Concave Measures and the Fuzzy Core of Exchange Economies with Heterogeneous Divisible Commodities

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Abstract

The main purpose of this paper is to prove the existence of the fuzzy core of an exchange economy with a heterogeneous divisible commodity in which preferences of individuals are given by nonadditive utility functions defined on a σ-algebra of admissible pieces of the total endowment of the commodity. The problem is formulated as the partitioning of a measurable space among finitely many individuals. Applying the Yosida–Hewitt decomposition theorem, we also demonstrate that partitions in the fuzzy core are supportable by prices in L^1.

Keywords: Nonatomic vector measure; Concave measure; Fuzzy coalition; Fuzzy core; Supporting price; Yosida–Hewitt decomposition.

1. Introduction

Cooperative fuzzy games proposed by [2, 3] allow for partial participation of individuals in coalitions. In defining the fuzzy core of exchange economies with homogeneous divisible commodities, individuals contribute only some portions of their initial endowments to coalitions they belong to. That is, a fuzzy coalition unlike the classical (crisp) coalitions, does not necessarily require its participants to contribute the whole of their initial endowments. A remarkable result for exchange economies established by [2] states that under the standard assumptions of continuous, convex preferences the fuzzy core and the set of Walrasian allocations coincide (see also [12, 14].)

In this paper we study the fuzzy core of an exchange economy with a heterogeneous divisible commodity in which preferences of individuals are given by set functions defined in a σ-algebra of admissible pieces of the total endowment of the commodity. Following the traditions of fair division literature along the lines of [9], a heterogeneous divisible commodity is modeled as a nonatomic finite measure space. The total endowment of the heterogeneous commodity is metaphorically called a "cake" in this literature and the problem of fair division consists in partitioning the cake among a finite number of individuals according some criteria of fairness and efficiency.

A common assumption in the theory of fair division is that the preferences of each individual are represented by a nonatomic probability measure. Under this additive utility hypothesis, Lyapunov's convexity theorem (see [19]) guarantees the convexity and compactness of the utility possibility set, crucial to establishing the existence and characterization of various solutions. However, here we assume preferences to be represented by nonadditive utility functions; hence the utility possibility set does not necessarily possess these properties. The utility functions assumed here contain concave measures introduced by [20, 21, 22].

[15, 16, 23] proved the existence of the core with nonadditive evaluations for exchange economies with heterogeneous divisible commodity under diverse assumptions. The corresponding result for the case of the fuzzy core is not straightforward; an adequate notion of fuzzy improvement must reflect awareness of agents of the bounds of the available heterogeneous divisible commodity. We propose here a such notion of the fuzzy improvement and the fuzzy core.

The organization of the paper is as follows: In Section 2 we present a representation result for concave measures, stating that an arbitrary concave measure can be represented as a composition of a concave function and a finite-dimensional nonatomic vector measure. From this characterization we derive the continuity of concave measures at the measurable set whose vector measure lies in the interior of Lyapunov's set. We also provide a core representation theorem for nonatomic vector measure games along the lines of [11].

Section 3 is devoted to the formulation of the fuzzy coalitions and fuzzy core. To this end, we focus our attention on the case where the single heterogeneous divisible commodity possesses a finite number of attributes, which can be evaluated objectively in terms of finite-dimensional nonatomic vector measures. We define the notions of the fuzzy coalitions and fuzzy core in an exchange economy with a heterogeneous divisible commodity, where each individual has a utility function represented by a nonadditive set function.

The main result of this paper, Theorem 4.1 on the existence of the fuzzy core, is stated in Section

4. To prove this theorem, we extend the commodity space from the set of measurable sets to the set of measurable functions taking values in the unit interval along the lines of [1, 7, 10]. If f in L^∞ is a characteristic function of a measurable set A, then an individual possessing f is fully entitled to set A and to nothing else. Thus, we can treat allocations in the extended economy with an L^∞-commodity space, which can be embedded into the framework of [5, 6]. We prove the existence of the fuzzy core of this extended economy by constructing a non-transferrable utility (NTU) game and showing that it satisfies the assumptions of Scarf's core existence theorem (see [24]). Exploiting a technique from [18], the existence of the fuzzy core in the original exchange economy follows from the observation that the extreme points in the fuzzy core of the extended economy are indeed measurable partitions.

Section 5 deals with the supportability of efficient partitions by prices. The argument is based on the effective use of the separation theorem under the convexity assumption. We demonstrate that partitions in the fuzzy core are supportable by prices in L^1, applying the Yosida–Hewitt decomposition theorem (see [25]), which is by now a standard method, having been established by [5, 6].

