Intersection and union of type-2 fuzzy sets and connection to (α_1, α_2)-double cuts

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Abstract

It is known that the standard intersection and union of type-1 fuzzy sets (i.e., the intersection and union under the minimum t-norm and maximum t-conorm) are the only cutworthy operations for type-1 fuzzy sets. The aim of this paper is to show that similar property holds also for type-2 fuzzy sets, with respect to some special cutting. As was already demonstrated, the intersection and union of type-2 fuzzy sets are not preserved in α-planes. Thus, we study another kind of cutting, so-called double cuts, and show that the intersection and union of type-2 fuzzy sets are preserved in these double cuts.

Keywords: Type-2 fuzzy sets, double cut, α-plane, intersection, union, fuzzy sets

1. Introduction

The standard intersection and union operations of type-1 fuzzy sets (T1 FSs, in abbreviation) are cutworthy operations, which means that they are preserved in α-cuts for all $\alpha \in [0,1]$ in the classical sense (see [1]). This is a very useful property of α-cuts for computation. Furthermore, it is a 'nice' relation between the intersection and union of fuzzy sets A, B and the intersection and union of crisp sets A_α, B_α:

$$(A \cap B)_\alpha = A_\alpha \cap B_\alpha\,, \qquad (A \cup B)_\alpha = A_\alpha \cup B_\alpha\,.$$

Recently has been published an α-plane representation for type-2 fuzzy sets (T2 FSs) in [2] and [3], which is some kind of generalization of α-cut representation for T1 FSs. As was pointed out in [4] the intersection and union of T2 FSs are not preserved in α-planes in general, i.e. they are not cuthworthy operations. The main results of [4] are summarized in this paper too.

The α-cuts of T1 FSs have two properties that make them very useful: 1. α-cut is a (crisp) subset of a universe of discourse, 2. α-cut emphasizes elements with high membership (higher or equal to α). The α-planes have no these two properties. Hamrawi et al. (see [5]) defined an α-cut of T2 FS, denoted by $\tilde{A}_{\tilde{\alpha},\alpha}$ (as α-cut of $\tilde{\alpha}$-plane). We independently defined (α_1, α_2)-double cut of T2 FS (see [6]), which is very similar notion to Hamrawi's one[1]. Furthermore, the (α_1, α_2)-double cut meets both over mentioned properties. The main aim of this paper is to show that intersection and union of T2 FSs are preserved in (α_1, α_2)-double cuts (under some special conditions). We study relations between the intersection (union) of (α_1, α_2)-double cuts of T2 FSs and (α_1, α_2)-double cuts of intersection (union) of T2 FSs, and provide relevant proofs. We study these relations for the intersection and union under various t-norms and t-conorms. Furthermore, we discuss differences between α-planes and (α_1, α_2)-double cuts in this context.

The rest of this paper is organized as follows. Section 2 contains basic definitions and notations that are used in the remaining parts of paper. We summarize properties of α-planes in connection with the intersection and union of T2 FSs under various t-norms and t-conorms in Section 3. In Section 4, we study properties of (α_1, α_2)-double cuts. The conclusions are discussed in Section 5.

2. Preliminaries

2.1. Type-2 fuzzy sets and vertical slices

A *T2 FS*, denoted by $\tilde{A}$, in a crisp set X is characterized by a *type-2 membership function* $\mu_{\tilde{A}}(x,u)$, i.e.,

$$\tilde{A} = \{((x,u), \mu_{\tilde{A}}(x,u)) | \forall x \in X, \forall u \in J_x \subseteq [0,1]\}\,,$$

where $0 \leq \mu_{\tilde{A}}(x,u) \leq 1$, u is a *primary grade* and $\mu_{\tilde{A}}(x,u)$ is a *secondary grade* ([7], [8], [9]).

The T2 FS $\tilde{A}$ can also be expressed as

$$\tilde{A} = \int_{x \in X} \int_{u \in J_x} \mu_{\tilde{A}}(x,u)/(x,u), \qquad J_x \subseteq [0,1],$$

where $\int\int$ denotes a union over all admissible x and u. For discrete universes of discourse, $\int$ is replaced with $\sum$ (e.g. [10], [9]).

At each value of x, say $x = x'$, the 2D plane whose axes are u and $\mu_{\tilde{A}}(x',u)$ is called a *vertical slice* of T2 FS $\tilde{A}$ (see [9]). Type-2 membership

[1]An α-cut of T2 FS is represented by α-cut of UMF and LMF of α-plane, an (α_1, α_2)-double cut is represented only by α-cut of UMF of α-plane. Hamrawi's α-cut leads to much more easier representation theorem as our (α_1, α_2)-double cut, but we consider our approach to be more natural in some point of view - if one can emphasizes elements of X with both high primary and secondary grades, then only α-cut of UMF is important.

function $\mu_{\tilde{A}}(x,u) : X \times [0,1] \to [0,1]$ is a function that assigns a secondary grade to each pair (x,u), where $x \in X$ and $u \in J_x \subseteq [0,1]$ is its primary grade. $f_x(u) : J_x \to [0,1]$ where $J_x \subseteq [0,1]$ is a function that assigns a secondary grade to each primary grade u for some fixed x. It is a T1 FS in $[0,1]$ (or in J_x) in the vertical slice. Secondary MF or vertical slice $\mu_{\tilde{A}}(x) : X \to [0,1]^{[0,1]}$ is a function that assigns a function $f_x(u)$ to each $x \in X$.

We can review a T2 FS set in a vertical-slice manner, as

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | \forall x \in X\}$$

or, as

$$\tilde{A} = \int_{x\in X} \mu_{\tilde{A}}(x)/x = \int_{x\in X} [\int_{u\in J_x} f_x(u)/u]/x = = \int_{x\in X} [\int_{u\in J_x} \mu_{\tilde{A}}(x,u)/u]/x,$$

where $J_x \subseteq [0,1]$.

Membership function $\mu_{\tilde{A}}(x)$ for some x is *convex* if $u_1 \le u_2 \le u_3$ implies $\mu_{\tilde{A}}(x,u_2) \ge \mu_{\tilde{A}}(x,u_1) \wedge \mu_{\tilde{A}}(x,u_3)$, for all $u_1, u_2, u_3 \in [0,1]$, and is *normal* if

$$\max_{u\in[0,1]} \{\mu_{\tilde{A}}(x,u)\} = 1 .$$

2.2. α-planes

A two dimensional *α-plane*, denoted by $\tilde{A}_\alpha$, is a union of all primary memberships whose secondary grades are greater or equal to a special value α ([2], [3], [9]), i.e.,

$$\tilde{A}_\alpha = \bigcup_{x\in X} (x,u) | \mu_{\tilde{A}}(x,u) \ge \alpha = \bigcup_{x\in X} (\mu_{\tilde{A}}(x))_\alpha , \quad (1)$$

where $(\mu_{\tilde{A}}(x))_\alpha$ is an α-cut of vertical slice $\mu_{\tilde{A}}(x)$. Both α-plane of T2 FS and α-cut of T1 FS are crisp sets (see Definition 1 in [2]), however, in spite of α-cut, α-plane is not a subset of an universe of discourse X.

