(X,Y)),$$

for all $X, Y, \lambda \in \mathbb{U}$.

□

4. Final remarks

In this paper, we consider the study of interval pseudo-homegeneous functions, but specifically the pseudo-homogeneous uninorms. It is studied two cases of interval pseudo-homogeneous uninorms, i.e., interval pseudo-homogeneous t-norms and interval pseudo-homogeneous t-conorms. It is proved a form of interval pseudo-homogeneous t-norms, i.e., $\mathbb{T}_M$ and we also proved that only interval t-conorm which is pseudo-homogeneous is $\mathbb{S}_M$ and that there are no interval pseudo-homogeneous proper uninorms.

Since in [23] it has been proved that exist two more forms of pseudo-homogeneous t-norms, but in work we proved only one for interval case. Then, in the future work, we will prove these two forms for interval case.

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