2. Representation and Continuity of Concave Measures

2.1. Representation of Concave Measures

Let $(\Omega, \mathscr{F})$ be a measurable space with a σ-algebra $\mathscr{F}$ of subsets of a nonempty set Ω. A measure μ on $\mathscr{F}$ is *nonatomic* if, for every $A \in \mathscr{F}$ with $\mu(A) > 0$, there exists some $E \in \mathscr{F}$ such that $0 < \mu(E) < \mu(A)$. For nonatomic finite measures $\mu_1, \dots, \mu_m$, we denote by $\vec{\mu} = (\mu_1, \dots, \mu_m)$ an $\mathbb{R}^m$-valued vector measure. Lyapunov's convexity theorem asserts that the range $\mathscr{R}(\vec{\mu})$ of $\vec{\mu}$ is a compact and convex set in $\mathbb{R}^m$ (see [19]).

For an arbitrarily given $A \in \mathscr{F}$ and $t \in [0,1]$, we define the family $\mathscr{K}_t^{\vec{\mu}}(A)$ of measurable subsets of A by:

$$\mathscr{K}_t^{\vec{\mu}}(A) = \{E \in \mathscr{F} \mid E \subset A \text{ and } \vec{\mu}(E) = t\vec{\mu}(A)\}.$$

By Lyapunov's convexity theorem, $\mathscr{K}_t^{\vec{\mu}}(A)$ is nonempty for every $A \in \mathscr{F}$ and $t \in [0,1]$. Furthermore, for an arbitrarily given $A, B \in \mathscr{F}$ and $t \in [0,1]$, we denote by $\mathscr{K}_t^{\vec{\mu}}(A,B)$ the family of sets $C \in \mathscr{F}$, which are written as the union of some disjoint sets $E \in \mathscr{K}_t^{\vec{\mu}}(A)$ and $F \in \mathscr{K}_{1-t}^{\vec{\mu}}(B)$. For a nonatomic scalar measure μ, we use $\mathscr{K}_t^{\mu}(A,B)$. It is evident that $C \in \mathscr{K}_t^{\vec{\mu}}(A,B)$ if and only if $C \in \mathscr{K}_t^{\mu_k}(A,B)$ for each $k = 1, \dots, m$, and hence, $\mathscr{K}_t^{\vec{\mu}}(A,B) = \bigcap_{k=1}^m \mathscr{K}_t^{\mu_k}(A,B)$ for every $A, B \in \mathscr{F}$ and $t \in [0,1]$. It can be shown that $\mathscr{K}_t^{\vec{\mu}}(A,B)$ is nonempty for every $A, B \in \mathscr{F}$ and $t \in [0,1]$ (see [22]).

The notion of concave measures on σ-algebras presented in the following definition bears an obvious resemblance to that of concave functions on real vector spaces. We extend here the definition of [20, 21, 22] to the vector measure case.

Definition 2.1. A set function $\nu : \mathscr{F} \to \mathbb{R}$ is a *concave measure* if $\nu(\emptyset) = 0$ and there exists a finite-dimensional nonatomic vector measure $\vec{\mu}$ such that for every $A, B \in \mathscr{F}$ and $t \in [0,1]$, we have:

$$t\nu(A) + (1-t)\nu(B) \leqslant \nu(C) \quad \forall C \in \mathscr{K}_t^{\vec{\mu}}(A,B).$$

When the underlying vector measure is $\vec{\mu}$ for the concave measure ν, we say that ν is *$\vec{\mu}$-concave.*

The following result presents a useful representation of concave measures in terms of nonatomic vector measure games along the lines of [4, 11].

Theorem 2.1. *A set function $\nu : \mathscr{F} \to \mathbb{R}$ is a concave measure if and only if there exist a finite-dimensional nonatomic vector measure $\vec{\mu}$ and a concave function $\varphi : \mathscr{R}(\vec{\mu}) \to \mathbb{R}$ with $\varphi(0) = 0$ such that $\nu = \varphi \circ \vec{\mu}$.*

Recall that a set function $\nu : \mathscr{F} \to \mathbb{R}$ is *submodular* if $\nu(A \cup B) + \nu(A \cap B) \leqslant \nu(A) + \nu(B)$ for every $A, B \in \mathscr{F}$. The next example is due to [23].

Example 2.1. Let μ be a nonatomic scalar measure. Define the set function $\nu_\varphi : \mathscr{F} \to \mathbb{R}$ for a continuous function $\varphi : \mathscr{R}(\mu) \to \mathbb{R}$ with $\varphi(0) = 0$ by $\nu_\varphi = \varphi \circ \mu$. The following conditions are equivalent:

(i) φ is concave;
(ii) ν_φ is μ-concave;
(iii) ν_φ is submodular.