Let $I_{\tilde{A}_\alpha}(x,u)$ be an *indicator function* of the α-plane $\tilde{A}_\alpha$, i.e.,

$$I_{\tilde{A}_\alpha}(x,u) = \begin{cases} 1, & (x,u) \in \tilde{A}_\alpha, \\ 0, & \text{otherwise.} \end{cases}$$

Then, the *associated T2 FS* (α-level T2 FS) of the α-plane, denoted by $\tilde{A}(\alpha)$, is defined ([2], [3]) as (see Figure 1)

$$\tilde{A}(\alpha) = \{((x,u), \alpha I_{\tilde{A}_\alpha}(x)) | \forall x \in X, \forall u \in J_x \subseteq [0,1]\}.$$

2.3. (α_1, α_2)-double cuts

An (α_1, α_2)-*double cut* of a T2 FS $\tilde{A}$ in a set X, denoted by $\tilde{A}_{\alpha_1,\alpha_2}$, is a crisp set containing all elements $x \in X$ with property (see [6]): there exists at least one primary grade $u \in J_x$ greater or equal to

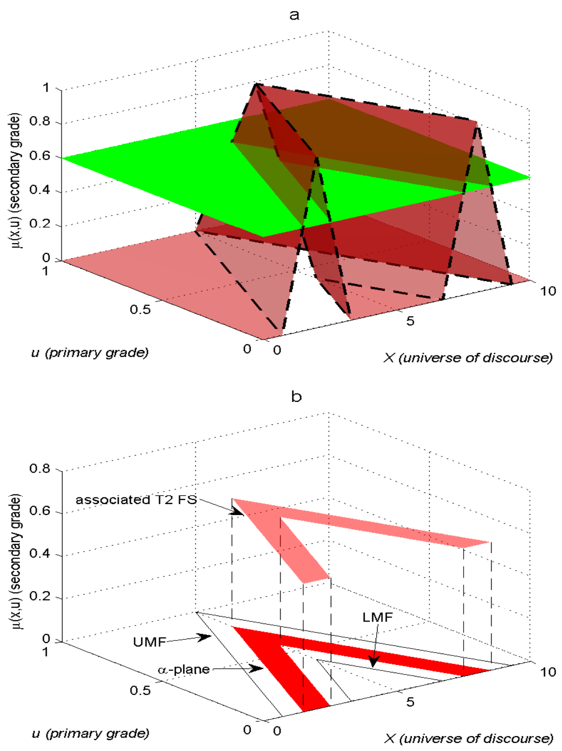

Figure 1: a) T2 FS with triangular secondary membership functions, triangular UMF and LMF. 'Cutting' plane in level $\alpha = 0.6$. b) α-plane for $\alpha = 0.6$ and associated T2 FS of the α-plane.

α_1 such that the secondary grade of (x,u) is greater or equal to α_2, i.e.,

$$\tilde{A}_{\alpha_1,\alpha_2} = \{x | \exists u \in J_x (u \ge \alpha_1 \& \mu_{\tilde{A}}(x,u) \ge \alpha_2)\}.$$

An (α_1, α_2)-double cut is a crisp subset of an universe of discourse X, so it meet both properties discussed in Introduction.

Let $I_{\tilde{A}_{\alpha_1,\alpha_2}}(x)$ be an indicator function of (α_1, α_2)-double cut $\tilde{A}_{\alpha_1,\alpha_2}$. Then, the *associated T2 FS* of the (α_1, α_2)-double cut, denoted by $\tilde{A}(\alpha_1, \alpha_2)$, is defined as (see Figure 2)

$$\tilde{A}(\alpha_1, \alpha_2) = = \{((x, \alpha_1 I_{\tilde{A}_{\alpha_1,\alpha_2}}(x)), \alpha_2 I_{\tilde{A}_{\alpha_1,\alpha_2}}(x)) | \forall x \in X\}.$$

2.4. Intersection and union of T2 FSs

Using Zadeh's Extension Principle ([7]), the membership grades for intersection and union of T2 FSs $\tilde{A}$ and $\tilde{B}$ are defined as follows:

$$\mu_{\tilde{A}\cap\tilde{B}}(x) = \mu_{\tilde{A}}(x) \sqcap \mu_{\tilde{B}}(x) = = \int_u \int_v (\mu_{\tilde{A}}(x,u) \star \mu_{\tilde{B}}(x,v))/(u * v), \quad (2)$$

$$\mu_{\tilde{A}\cup\tilde{B}}(x) = \mu_{\tilde{A}}(x) \sqcup \mu_{\tilde{B}}(x) = = \int_u \int_v (\mu_{\tilde{A}}(x,u) \star \mu_{\tilde{B}}(x,v))/(u \circ v), \quad (3)$$

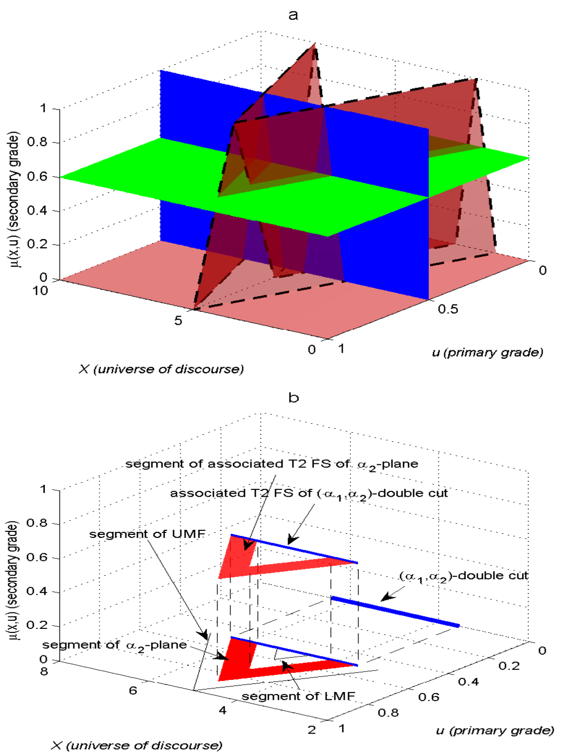

Figure 2: a) T2 FS with triangular secondary membership functions, triangular UMF and LMF. 'Cutting' planes in level $\alpha_1 = 0.5, \alpha_2 = 0.6$. b) (α_1, α_2)-double cut for $\alpha_1 = 0.5, \alpha_2 = 0.6$ and associated T2 FS of the (α_1, α_2)-double cut.

where $*$ and $\star$ represent t-norms (both mostly minimum t-norm), $\circ$ represents a t-conorm (mostly maximum t-conorm) and $\sqcap, \sqcup$ denote *meet* and *join* operations, respectively ([11], [12]). It is not usual to distinguish between the t-norms $*, \star$ in (2), but there is no reason not to do it (see [13], [14]). Note also, that we do not use symbols $\wedge$ and $\vee$ for t-norms and t-conorms, because we will use them hereafter for special cases - for minimum t-norm and maximum t-conorm, respectively.