2.2. Continuity of Concave Measures

Sets A and B in $\mathscr{F}$ are *$\vec{\mu}$-equivalent* if $\vec{\mu}(A \triangle B) = 0$, where $A \triangle B = (A \cup B) \setminus (A \cap B)$ is the symmetric difference of A and B. The $\vec{\mu}$-equivalence defines an equivalence relation (reflexive, symmetric, transitive binary relation) on $\mathscr{F}$. We denote the $\vec{\mu}$-equivalence class of $A \in \mathscr{F}$ by $[A]$ and denote the set of $\vec{\mu}$-equivalence classes in $\mathscr{F}$ by $\mathscr{F}_{\vec{\mu}}$. Define the metric $d_{\vec{\mu}}$ on $\mathscr{F}_{\vec{\mu}}$ by $d_{\vec{\mu}}([A],[B]) = \|\vec{\mu}(A \triangle B)\|$, where $\|\cdot\|$ is the Euclidean norm of $\mathbb{R}^m$. If $\mathscr{F}$ is countably generated, then the metric space $(\mathscr{F}_{\vec{\mu}}, d_{\vec{\mu}})$ is complete and separable (see [8, Lemma III.7.1]; [13, Theorem 40.B]).

Continuous functions on $(\mathscr{F}_{\vec{\mu}}, d_{\vec{\mu}})$ arise in a natural way from the set functions on $\mathscr{F}$. The following definition is a straightforward generalization of [20, 21] to the vector measure case.

Definition 2.2. A set function $\nu : \mathscr{F} \to \mathbb{R}$ is *$\vec{\mu}$-continuous* at $A \in \mathscr{F}$ if for every $\varepsilon > 0$ there exists some $\delta > 0$ such that $\|\vec{\mu}(A \triangle B)\| < \delta$ implies $|\nu(A) - \nu(B)| < \varepsilon$. When ν is $\vec{\mu}$-continuous at every element of $\mathscr{F}$, we say that ν is $\vec{\mu}$-continuous.

We denote by $\operatorname{int}\mathscr{R}(\vec{\mu})$ and $\operatorname{bd}\mathscr{R}(\vec{\mu})$ the interior of $\mathscr{R}(\vec{\mu})$ and the boundary of $\mathscr{R}(\vec{\mu})$, respectively.

Corollary 2.1. *Every concave measure is $\vec{\mu}$-continuous at every $A \in \mathscr{F}$ with $\vec{\mu}(A) \in \operatorname{int}\mathscr{R}(\vec{\mu})$ for some finite-dimensional nonatomic vector measure $\vec{\mu}$.*

We denote by $ba(\Omega, \mathscr{F})$ the space of bounded, finitely additive, signed measures on $\mathscr{F}$. A set function $\nu : \mathscr{F} \to \mathbb{R}$ is a *game* if $\nu(\emptyset) = 0$. A *feasible payoff* of a game ν is an element μ in $ba(\Omega, \mathscr{F})$ satisfying $\mu(\Omega) = \nu(\Omega)$. The *core* of a game ν is defined by:

$$\mathscr{C}(\nu) = \{\mu \in ba(\Omega, \mathscr{F}) \mid \nu \leqslant \mu \text{ and } \mu(\Omega) = \nu(\Omega)\},$$

that is, the core is the set of feasible payoffs upon which no coalition can improve.

Recall that a *supergradient* of a concave function $\varphi : \mathscr{R}(\vec{\mu}) \to \mathbb{R}$ at $x \in \mathscr{R}(\vec{\mu})$ is a vector $p \in \mathbb{R}^m$ satisfying $\varphi(y) - \varphi(x) \leqslant \langle p, y - x\rangle$ for every $y \in \mathscr{R}(\vec{\mu})$, where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^m$. The *superdifferential* $\partial\varphi(x)$ of φ at x is the set of supergradients of φ at x.

Theorem 2.2. *If $\nu : \mathscr{F} \to \mathbb{R}$ is a $\vec{\mu}$-concave measure that is $\vec{\mu}$-continuous at Ω, then there exists a concave function $\varphi : \mathscr{R}(\vec{\mu}) \to \mathbb{R}$ with $\varphi(0) = 0$ such that:*

$$\mathscr{C}(\nu) = \left\{ \langle p, \vec{\mu}\rangle \in ba(\Omega, \mathscr{F}) \;\middle|\; \begin{array}{l} p \in \partial\varphi(\vec{\mu}(\Omega)) \\ \langle p, \vec{\mu}(\Omega)\rangle = \varphi(\vec{\mu}(\Omega)) \end{array} \right\}.$$

Theorem 2.2 involves a "core representation" result for $\vec{\mu}$-concave measures. Indeed, the core of a $\vec{\mu}$-concave measure $\nu = \varphi \circ \vec{\mu}$ that is $\vec{\mu}$-continuous at Ω can be characterized by the local behavior of the superdifferential of φ at $\vec{\mu}(\Omega)$. A similar characterization of the core of a game with the form $\nu = \varphi \circ \vec{\mu}$ is obtained by [11] under the alternative continuity hypothesis.

3. Fuzzy Coalitions in Exchange Economies

3.1. Partitioning of a Measurable Space

The problem of dividing of a heterogeneous commodity among a finite number of individuals is formulated as partitioning a measurable space $(\Omega, \mathscr{F})$. Here, set Ω is a heterogeneous divisible commodity, and σ-algebra $\mathscr{F}$ of subsets of Ω describes the collection of possible pieces of Ω. There are m attributes for the heterogeneous divisible commodity Ω, each of which has a cardinal evaluation represented by a nonatomic finite measure μ_k on $(\Omega, \mathscr{F})$ for $k = 1, \dots, m$. Let $\vec{\mu} = (\mu_1, \dots, \mu_m)$ be an m-dimensional nonatomic vector measure.

There are n individuals, indexed by $i = 1, \dots, n$, with the set $N = \{1, \dots, n\}$ of all individuals, whose preferences on $\mathscr{F}$ are given by *utility functions* $\nu_i : \mathscr{F} \to \mathbb{R}$ for $i \in N$. A *partition* of Ω is an ordered n-tuple $(A_1, \dots, A_n)$ of mutually disjoint elements $A_1, \dots, A_n$ in $\mathscr{F}$ whose union is Ω, where each A_i is a piece of the cake given to individual i. Let individual i be initially endowed with $\Omega_i \in \mathscr{F}$ for $i \in N$. So $(\Omega_1, \dots, \Omega_n)$ is an *initial partition* of Ω. An *exchange economy* $\mathscr{E} = \langle(\Omega, \mathscr{F}), \Omega_i, \nu_i\rangle_{i \in N}$ for the partitioning problem under study is the primitive consisting of a common consumption set $(\Omega, \mathscr{F})$ and the individuals' profile of initial endowments Ω_i and utility functions ν_i.

We formulate partial participation of individuals to coalitions as proposed by [2, 3]. A nonzero vector $\alpha = (\alpha_1, \dots, \alpha_n)$ in the unit cube $[0,1]^n$ is called a *fuzzy coalition*, whose component $\alpha_i \in [0,1]$ denotes the degree of participation of individual i in this coalition. For each nonempty set $S \subset N$, a fuzzy coalition with support S is a vector $\alpha^S = (\alpha^S_1, \dots, \alpha^S_n) \in [0,1]^n$, satisfying $\alpha^S_i > 0$ for each $i \in S$ and $\alpha^S_i = 0$ otherwise; S is the set of 'active individuals' in the fuzzy coalition α^S. The vector e^S in $\{0,1\}^n$ defined as $e^S_i = 1$ for each $i \in S$ and $e^S_i = 0$ otherwise is called a *crisp coalition*, and is identified with an ordinary (nonfuzzy) coalition S.

Definition 3.1. A partition $(A_1, \dots, A_n)$ is an α^S-*partition* if for each $i \in S$ there exist $E_i \in \mathscr{K}^{\vec{\mu}}_{\alpha^S_i}(A_i)$ and $F_i \in \mathscr{K}^{\vec{\mu}}_{\alpha^S_i}(\Omega_i)$ such that:

$$\vec{\mu}\left(\bigcup_{i \in S} E_i \,\triangle\, \bigcup_{i \in S} F_i\right) = 0.$$

An e^S-partition is simply said to be an S-*partition.*

It follows from the definition that an S-partition $(A_1, \dots, A_n)$ satisfies the coalitional feasibility constraint characterwise, that is:

$$\mu_k\left(\bigcup_{i \in S} A_i \,\triangle\, \bigcup_{i \in S} \Omega_i\right) = 0$$

for each $k = 1, \dots, m$.

Definition 3.2. A fuzzy coalition α^S *improves upon* a partition $(B_1, \dots, B_n)$ if there exists an α^S-partition $(A_1, \dots, A_n)$ such that $\nu_i(A_i) > \nu_i(B_i)$ for each $i \in S$. A partition that cannot be improved upon by any fuzzy coalition is a *fuzzy core partition.*

3.2. Allocations in L^∞-Spaces

Define $\mu = \sum_{k=1}^m \mu_k$. Let $L^\infty(\Omega, \mathscr{F}, \mu)$ be the space of μ-essentially bounded measurable functions on Ω with the sup norm. Denote by $\chi_A \in L^\infty(\Omega, \mathscr{F}, \mu)$ the characteristic function of $A \in \mathscr{F}$.