Let $\tilde{A}, \tilde{B}$ be T2 FSs in X with convex, normal secondary MFs $\mu_{\tilde{A}}(x'), \mu_{\tilde{B}}(x')$ for some $x' \in X$, respectively (recall that $\mu_{\tilde{A}}(x')$ is in fact a function of u). Let $v_1, v_2 \in [0,1], v_1 \le v_2$ and $\mu_{\tilde{A}}(x', v_1) = \mu_{\tilde{B}}(x', v_2) = 1$. Then, for intersection and union under minimum t-norm and maximum t-conorm following holds ([11]),

$$\mu_{\tilde{A}\cap\tilde{B}}(x', u) = \\ = \begin{cases} \mu_{\tilde{A}}(x', u) \vee \mu_{\tilde{B}}(x', u), & u < v_1, \\ \mu_{\tilde{A}}(x', u), & v_1 \le u < v_2, \\ \mu_{\tilde{A}}(x', u) \wedge \mu_{\tilde{B}}(x', u), & u \ge v_2 \end{cases} \quad (4)$$

and

$$\mu_{\tilde{A}\cup\tilde{B}}(x', u) = \\ = \begin{cases} \mu_{\tilde{A}}(x', u) \wedge \mu_{\tilde{B}}(x', u), & u < v_1, \\ \mu_{\tilde{B}}(x', u), & v_1 \le u < v_2, \\ \mu_{\tilde{A}}(x', u) \vee \mu_{\tilde{B}}(x', u), & u \ge v_2, \end{cases} \quad (5)$$

where $\wedge$ denotes minimum and $\vee$ maximum (see Figure 3 and Figure 4).

3. Properties of α-planes

3.1. Inclusion

Following is a well known property of α-cuts of T1 FS A:

$$\text{if } \alpha_1 \le \alpha_2 \text{ then } A_{\alpha_2} \subseteq A_{\alpha_1}, \quad (6)$$

where α_1 and α_2 denote specific values of grade of T1 FS A. It shows that an α-cut for greater α is included in an α-cut for smaller α .

For α-planes of T2 FS $\tilde{A}$ similar property holds:

$$\text{if } \alpha_1 \le \alpha_2 \text{ then } \tilde{A}_{\alpha_2} \subseteq \tilde{A}_{\alpha_1}, \quad (7)$$

where α_1 and α_2 denote specific values of secondary grade of T2 FS $\tilde{A}$. Liu (see [2, p. 11], [3]) states this property of α-planes of T2 FS $\tilde{A}$ without proof, which was not necessary with respect to his work, and which is very simple: if $(x, u) \in \tilde{A}_{\alpha_2}$ for some $x \in X, u \in J_x, \alpha_2 \in [0,1]$, then $\mu_{\tilde{A}}(x, u) \ge \alpha_2$, and for any $\alpha_1 \le \alpha_2$ it follows $\mu_{\tilde{A}}(x, u) \ge \alpha_1$, so $(x, u) \in \tilde{A}_{\alpha_1}$ and consequently $A_{\alpha_2} \subseteq A_{\alpha_1}$.

3.2. Intersection and union

3.2.1. α-cuts of T1 FSs

We recapitulate and remind properties of intersection and union of T1 FSs with respect to α-cuts first. Following are well known properties of α-cuts of T1 FSs A, B:

$$(A \cap B)_\alpha = A_\alpha \cap B_\alpha, \quad (8)$$

$$(A \cup B)_\alpha = A_\alpha \cup B_\alpha, \quad (9)$$

where $\cap$ and $\cup$ on the left sides of equalities denote intersection and union, respectively, of T1 FSs and the same symbols on the right sides of the equalities denote intersection and union, respectively, of crisp sets.

Note that these properties hold only for minimum t-norm and maximum t-conorm within definition of intersection and union of T1 FSs. These standard intersection and union operations are the only cutworthy operations among the t-norms and t-conorms, which means that they are preserved in α-cuts for all $\alpha \in [0,1]$ in the classical sense (see e.g. [1]).

Proposition 1 *Formulas (8) and (9) hold for the standard intersection and union of T1 FSs, i.e., for* $\mu_{A\cap B}(x) = \mu_A(x) \wedge \mu_B(x)$ *and* $\mu_{A\cup B}(x) = \mu_A(x) \vee \mu_B(x)$ *.*

Proposition 2 *Formulas (8) and (9) do not hold for the intersection and union of T1 FSs under t-norm, t-conorm different from minimum t-norm, maximum t-conorm, respectively, i.e., for* $\mu_{A\cap B}(x) = \mu_A(x) * \mu_B(x)$ *and* $\mu_{A\cup B}(x) = \mu_A(x) \circ \mu_B(x)$ *, where t-norm* $*$ *is not minimum and t-conorm* $\circ$ *is not maximum.*

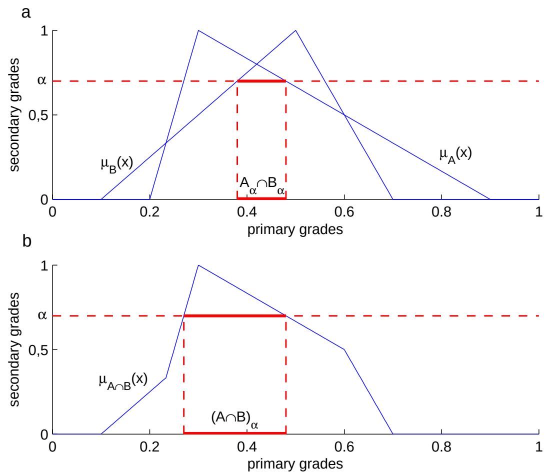

Figure 3: a) Convex, normal secondary membership functions $\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)$ of T2 FSs $\tilde{A}, \tilde{B}$ for some $x \in X$. b) Secondary membership function $\mu_{\tilde{A}\cap\tilde{B}}(x)$ of the intersection of the T2 FSs $\tilde{A}, \tilde{B}$ for the same $x \in X$ as in a).

Proposition 3 *Inclusions $(A \cap B)_\alpha \subseteq A_\alpha \cap B_\alpha$ and $A_\alpha \cup B_\alpha \subseteq (A \cup B)_\alpha$ hold for intersection and union of T1 FSs under each t-norm and t-conorm, respectively, i.e., for $\mu_{A\cap B}(x) = \mu_A(x) * \mu_B(x)$ and $\mu_{A\cup B}(x) = \mu_A(x) \circ \mu_B(x)$.*

Similar properties of intersection and union of T2 FSs with respect to α-planes are discussed in detail in [4]. We summarize the main results of the article without proofs here.