Let $X = \{f \in L^\infty(\Omega, \mathscr{F}, \mu) \mid 0 \leqslant f \leqslant 1, \mu\text{-a.e.}\}$. Then, X is a weakly* compact, convex subset of $L^\infty(\Omega, \mathscr{F}, \mu)$. We identify $\mathscr{F}$ with the subset of characteristic functions in X. An n-tuple $(f_1, \dots, f_n)$ of elements in $L^\infty(\Omega, \mathscr{F}, \mu)$ is an *allocation* of Ω if $\sum_{i=1}^n f_i = 1$ and $f_i \in X$ for each $i \in N$. Note that $(A_1, \dots, A_n)$ is a partition of Ω if and only if $\sum_{i=1}^{n} \chi_{A_i} = 1$. We denote by $\mathscr{A}$ the set of allocations of Ω.

For $f \in X$, set $\mu_k(f) = \int f d\mu_k$ for $k = 1, \dots, m$. Given a utility function ν_i of the form $\nu_i = \varphi_i \circ \vec{\mu}$ with $\varphi_i : \mathscr{R}(\vec{\mu}) \to \mathbb{R}$, we will denote by $\hat{\nu}_i$ the extension of ν_i to X defined as $\hat{\nu}_i(f) = \varphi_i(\vec{\mu}(f))$. This extension is indeed well defined because $\mathscr{R}(\vec{\mu})$ coincides with the set $\{\vec{\mu}(f) \mid f \in X\}$ by Lyapunov's convexity theorem. If φ_i is continuous and quasiconcave on $\mathscr{R}(\vec{\mu})$, then $\hat{\nu}_i$ is weakly* continuous and quasiconcave on X.

An exchange economy $\widehat{\mathscr{E}} = \langle X, \chi_{\Omega_i}, \hat{\nu}_i \rangle_{i \in N}$ for the allocation problem is the primitive consisting of a common consumption set X and the individuals' profile of initial endowments χ_{Ω_i} and utility functions $\hat{\nu}_i$, which is an extension of the original economy $\mathscr{E} = \langle (\Omega, \mathscr{F}), \Omega_i, \nu_i \rangle_{i \in N}$ whenever $\nu_i = \varphi_i \circ \vec{\mu}$ for each $i \in N$.

Lemma 3.1. *$\mathscr{A}$ is a weakly* compact, convex subset of* $[L^\infty(\Omega, \mathscr{F}, \mu)]^n$.

To explore the notion of fuzzy core allocations for $\widehat{\mathscr{E}} = \langle X, \chi_{\Omega_i}, \hat{\nu}_i \rangle_{i \in N}$, we introduce a set-valued mapping $\widehat{\mathscr{K}}_t^{\vec{\mu}} : X \to 2^X$, an eligible extension of $\mathscr{K}_t^{\vec{\mu}} : \mathscr{F} \to 2^{\mathscr{F}}$, as follows. For $f \in X$ and $t \in [0,1]$, we define

$$\widehat{\mathscr{K}}_t^{\vec{\mu}}(f) = \{ v \in X \mid \vec{\mu}(v) = t\vec{\mu}(f),\, v \leqslant f \}.$$

It follows from the definition that $\mathscr{K}_t^{\vec{\mu}}(A) \subset \widehat{\mathscr{K}}_t^{\vec{\mu}}(\chi_A)$ for every $A \in \mathscr{F}$ and $t \in [0,1]$. Moreover, $\chi_E \in \widehat{\mathscr{K}}_t^{\vec{\mu}}(\chi_A)$ if and only if $E \in \mathscr{K}_t^{\vec{\mu}}(A)$.

Definition 3.3. An allocation $(f_1, \dots, f_n)$ is an α^S*-allocation* if for each $i \in S$ there exist $v_i \in \widehat{\mathscr{K}}_{\alpha_i^S}^{\vec{\mu}}(f_i)$ and $w_i \in \widehat{\mathscr{K}}_{\alpha_i^S}^{\vec{\mu}}(\chi_{\Omega_i})$ such that $\sum_{i \in S} v_i = \sum_{i \in S} w_i$. An e^S-allocation is simply said to be an *S-allocation.*

Note that $(\chi_{A_1}, \dots, \chi_{A_n})$ is an α^S-allocation with $(\chi_{E_i}, \chi_{F_i}) \in \widehat{\mathscr{K}}_{\alpha_i^S}^{\vec{\mu}}(\chi_{A_i}) \times \widehat{\mathscr{K}}_{\alpha_i^S}^{\vec{\mu}}(\chi_{\Omega_i})$ for each $i \in S$ if and only if $(A_1, \dots, A_n)$ is an α^S-partition with $(E_i, F_i) \in \mathscr{K}_{\alpha_i^S}^{\vec{\mu}}(A_i) \times \mathscr{K}_{\alpha_i^S}^{\vec{\mu}}(\Omega_i)$ for each $i \in S$. Thus, the notion of α^S-allocations introduced here is a consistent extension of that of α^S-partitions to L^∞-spaces.

Definition 3.4. A fuzzy coalition α^S *improves upon* an allocation $(f_1, \dots, f_n)$ if there exists an α^S-allocation $(g_1, \dots, g_n)$ such that $\hat{\nu}_i(f_i) < \hat{\nu}_i(g_i)$ for each $i \in S$. An allocation that cannot be improved upon by any fuzzy coalition is a *fuzzy core allocation.*

When commodities are homogeneous and divisible as in classical exchange economies, the usual definition of an α^S-allocation is an allocation $(f_1, \dots, f_n)$ such that:

$$\sum_{i \in S} \alpha_i^S f_i = \sum_{i \in S} \alpha_i^S \chi_{\Omega_i}. \tag{3.1}$$

(See [2, 3, 12, 14].) Although the definition of α^S-allocations in the sense of (3.1) seems to make sense in the extended economy $\widehat{\mathscr{E}}$, it is inadequate in that it cannot be reduced to the corresponding definition of α^S-partitions in the original economy $\mathscr{E}$.

To illustrate how this definition malfunctions, consider for $n = 2$ any fuzzy coalition $\alpha = (\alpha_1, \alpha_2) \in [0,1]^2$ with $\alpha_1 \neq \alpha_2$. Restricting the definition of α-allocations in the sense of (3.1) to the characteristic functions (χ_{A_1}, χ_{A_2}) with $\chi_{A_1} + \chi_{A_2} = 1$ yields $\alpha_1(\chi_{A_1} - \chi_{\Omega_1}) = \alpha_2(\chi_{\Omega_2} - \chi_{A_2})$, which is true if and only if $A_1 = \Omega_1$ and $A_2 = \Omega_2$. So, every α-partition with $\alpha_1 \neq \alpha_2$ in the sense of (3.1) consists of only the initial partition (Ω_1, Ω_2). Fuzziness entirely disappears from the definition.

4. Existence of Fuzzy Core Allocations

4.1. The NTU Game for Exchange Economies

Let $\mathscr{N} = 2^N \setminus \{\emptyset\}$. The market game $V : \mathscr{N} \to 2^{\mathbb{R}^n}$ with NTU for the exchange economy $\widehat{\mathscr{E}} = \langle X, \chi_{\Omega_i}, \hat{\nu}_i \rangle_{i \in N}$ is given by:

$$V(S) = \left\{ x \in \mathbb{R}^n \,\middle|\, \begin{array}{l} \exists \alpha^S\text{-allocation } (f_1, \dots, f_n) \\ \text{such that } x_i \leqslant \hat{\nu}_i(f_i),\ \forall i \in S \end{array} \right\}.$$

By construction, $V(S)$ is the *utility possibility set* of the players (individuals) in which payoff vectors are attainable via some fuzzy coalition α^S. The *core* $C(V)$ of V is given by:

$$C(V) = \left\{ x \in V(N) \,\middle|\, \begin{array}{l} \nexists (S, y) \in \mathscr{N} \times V(S) \\ \text{such that } x_i < y_i,\ \forall i \in S \end{array} \right\}.$$

Proposition 4.1. *For an exchange economy $\mathscr{E} = \langle (\Omega, \mathscr{F}), \Omega_i, \nu_i \rangle_{i \in N}$, if ν_i is of the form $\nu_i = \varphi_i \circ \vec{\mu}$ such that $\varphi_i : \mathscr{R}(\vec{\mu}) \to \mathbb{R}$ is continuous and quasiconcave for each $i \in N$, then $C(V)$ is nonempty.*

Corollary 4.1. *For an exchange economy $\mathscr{E} = \langle (\Omega, \mathscr{F}), \Omega_i, \nu_i \rangle_{i \in N}$, if ν_i is of the form $\nu_i = \varphi_i \circ \vec{\mu}$ such that $\varphi_i : \mathscr{R}(\vec{\mu}) \to \mathbb{R}$ is continuous and quasiconcave for each $i \in N$, then there exists a fuzzy core allocation for $\widehat{\mathscr{E}} = \langle X, \chi_{\Omega_i}, \hat{\nu}_i \rangle_{i \in N}$.*

4.2. Existence of Fuzzy Core Partitions

Let $x = (x_1, \dots, x_n)$ be in $C(V)$ and define the set $\mathscr{C}_x$ by:

$$\mathscr{C}_x = \{ (f_1, \dots, f_n) \in \mathscr{A} \mid x_i \leqslant \hat{\nu}_i(f_i),\ \forall i \in N \}.$$

It is easy to verify that $\mathscr{C}_x$ is a subset of the set of fuzzy core allocations for $\widehat{\mathscr{E}} = \langle X, \chi_{\Omega_i}, \hat{\nu}_i \rangle_{i \in N}$.

The proof of the next result is essentially based on the ingenious technique of [18], yielding that an extreme point of $\mathscr{C}_x$ is indeed a measurable partition of Ω (see also [1]).