3.2.2. Intersection of α-planes

We will investigate two following inclusions first:

$$\tilde{A}_\alpha \cap \tilde{B}_\alpha \subseteq (\tilde{A} \cap \tilde{B})_\alpha, \tag{10}$$

$$(\tilde{A} \cap \tilde{B})_\alpha \subseteq \tilde{A}_\alpha \cap \tilde{B}_\alpha. \tag{11}$$

Theorem 1 *Let $\tilde{A}$ and $\tilde{B}$ be T2 FSs in a set X, let $*$ and $\star$ are t-norms from the definition (2) of intersection of T2 FSs. Then*

i) *(10) holds, if $*$ and $\star$ are both minimum t-norm,*

ii) *(10) does not hold, if $*$ is a t-norm different from minimum t-norm,*

iii) *(10) does not hold, if $\star$ is a t-norm different from minimum t-norm.*

Theorem 2 *Let $\tilde{A}$ and $\tilde{B}$ be T2 FSs in a set X, let $*$ and $\star$ are t-norms from the definition (2) of intersection of T2 FSs. Then (11) does not hold for any t-norms $*, \star$.*

Using (4) it is easy to see that (11) does not hold under minimum t-norm for convex, normal secondary MFs. Situation is depicted in Figure 3.

Example 1 *Let $\tilde{A}$, $\tilde{B}$ be T2 FSs in $X = \{x_1, x_2, x_3, x_4\}$ given as follows (we will focus just on x_1): $\mu_{\tilde{A}}(x_1) = \{(0.3, 0.2), (0.4, 0.8), (0.5, 0.3)\}$; $\mu_{\tilde{B}}(x_1) = \{(0.4, 0.4), (0.5, 0.6), (0.6, 0.3)\}$. Let $\alpha = 0.5$ and $u = 0.4$. Then $(x_1, u) \in \tilde{A}_\alpha$ and $(x_1, u) \notin \tilde{B}_\alpha$, so consequently $(x_1, u) \notin (\tilde{A}_\alpha \cap \tilde{B}_\alpha)$ and $(x_1, u) \in (\tilde{A}_\alpha \cup \tilde{B}_\alpha)$.*

Using (2) we compute $\tilde{A} \cap \tilde{B}$ (under minimum t-norms), for which $\mu_{\tilde{A}\cap\tilde{B}}(x_1) = \{(0.3, 0.2), (0.4, 0.6), (0.5, 0.3)\}$. Thus, $(x_1, u) \in (\tilde{A} \cap \tilde{B})_\alpha$. Hence, $(\tilde{A} \cap \tilde{B})_\alpha \not\subseteq \tilde{A}_\alpha \cap \tilde{B}_\alpha$ (see the proof of Theorem 1).

Using (3) we compute $\tilde{A} \cup \tilde{B}$, for which $\mu_{\tilde{A}\cup\tilde{B}}(x_1) = \{(0.4, 0.4), (0.5, 0.6), (0.6, 0.3)\}$. Thus, $(x_1, u) \notin (\tilde{A} \cup \tilde{B})_\alpha$. Hence, $\tilde{A}_\alpha \cup \tilde{B}_\alpha \not\subseteq (\tilde{A} \cup \tilde{B})_\alpha$ (see the proof of Theorem 3). □

Example 2 *We show that the restriction to minimum t-norm in the first part of Theorem 1 is reasonable. Let $*$ and $\star$ both denote product t-norm in the definition of intersection of T2 FSs (2). Let $\tilde{A}$ and $\tilde{B}$ be T2 FSs in a set X with constant secondary grades equal to 0.8. Then, for all (x, u) and (x, v), where $x \in X, u \in J_x^{\tilde{A}} \subseteq [0, 1], v \in J_x^{\tilde{B}} \subseteq [0, 1]$, secondary grades satisfy $\mu_{\tilde{A}}(x, u) = \mu_{\tilde{B}}(x, v) = 0.8$. Furthermore, for all (x, w), where $x \in X, w = u \cdot v$, secondary grades satisfy $\mu_{\tilde{A}}(x, w) = 0.8^2 = 0.64$. So, e.g. for $\alpha = 0.7$, $(\tilde{A} \cap \tilde{B})_\alpha$ has to be an empty set, but $\tilde{A}_\alpha \cap \tilde{B}_\alpha$ may be a nonempty set. Hence, (10) does not hold in general under product t-norms.*

Note that this take place also if $*$ denotes product t-norm and $\star$ denotes minimum t-norm. □

3.2.3. Union of α-planes

We will investigate two following inclusions next:

$$(\tilde{A} \cup \tilde{B})_\alpha \subseteq \tilde{A}_\alpha \cup \tilde{B}_\alpha. \tag{12}$$

$$\tilde{A}_\alpha \cup \tilde{B}_\alpha \subseteq (\tilde{A} \cup \tilde{B})_\alpha, \tag{13}$$

Theorem 3 *Let $\tilde{A}$ and $\tilde{B}$ be T2 FSs in a set X, let $*$ and $\circ$ are t-norm and t-conorm, respectively, from the definition (3) of union of T2 FSs. Then*

i) *(12) holds, if $\circ$ is maximum t-conorm (and $*$ is arbitrary t-norm),*

ii) *(12) does not hold, if $\circ$ is a t-conorm different from maximum t-conorm (and $*$ is arbitrary t-norm).*

Theorem 4 *Let $\tilde{A}$ and $\tilde{B}$ be T2 FSs in a set X, let $*$ and $\circ$ are t-norm and t-conorm, respectively, from the definition (3) of union of T2 FSs. Then (13) does not hold for any t-norm $*$ and any t-conorm $\circ$.*

Using (5) it is easy to see that (13) does not hold under minimum t-norm and maximum t-conorm for convex, normal secondary MFs. Situation is depicted in Figure 4.

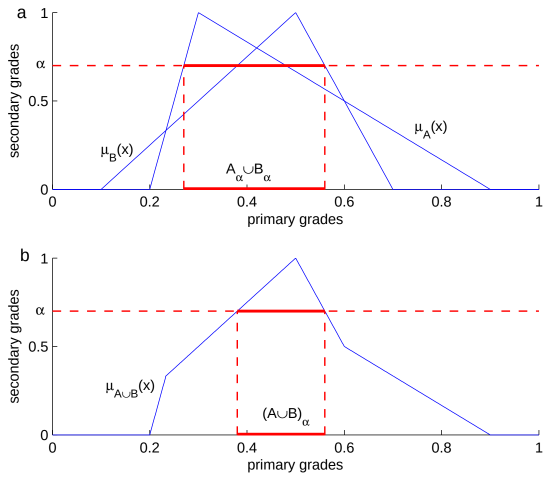

Figure 4: a) Convex, normal secondary membership functions $\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)$ of T2 FSs $\tilde{A}, \tilde{B}$ for some $x \in X$. b) Secondary membership function $\mu_{\tilde{A}\cup\tilde{B}}(x)$ of the union of the T2 FSs $\tilde{A}, \tilde{B}$ for the same $x \in X$ as in a).

4. Properties of (α_1, α_2)-double cuts

4.1. Inclusion

We stated well known property of α-cuts of T1 FSs (6) and then we proved property of α-planes of T2 FSs (7). A similar property holds also for (α_1, α_2)-double cuts of T2 FSs. Lemma 5, Lemma 6 and Theorem 7 show this fact.