Proposition 4.2. *For an exchange economy $\mathscr{E} = \langle (\Omega, \mathscr{F}), \Omega_i, \nu_i \rangle_{i \in N}$, if ν_i is of the form $\nu_i = \varphi_i \circ \vec{\mu}$*

such that $\varphi_i : \mathscr{R}(\vec{\mu}) \to \mathbb{R}$ is continuous and quasiconcave for each $i \in N$, then there exists a partition $(A_1, \dots, A_n)$ of Ω such that $(\chi_{A_1}, \dots, \chi_{A_n}) \in \mathscr{C}_x$.

The next theorem is now an immediate consequence of Proposition 4.2 and Corollary 4.1.

Theorem 4.1. *For an exchange economy $\mathscr{E} = \langle(\Omega, \mathscr{F}), \Omega_i, \nu_i\rangle_{i \in N}$, if ν_i is of the form $\nu_i = \varphi_i \circ \vec{\mu}$ such that $\varphi_i : \mathscr{R}(\vec{\mu}) \to \mathbb{R}$ is continuous and quasiconcave for each $i \in N$, then there exists a fuzzy core partition.*

Corollary 4.2. *For an exchange economy $\mathscr{E} = \langle(\Omega, \mathscr{F}), \Omega_i, \nu_i\rangle_{i \in N}$, if ν_i is $\vec{\mu}$-concave and $\vec{\mu}$-continuous at every $A \in \mathscr{F}$ with $\vec{\mu}(A) \in \operatorname{bd} \mathscr{R}(\vec{\mu})$ for each $i \in N$, then there exists a fuzzy core partition.*

Since a fuzzy core partition is a core partition, Theorem 4.1 is an extension of [15], and [23], the former proved the existence of core partitions for the case where utility functions of the individuals are continuous quasiconcave transformations of a finite-dimensional nonatomic vector measure and the latter for the case of concave measures.

5. Supporting Prices

5.1. Continuous, Quasiconcave, Strictly Monotonic Extensions

For vectors $x, y \in \mathbb{R}^m$, denote $x \leqslant y$ to mean that $x_k \leqslant y_k$ for each $k = 1, \dots, m$ and $x < y$ to mean that $x_k < y_k$ for each $k = 1, \dots, m$. The positive and strictly positive orthants of $\mathbb{R}^m$ are given respectively by $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m \mid x \geqslant 0\}$ and $\mathbb{R}^m_{++} = \{x \in \mathbb{R}^m \mid x > 0\}$. A function $\varphi : \mathscr{R}(\vec{\mu}) \to \mathbb{R}$ is *monotonic* if $x \leqslant y$ and $x, y \in \mathscr{R}(\vec{\mu})$ imply $\varphi(x) \leqslant \varphi(y)$; φ is *strictly monotonic* if $x < y$ and $x, y \in \mathscr{R}(\vec{\mu})$ imply $\varphi(x) < \varphi(y)$.

Lemma 5.1. *If $\varphi : \mathscr{R}(\vec{\mu}) \to \mathbb{R}$ is continuous, quasiconcave and strictly monotonic, then φ has an extension $\hat{\varphi} : \mathbb{R}^m_+ \to \mathbb{R}$ preserving its properties.*

Since we have assumed that the component measures of $\vec{\mu} = (\mu_1, \dots, \mu_m)$ are mutually absolutely continuous, for every $x \in \mathscr{R}(\vec{\mu})$ with $x \neq \vec{\mu}(\Omega)$, we have $x < \vec{\mu}(\Omega)$. Taking into account this observation, it is helpful to deduce the above lemma from the more general assertion below.

Proposition 5.1. *Let $c = (c_1, \dots, c_m) \in \mathbb{R}^m_{++}$ and C be a compact convex subset of $\prod_{k=1}^m [0, c_k]$ that contains 0 and c satisfying $x < c$ for every $x \in C$ with $x \neq c$. If $\varphi : C \to \mathbb{R}$ is continuous, quasiconcave and strictly monotonic, then φ has an extension $\hat{\varphi} : \mathbb{R}^m_+ \to \mathbb{R}$ preserving its properties.*

If $\varphi : C \to \mathbb{R}$ is continuous, concave and monotonic, then an extension $\tilde{\varphi} : \mathbb{R}^m_+ \to \mathbb{R}$ of φ, preserving its properties, is given by:

$$\tilde{\varphi}(x) = \max\{\varphi(y) \mid y \in C,\, y \leqslant x\}.$$

This extension was suggested by [11]. Unfortunately, when φ is continuous, quasiconcave and strictly monotonic, the extension $\tilde{\varphi}$ to $\mathbb{R}^m_+$ does not necessarily preserve its properties (strict monotonicity might be violated). We have exploited this extension not on the entire domain $\mathbb{R}^m_+$, but on D in the proof of Proposition 5.1. For a general treatment of extensions of continuous, (strictly) monotonic functions, see [17].

5.2. Existence of Supporting Prices in L^1

Let $ba(\Omega, \mathscr{F}, \mu)$ be the vector subspace of $ba(\Omega, \mathscr{F})$ whose elements vanish at every $A \in \mathscr{F}$ with $\mu(A) = 0$. Then, $ba(\Omega, \mathscr{F}, \mu)$ is the dual space of $L^\infty(\Omega, \mathscr{F}, \mu)$ (see [8, Theorem IV.8.16]). A nonzero element $\pi \in ba(\Omega, \mathscr{F})$ is *positive* if $\pi(A) \geqslant 0$ for every $A \in \mathscr{F}$. A positive element $\pi \in ba(\Omega, \mathscr{F})$ is *purely finitely additive* if every countably additive measure λ satisfying $0 \leqslant \lambda \leqslant \pi$ is identically zero.

Definition 5.1. A nonzero element $\pi \in ba(\Omega, \mathscr{F}, \mu)$ is a *supporting price* for an allocation $(f_1, \dots, f_n)$ for $\widehat{\mathscr{E}} = \langle X, \chi_{\Omega_i}, \hat{\nu}_i\rangle_{i \in N}$ if $\hat{\nu}_i(f_i) \leqslant \hat{\nu}_i(f)$ implies $\pi(f_i) \leqslant \pi(f)$.

As observed by [5, p. 516], "one could call any element of ba a price system, but since those elements of ba not belonging to L^1 have no economic interpretation, we will be interested only in equilibria with price systems in L^1." If a supporting price happens to be countably additive, then it has the Radon–Nikodym derivative with respect to μ. In such a case, it is identified with an element in $L^1(\Omega, \mathscr{F}, \mu)$.

Definition 5.2. An allocation $(f_1, \dots, f_n)$ in $\widehat{\mathscr{E}} = \langle X, \chi_{\Omega_i}, \hat{\nu}_i\rangle_{i \in N}$ is *Pareto optimal* if there exists no allocation $(g_1, \dots, g_n)$ such that $\hat{\nu}_i(f_i) \leqslant \hat{\nu}_i(g_i)$ for each $i \in N$ and $\hat{\nu}_j(f_j) \leqslant \hat{\nu}_j(g_j)$ for some $j \in N$.

Theorem 5.1. *For an exchange economy $\mathscr{E} = \langle(\Omega, \mathscr{F}), \Omega_i, \nu_i\rangle_{i \in N}$, if ν_i is of the form $\nu_i = \varphi_i \circ \vec{\mu}$ such that $\varphi_i : \mathscr{R}(\vec{\mu}) \to \mathbb{R}$ is continuous, quasiconcave and strictly monotonic for each $i \in N$, then every Pareto optimal allocation in $\widehat{\mathscr{E}} = \langle X, \chi_{\Omega_i}, \hat{\nu}_i\rangle_{i \in N}$ has a positive supporting price.*

"[T]heorem [5.1] ... would be of little interest if one could not find interesting conditions under which equilibrium price systems could be chosen from L^1." (See [5, p. 523].) To obtain supporting prices in L^1, we need the following lemma.

Lemma 5.2. *For an exchange economy $\mathscr{E} = \langle(\Omega, \mathscr{F}), \Omega_i, \nu_i\rangle_{i \in N}$, if ν_i is of the form $\nu_i = \varphi_i \circ \vec{\mu}$ such that $\varphi_i : \mathscr{R}(\vec{\mu}) \to \mathbb{R}$ is strictly quasiconcave for each $i \in N$, then $\pi \in ba(\Omega, \mathscr{F}, \mu)$ is a supporting price for an allocation $(f_1, \dots, f_n)$ in $\widehat{\mathscr{E}} = \langle X, \chi_{\Omega_i}, \hat{\nu}_i\rangle_{i \in N}$ if and only if $\hat{\nu}_i(f_i) < \hat{\nu}_i(f)$ implies $\pi(f_i) \leqslant \pi(f)$.*

Theorem 5.2. *For an exchange economy* $\mathscr{E} = \langle(\Omega,\mathscr{F}),\Omega_i,\nu_i\rangle_{i\in N}$*, if* ν_i *is of the form* $\nu_i = \varphi_i \circ \vec{\mu}$ *such that* $\varphi_i : \mathscr{R}(\vec{\mu}) \to \mathbb{R}$ *is continuous, strictly quasiconcave and strictly monotonic for each* $i \in N$*, then every Pareto optimal allocation (in particular, every fuzzy core partition) in* $\widehat{\mathscr{E}} = \langle X, \chi_{\Omega_i}, \hat{\nu}_i\rangle_{i\in N}$ *has a positive supporting price in* $L^1(\Omega,\mathscr{F},\mu)$.

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