Lemma 5 *Let $\alpha_1^1, \alpha_1^2 \in [0,1]$ be two specific values of α_1, and let $\tilde{A}$ be a T2 FS. Then following holds:*

$$\text{if } \alpha_1^1 \leq \alpha_1^2 \text{ then } \tilde{A}_{\alpha_1^2,\alpha_2} \subseteq \tilde{A}_{\alpha_1^1,\alpha_2}.$$

□

Proof. Let x be an arbitrary element of the crisp set $\tilde{A}_{\alpha_1^2,\alpha_2}$. From the definition of $\tilde{A}_{\alpha_1^2,\alpha_2}$ it follows that there exists $u \in J_x \subseteq [0,1]$ such that

$$u \geq \alpha_1^2 \qquad \text{and} \qquad \mu_{\tilde{A}}(x,u) \geq \alpha_2.$$

Because $\alpha_1^1 \leq \alpha_1^2$, there exists $u \in J_x \subseteq [0,1]$ such that

$$u \geq \alpha_1^1 \qquad \text{and} \qquad \mu_{\tilde{A}}(x,u) \geq \alpha_2.$$

Hence, $x \in \tilde{A}_{\alpha_1^1,\alpha_2}$ and finally $\tilde{A}_{\alpha_1^2,\alpha_2} \subseteq \tilde{A}_{\alpha_1^1,\alpha_2}$. q.e.d.

Lemma 6 *Let $\alpha_2^1, \alpha_2^2 \in [0,1]$ be two specific values of α_2, and let $\tilde{A}$ be a T2 FS. Then following holds:*

$$\text{if } \alpha_2^1 \leq \alpha_2^2 \text{ then } \tilde{A}_{\alpha_1,\alpha_2^2} \subseteq \tilde{A}_{\alpha_1,\alpha_2^1}.$$

□

Proof. Proof is similar to the proof of Lemma 5. q.e.d.

Theorem 7 *Let $\alpha_1^1, \alpha_1^2 \in [0,1]$ be two specific values of α_1, let $\alpha_2^1, \alpha_2^2 \in [0,1]$ be two specific values of α_2, and let $\tilde{A}$ be a T2 FS in a set X. Then following holds:*

$$\text{if } \alpha_1^1 \leq \alpha_1^2 \text{ and } \alpha_2^1 \leq \alpha_2^2 \text{ then } \tilde{A}_{\alpha_1^2,\alpha_2^2} \subseteq \tilde{A}_{\alpha_1^1,\alpha_2^1}.$$

□

Proof. Immediately follows from Lemma 5 and Lemma 6. q.e.d.

4.2. Intersection and union of (α_1, α_2)-double cuts

4.2.1. Intersection of (α_1, α_2)-double cuts

In Section 3 we studied inclusions (10) - (13) for α-planes of T2 FSs. Now we focus on (α_1, α_2)-double cuts of T2 FSs and investigate:

$$(\tilde{A} \cap \tilde{B})_{\alpha_1,\alpha_2} \subseteq \tilde{A}_{\alpha_1,\alpha_2} \cap \tilde{B}_{\alpha_1,\alpha_2}, \qquad (14)$$

$$\tilde{A}_{\alpha_1,\alpha_2} \cap \tilde{B}_{\alpha_1,\alpha_2} \subseteq (\tilde{A} \cap \tilde{B})_{\alpha_1,\alpha_2}. \qquad (15)$$

Theorem 8 *Let $\tilde{A}$ and $\tilde{B}$ be T2 FSs in a set X. Then (14) holds.*

Proof. Let x be an arbitrary fixed element of $(\tilde{A} \cap \tilde{B})_{\alpha_1,\alpha_2}$. Using (2) one can deduce that there exists $u \in J_x^{\tilde{A}}, v \in J_x^{\tilde{B}}$ such that $u * v \geq \alpha_1$ and $\mu_{\tilde{A}}(x,u) \star \mu_{\tilde{B}}(x,v) \geq \alpha_2$. From well known property of t-norms ($x, y \geq T(x,y)$ for each $x, y \in [0,1]$ and for each t-norm T), it follows $u \geq \alpha_1, v \geq \alpha_1$ and $\mu_{\tilde{A}}(x,u) \geq \alpha_2, \mu_{\tilde{B}}(x,v) \geq \alpha_2$. Hence, $x \in \tilde{A}_{\alpha_1,\alpha_2}$ and $x \in \tilde{B}_{\alpha_1,\alpha_2}$, so $x \in \tilde{A}_{\alpha_1,\alpha_2} \cap \tilde{B}_{\alpha_1,\alpha_2}$. Finally $(\tilde{A} \cap \tilde{B})_{\alpha_1,\alpha_2} \subseteq \tilde{A}_{\alpha_1,\alpha_2} \cap \tilde{B}_{\alpha_1,\alpha_2}$. q.e.d.

Thus, formula (14) holds for intersection of T2 FSs under each t-norm $\star$ and $*$. Remind that similar formula (11) for α-planes does not hold under any t-norms.

Theorem 9 *Let $\tilde{A}$ and $\tilde{B}$ be T2 FSs in a set X, let $*$ and $\star$ are t-norms from the definition (2) of intersection of T2 FSs. Then*

i) *(15) holds, if $*$ and $\star$ are both minimum t-norm,*

ii) *(15) does not hold, if $*$ is a t-norm different from minimum t-norm,*

iii) *(15) does not hold, if $\star$ is a t-norm different from minimum t-norm.*

Proof. i) We prove that formula (15) holds, if $*$ and $\star$ both denote minimum t-norm. Let x be an arbitrary fixed element of $\tilde{A}_{\alpha_1,\alpha_2} \cap \tilde{B}_{\alpha_1,\alpha_2}$, i.e., $x \in \tilde{A}_{\alpha_1,\alpha_2}$ and $x \in \tilde{B}_{\alpha_1,\alpha_2}$. Then, according to (2), there exists $u \in J_x^{\tilde{A}} \subseteq [0,1]$ such that $u \geq \alpha_1$ and $\mu_{\tilde{A}}(x,u) \geq \alpha_2$, and there exists $v \in J_x^{\tilde{B}} \subseteq [0,1]$ such that $v \geq \alpha_1$ and $\mu_{\tilde{B}}(x,v) \geq \alpha_2$. Because $*$ and $\star$ denote minimum t-norms, it follows $u * v \geq \alpha_1$ and $\mu_{\tilde{A}}(x,u) \star \mu_{\tilde{B}}(x,v) \geq \alpha_2$, and consequently $x \in (\tilde{A} \cap \tilde{B})_{\alpha_1,\alpha_2}$. Finally $\tilde{A}_{\alpha_1,\alpha_2} \cap \tilde{B}_{\alpha_1,\alpha_2} \subseteq (\tilde{A} \cap \tilde{B})_{\alpha_1,\alpha_2}$.

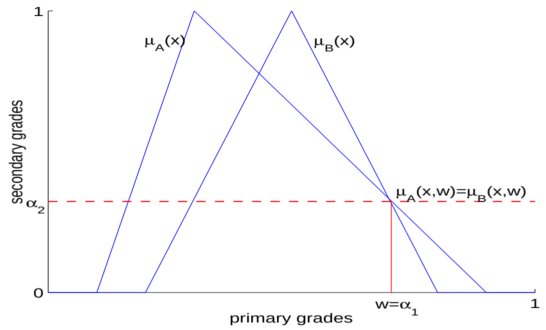

Figure 5: Secondary membership functions $\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)$ of T2 FSs $\tilde{A}, \tilde{B}$ for some $x \in X$.

ii) We prove failure of (15), if $*$ is a t-norm different from minimum t-norm by counterexample. Remind that minimum t-norm is the only t-norm, whose set of idempotents is equal to $[0, 1]$, i.e., it is the only t-norm, which has not any non-idempotents (e.g. [15]). Thus, for $*$ there exists at least one non-idempotent in $(0, 1)$.

Let $w \in (0, 1)$ be a non-idempotent of $*$ and let $\tilde{A}, \tilde{B}$ be T2 FSs in X, whose secondary membership functions $\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)$, respectively, for some $x \in X$ satisfy following conditions: let $\mu_{\tilde{A}}(x, w) = \mu_{\tilde{B}}(x, w)$ and let $\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)$ are decreasing in $[w, 1] \cap J_x^{\tilde{A}}$, $[w, 1] \cap J_x^{\tilde{B}}$, respectively (see Figure 5). Let $\alpha_1 = w$ and $\alpha_2 = \mu_{\tilde{A}}(x, w) = \mu_{\tilde{B}}(x, w)$. Then obviously $x \in \tilde{A}_{\alpha_1,\alpha_2}$ and $x \in \tilde{B}_{\alpha_1,\alpha_2}$, so $x \in \tilde{A}_{\alpha_1,\alpha_2} \cap \tilde{B}_{\alpha_1,\alpha_2}$. Now we show that $x \notin (\tilde{A} \cap \tilde{B})_{\alpha_1,\alpha_2}$. Let $u * v \geq w = \alpha_1$ for some $u, v \in [0, 1]$. Then $u > w$ or $v > w$ (w is non-idempotent). Let the former is true. Because $\mu_{\tilde{A}}(x)$ is decreasing in $[w, u]$, it follows $\mu_{\tilde{A}}(x, u) < \mu_{\tilde{A}}(x, w)$ and consequently $\mu_{\tilde{A}}(x, u) \star \mu_{\tilde{B}}(x, v) < \mu_{\tilde{A}}(x, w) = \alpha_2$. Finally, $x \notin (\tilde{A} \cap \tilde{B})_{\alpha_1,\alpha_2}$.

iii) We prove failure of (15), if $\star$ is a t-norm different from minimum t-norm by counterexample. Let $\tilde{A}, \tilde{B}$ be T2 FSs in X, whose secondary membership functions $\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)$, respectively, for some $x \in X$ satisfy following conditions: let $\mu_{\tilde{A}}(x, w) = \mu_{\tilde{B}}(x, w)$ be a non-idempotent of $\star$ for some $w \in (0, 1)$ and let $\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)$ are both decreasing in $[w, 1] \cap J_x^{\tilde{A}}$, $[w, 1] \cap J_x^{\tilde{B}}$, respectively (see Figure 5). Let $\alpha_1 = w$ and $\alpha_2 = \mu_{\tilde{A}}(x, w) = \mu_{\tilde{B}}(x, w)$. Then obviously $x \in \tilde{A}_{\alpha_1,\alpha_2} \cap \tilde{B}_{\alpha_1,\alpha_2}$ - see item ii of this proof. Now we show that $x \notin (\tilde{A} \cap \tilde{B})_{\alpha_1,\alpha_2}$. Let $u * v \geq w = \alpha_1$ for some $u, v \in [0, 1]$. Then there are two options:

1. $u > w$ or $v > w$ (or both) - it is easy to see that in this case following holds $\mu_{\tilde{A}}(x, u) \star \mu_{\tilde{B}}(x, v) < \mu_{\tilde{A}}(x, w) = \alpha_2$ (see item ii of this proof).

2. $u = v = w$ - from $\mu_{\tilde{A}}(x, u) = \mu_{\tilde{B}}(x, v) = \alpha_2$ and fact that α_2 is a non-idempotent it follows $\mu_{\tilde{A}}(x, u) \star \mu_{\tilde{B}}(x, v) < \alpha_2$.

Thus, in both cases $x \notin (\tilde{A} \cap \tilde{B})_{\alpha_1,\alpha_2}$. q.e.d.

Example 3 *We demonstrate failure of (15) under product t-norm by counterexample.*

Let $\tilde{A}, \tilde{B}$ be T2 FSs in $X = \{x_1, x_2, x_3, x_4\}$ given as follows (we will focus just on x_3): $\mu_{\tilde{A}}(x_3) = \{(0.5, 0.9), (0.6, 0.8), (0.8, 0.6)\}$; $\mu_{\tilde{B}}(x_3) = \{(0.7, 0.7), (0.8, 0.4)\}$. Let $\alpha_1 = 0.6$ and $\alpha_2 = 0.7$. Then $x_3 \in \tilde{A}_{\alpha_1,\alpha_2}$ and $x_3 \in \tilde{B}_{\alpha_1,\alpha_2}$, and consequently $x_3 \in (\tilde{A}_{\alpha_1,\alpha_2} \cap \tilde{B}_{\alpha_1,\alpha_2})$. We compute $\tilde{A} \cap \tilde{B}$ under different t-norms:

1. $*$ and $\star$ denote minimum t-norms:
$\mu_{\tilde{A}\cap\tilde{B}}(x_3) = \{(0.5, 0.7), (0.5, 0.4), (0.6, 0.7), (0.6, 0.4), (0.7, 0.6), (0.8, 0.4)\} = \{(0.5, 0.7), (0.6, 0.7), (0.7, 0.6), (0.8, 0.4)\}$,

2. $*$ and $\star$ denote product t-norms:
$\mu_{\tilde{A}\cap\tilde{B}}(x_3) = \{(0.35, 0.63), (0.4, 0.36), (0.42, 0.56), (0.48, 0.32), (0.56, 0.42), (0.64, 0.24)\}$,

3. $\star$ denotes product t-norm and $*$ denotes minimum t-norm:
$\mu_{\tilde{A}\cap\tilde{B}}(x_3) = \{(0.5, 0.63), (0.5, 0.36), (0.6, 0.56), (0.6, 0.32), (0.7, 0.42), (0.8, 0.24)\} = \{(0.5, 0.63), (0.6, 0.56), (0.7, 0.42), (0.8, 0.24)\}$.

Thus, in case 1, under minimum t-norms, $x_3 \in (\tilde{A} \cap \tilde{B})_{\alpha_1,\alpha_2}$, which corresponds to the Theorem 9. On the other hand, in case 2, under product t-norms, $x_3 \notin (\tilde{A} \cap \tilde{B})_{\alpha_1,\alpha_2}$, which shows that the restriction in the theorem is reasonable. Similarly, in case 3, under product t-norm $\star$ and minimum t-norm $*$, $x_3 \notin (\tilde{A} \cap \tilde{B})_{\alpha_1,\alpha_2}$. □

Theorems 8 and 9 lead to an obvious corollary - equality between intersection of (α_1, α_2)-double cuts of T2 FSs and (α_1, α_2)-double cuts of intersection of T2 FSs holds (just) under minimum t-norms $*$ and $\star$.

Corollary 10 *Let $\tilde{A}$ and $\tilde{B}$ be T2 FSs in a set X, let $*$ and $\star$ denote minimum t-norms (in the definition (2) of intersection of T2 FSs). Then following holds:*

$$\tilde{A}_{\alpha_1,\alpha_2} \cap \tilde{B}_{\alpha_1,\alpha_2} = (\tilde{A} \cap \tilde{B})_{\alpha_1,\alpha_2}.$$

Proof: Immediately follows from Theorem 8 and Theorem 9. q.e.d.

4.2.2. Union of (α_1, α_2)-double cuts

Now we are going to investigate following inclusions:

$$(\tilde{A} \cup \tilde{B})_{\alpha_1,\alpha_2} \subseteq \tilde{A}_{\alpha_1,\alpha_2} \cup \tilde{B}_{\alpha_1,\alpha_2}, \qquad (16)$$

$$\tilde{A}_{\alpha_1,\alpha_2} \cup \tilde{B}_{\alpha_1,\alpha_2} \subseteq (\tilde{A} \cup \tilde{B})_{\alpha_1,\alpha_2}. \qquad (17)$$

Theorem 11 *Let $\tilde{A}$ and $\tilde{B}$ be T2 FSs in a set X, let $\star$ is t-norm and $\circ$ is t-conorm from the definition (3) of union of T2 FSs. Then*

i) *(16) holds, if $\circ$ is maximum t-conorm (and $\star$ is arbitrary t-norm),*

ii) *(16) does not hold, if $\circ$ is a t-conorm different from maximum t-conorm.*

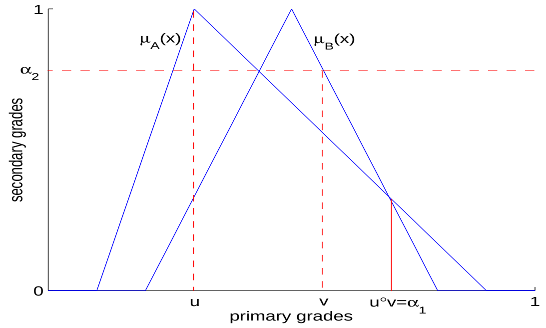

Figure 6: Secondary membership functions $\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)$ of T2 FSs $\tilde{A}, \tilde{B}$ for some $x \in X$.

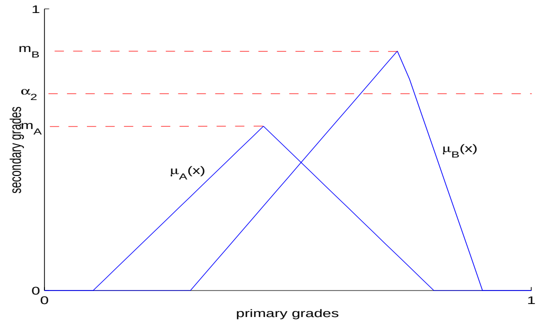

Figure 7: Secondary membership functions $\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)$ of T2 FSs $\tilde{A}, \tilde{B}$ for some $x \in X$.

Proof. i) We prove that formula (16) holds, if $\circ$ denotes maximum t-norm. Let x be an arbitrary fixed element of $(\tilde{A} \cup \tilde{B})_{\alpha_1,\alpha_2}$. Then, according to (3), there exists $u \in J_x^{\tilde{A}}$ and $v \in J_x^{\tilde{B}}$ such that $u \circ v \geq \alpha_1$ and $\mu_{\tilde{A}}(x,u) \star \mu_{\tilde{B}}(x,v) \geq \alpha_2$. Then $u = \alpha_1$ or $v = \alpha_1$ (remind that $\circ$ is maximum). Furthermore $\mu_{\tilde{A}}(x,u) \geq \alpha_2$ and $\mu_{\tilde{B}}(x,v) \geq \alpha_2$. Hence, $x \in \tilde{A}_{\alpha_1,\alpha_2}$ or $x \in \tilde{B}_{\alpha_1,\alpha_2}$ and finally $x \in \tilde{A}_{\alpha_1,\alpha_2} \cup \tilde{B}_{\alpha_1,\alpha_2}$.

ii) We prove failure of (16), if $\circ$ is a t-conorm different from maximum t-conorm by counterexample. Because of maximum is the pointwise smallest t-conorm, there exist $u, v \in (0,1)$ such that $u \circ v > u \wedge v$. Let $\tilde{A}, \tilde{B}$ be T2 FSs in X, whose secondary membership functions $\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)$, respectively, for some $x \in X$ satisfy following conditions: let $\mu_{\tilde{A}}(x, u \circ v) = \mu_{\tilde{B}}(x, u \circ v)$, let $\mu_{\tilde{A}}(x,u) = 1$, $\mu_{\tilde{B}}(x,v) > \mu_{\tilde{B}}(x, u \circ v)$ and $\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)$ are both decreasing in $[u,1] \cap J_x^{\tilde{A}}, [v,1] \cap J_x^{\tilde{B}}$, respectively (see Figure 6). Let $\alpha_1 = u \circ v$ and $\alpha_2 = \mu_{\tilde{B}}(x,v)$. Then obviously $x \notin \tilde{A}_{\alpha_1,\alpha_2}$ and $x \notin \tilde{B}_{\alpha_1,\alpha_2}$, so $x \notin \tilde{A}_{\alpha_1,\alpha_2} \cup \tilde{B}_{\alpha_1,\alpha_2}$. Furthermore, from $u \circ v = \alpha_1$ and $\mu_{\tilde{A}}(x,u) \star \mu_{\tilde{B}}(x,v) = 1 \star \mu_{\tilde{B}}(x,v) = \mu_{\tilde{B}}(x,v) = \alpha_2$ it follows $x \in (\tilde{A} \cup \tilde{B})_{\alpha_1,\alpha_2}$. q.e.d.

In the next theorem we can see that truth of (17) does not depend on t-norm and t-conorm. It depends on normality or subnormality of secondary membership functions of T2 FSs.

Theorem 12 *Let $\tilde{A}$ and $\tilde{B}$ be T2 FSs in a set X. Then*

i) *(17) holds, if secondary membership functions of $\tilde{A}$ and $\tilde{B}$ are normal for all $x \in X$,*

ii) *(17) does not hold, if secondary membership function of $\tilde{A}$ or $\tilde{B}$ is not normal for some $x \in X$.*

Proof. i) Let x be an arbitrary fixed element of $\tilde{A}_{\alpha_1,\alpha_2} \cup \tilde{B}_{\alpha_1,\alpha_2}$. Hence, $x \in \tilde{A}_{\alpha_1,\alpha_2}$ or $x \in \tilde{A}_{\alpha_1,\alpha_2}$. Let the former is true. Then there exists $u \in J_x^{\tilde{A}}$ such that $u \geq \alpha_1$ and $\mu_{\tilde{A}}(x,u) \geq \alpha_2$. Furthermore, because secondary membership function is normal, there exists $v \in [0,1]$ such that $\mu_{\tilde{B}}(x,v) = 1$. Thus, $u \circ v \geq \alpha_1$ (for each t-conorm $\circ$) and $\mu_{\tilde{A}}(x,u) \star \mu_{\tilde{B}}(x,v) = \mu_{\tilde{A}}(x,u) \star 1 = \mu_{\tilde{A}}(x,u) \geq \alpha_2$, from which follows that $x \in (\tilde{A} \cup \tilde{B})_{\alpha_1,\alpha_2}$.

ii) Let there exists $x \in X$ such that secondary membership function $\mu_{\tilde{A}}(x)$ is not normal, i.e., $\max\{\mu_{\tilde{A}}(x,u) | u \in J_x^{\tilde{A}}\} = m_{\tilde{A}} < 1$. Let $\max\{\mu_{\tilde{B}}(x,v) | v \in J_x^{\tilde{B}}\} = m_{\tilde{B}} > m_{\tilde{A}}$. Let $\alpha_2 \in (m_{\tilde{A}}, m_{\tilde{B}})$ and let $\alpha_1 = 0$ (see Figure 7). Then obviously $x \in \tilde{B}_{\alpha_1,\alpha_2}$ and consequently $x \in \tilde{A}_{\alpha_1,\alpha_2} \cup \tilde{B}_{\alpha_1,\alpha_2}$. Furthermore, $\mu_{\tilde{A}}(x,u) \star \mu_{\tilde{B}}(x,v) \leq m_{\tilde{A}} < \alpha_2$ for all $u, v \in [0,1]$, so $\mu_{\tilde{A} \cup \tilde{B}}(x,w) < \alpha_2$ for all $w \in [0,1]$ and finally $x \notin (\tilde{A} \cup \tilde{B})_{\alpha_1,\alpha_2}$. q.e.d.

Example 4 *Let $\tilde{A}, \tilde{B}$ be T2 FSs in $X = \{x_1, x_2, x_3, x_4\}$ given as follows (we will focus just on x_4): $\mu_{\tilde{A}}(x_4) = \{(0.5, 0.9), (0.8, 0.6)\}$; $\mu_{\tilde{B}}(x_4) = \{(0.7, 0.4), (0.8, 0.4)\}$. Let $\alpha_1 = 0.6$ and $\alpha_2 = 0.5$. Then $x_4 \in \tilde{A}_{\alpha_1,\alpha_2}$ and $x_4 \notin \tilde{B}_{\alpha_1,\alpha_2}$, and consequently $x_4 \in \tilde{A}_{\alpha_1,\alpha_2} \cup \tilde{B}_{\alpha_1,\alpha_2}$. We compute $\tilde{A} \cup \tilde{B}$ under maximum t-conorm and two different t-norms (see (2)):*

1. min t-norm: $\mu_{\tilde{A} \cup \tilde{B}}(x_4) = \{(0.7, 0.4), (0.8, 0.4), (0.8, 0.4), (0.8, 0.4)\} = \{(0.7, 0.4), (0.8, 0.4)\}$,

2. product t-norm: $\mu_{\tilde{A} \cap \tilde{B}}(x_4) = \{(0.7, 0.36), (0.8, 0.36), (0.8, 0.24), (0.8, 0.24)\} = \{(0.7, 0.36), (0.8, 0.36)\}$.

Thus, in both cases, under min *t-norm and under product t-norm, $x_4 \notin (\tilde{A} \cup \tilde{B})_{\alpha_1,\alpha_2}$, which shows that $\tilde{A}_{\alpha_1,\alpha_2} \cup \tilde{B}_{\alpha_1,\alpha_2} \subseteq (\tilde{A} \cup \tilde{B})_{\alpha_1,\alpha_2}$ does not hold in general.* □

Theorems 11 and 12 lead to an obvious corollary - equality between union of (α_1, α_2)-double cuts of T2 FSs and (α_1, α_2)-double cuts of union of T2 FSs holds (just) for normal secondary MFs under maximum t-conorm $\circ$ (and arbitrary t-norm $\star$).

Corollary 13 *Let $\tilde{A}$ and $\tilde{B}$ be T2 FSs in a set X with normal secondary MFs, let $\circ$ denotes maximum t-conorm (in the definition (3) of union of T2 FSs). Then following holds:*

$$\tilde{A}_{\alpha_1,\alpha_2} \cup \tilde{B}_{\alpha_1,\alpha_2} = (\tilde{A} \cup \tilde{B})_{\alpha_1,\alpha_2}.$$

Proof: Immediately follows from Theorem 11 and Theorem 12. q.e.d.

Table 1: Summary of obtained results

Property	α-cut	α-plane	double cut
$(A \cap B)_\alpha \subseteq A_\alpha \cap B_\alpha$	T	—	$\star = T$ $* = T$
$A_\alpha \cap B_\alpha \subseteq (A \cap B)_\alpha$	$\wedge$	$\star = \wedge$ $* = \wedge$	$\star = \wedge$ $* = \wedge$
$(A \cap B)_\alpha = A_\alpha \cap B_\alpha$	$\wedge$	—	$\star = \wedge$ $* = \wedge$
$(A \cup B)_\alpha \subseteq A_\alpha \cup B_\alpha$	$\vee$	$\star = T$ $\circ = \vee$	$\star = T$ $\circ = \vee$
$A_\alpha \cup B_\alpha \subseteq (A \cup B)_\alpha$	S	—	$\star = T$ $\circ = \vee$ normal
$(A \cup B)_\alpha = A_\alpha \cup B_\alpha$	$\vee$	—	$\star = T$ $\circ = \vee$ normal

5. Conclusion

It is known, that the standard intersection and union of T1 FSs are (the only) cutworthy operations for T1 FSs (e.g. [1]). In [4] we showed that α-planes do not meet this useful property for T2 FSs. The reason is that computation of intersection (union) of type-2 FSs requires an interval arithmetic, while intersection (union) of α-planes is just an intersection (union) of crisp sets. Thus, we studied another kind of cutting of T2 FSs, so-called (α_1, α_2)-double cuts, and showed that intersection and union of T2 FSs are preserved in these double cuts. The existential quantifier in the definition of (α_1, α_2)-double cut is significant for this fact.

The results are summarized in Table 1. We see that for intersection as well as for union, one inclusion holds for both α-planes and (α_1, α_2)-double cuts, but inverse inclusion holds only for (α_1, α_2)-double cuts. Thus, for (α_1, α_2)-double cuts hold similar equalities (see Corollary 10 and Corollary 13) as for α-cuts of T1 FSs, though these do not hold for α-planes.

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