Nonexistence of Positive Solutions to an Elliptic System and Blow-Up Rate for a Parabolic System

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***Abstract*--We first get the conditions under which the elliptic system $-\Delta u_i = u_{i+1}^{p_i}$, $u_{s+1} := u_1$ ($i = 1,2,\cdots,s$) has no positive radially symmetric solutions. Then by using this nonexistence result, we establish blow-up estimates for semilinear reaction-diffusion system $u_{it} = \Delta u_i + u_{i+1}^{p_i}$, $u_{s+1} := u_1$ ($i = 1,2,\cdots,s$) with null Dirichlet boundary conditions. The results of our paper with the those in Wang (Comp and Math with Appl, 44, 573-585, 2002) are same, but our methods of proofs are entirely different, even easier than that used.**

Keywords-elliptic system; parabolic system; blow-up; blow-up rate; nonexistence

I INTRODUCTION

In the past decades, many physical phenomena have been formulated into local or nonlocal mathematical models of partial differential equation and studied by many authors, see [1-5] and the references therein. In particular, there exist many articles dealing with properties of positive solutions such as blow-up and blow-up rate to semilinear or degenerate parabolic equations. For example, Wang in [6] considered the following semilinear reaction-diffusion system

$$(u_i)_t = \Delta u_i + u_{i+1}^{p_i},\ u_{s+1} := u_1,\ i = 1,2,\cdots,s,\ (x,t) \in \Omega \times (0,T) \quad (1)$$

And obtained the blow-up rate estimate

$$c(T-t)^{-\lambda_i} \le \max\ u_i(\cdot,t) \le C(T-t)^{-\lambda_i}, \quad (2)$$

Where

$$\lambda_i = (1 + p_i + \sum_{l=i+1}^{i+s-2} p_i \cdots p_l / (p_1 \cdots p_s - 1), i = 1,2,\cdots,s.$$

At the same time, the initial value problem and localized problem of (1) were studied by Pedersen [7] and Fila [8], respectively; they also obtained the same estimate (2) of blow-up rate.

By the motivations of the above cited works, in this paper, we will continue to study the blow-up rate of positive solutions for (1) with the following initial and boundary value conditions

$$u_i(x,0) = u_{i0}(x), \qquad x \in \Omega,\ i = 1,2,\cdots s, \quad (3)$$

$$u_i(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \quad i = 1,2,\cdots s, \quad (4)$$

Where $\Omega \subseteq R^N, p_1, \cdots, p_s \ge 1, T > 0$, and $u_{10}(x), \cdots, u_{s0}(x)$ are continuous, nonnegative functions and vanish on $\partial\Omega$. By means of the non-existence result of the related elliptic system of (1) which obtained in next section, we establish the blow-up estimate (2). The results of this paper with those of [6] are same, but our methods of proofs are entirely different, even easier than that used. In addition, we always let $\Omega = B(0,R) = \{x \in R^N; |x| < R\}, R > 0$ denotes a open ball of R^N centered at the origin of radius R since we deal with positive radially symmetric solutions of system (1).

Before ending this section, we give the following lemma which will be used in our proofs. For convenience, we denote $p_{s+l} = p_l$ for all integers l and

$$A = \begin{pmatrix} 1 & -p_1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -p_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & -p_{s-1} \\ -p_s & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \quad (5)$$

Lemma 1 For the matrix A which is defined by (5) and any constant ε_0, if det $A \ne 0$, then the following linear system

$$A(y_1, y_2, \cdots, y_{s-1}, y_s)^T = (\varepsilon_0, \varepsilon_0, \cdots, \varepsilon_0, \varepsilon_0)^T \quad (6)$$

has an unique solution $(y_1, y_2, \cdots, y_{s-1}, y_s)^T$ which is given by

$$y_i = \frac{-\varepsilon_0(1 + p_i + \sum_{j=1}^{s-2} p_i \cdots p_{i=j})}{p_1 p_2 \cdots p_s - 1},\ y_{s+1} := y_1,\ i = 1,2,\cdots,s.$$

II NONEXISTENCE FOR ELLIPTIC SYSTEM

In this section, we investigate the following related elliptic system of (1)

$$-\Delta u_i(x) = u_{i+1}^{p_i}(x),\ u_{s+1} := u_1,\ i = 1,2,\cdots,s, \quad x \in \Omega \quad (7)$$

And consider the radially symmetric solutions of (7), i.e., suppose $u_i(x)=u_i(r)$ with $r=|x|$, $i=1,2,\cdots,s$. We have the following theorem.

Theorem 1 Assume that $N\geq 3$ and there exists $i\in\{1,2,\cdots,s\}$ such that $\delta_i>0$, or $\delta_1=\cdots=\delta_s=0$. Then (7) has no positive radially symmetric solution, where $\delta_i=2\alpha_i-(N-2)$, $i=1,2,\cdots,s$, here and next section, denote $(\alpha_1,\alpha_2,\cdots,\alpha_s)^T$ be a solution of linear system (6) with constant $\varepsilon_0=-1$.

To prove the Theorem 1, the system (7) can be rewritten in radial coordinates as

$$(u_i)''+\frac{N-1}{r}(u_i)'+u_{i+1}^{p_i}=0,\ u_{s+1}:=u_1,\ i=1,2,\cdots,s \tag{8}$$

With $u_i'(0)=0(i=1,2,\cdots,s)$. Secondly, similar to the Lemma 3.1 of [9], we have

Lemma 2 Let $(u_1,u_2,\cdots,u_s)$ be a positive, radially symmetric, decreasing solution of (8). Then for any $r>0$, we have

$$\frac{r^2}{N}u_{i+1}^{p_i}\leq -ru_i'\leq (N-2)u_i,\quad u_{s+1}:=u_1,\quad i=1,2,\cdots,s. \tag{9}$$

Lemma 3 Suppose the conditions in Theorem 1 are satisfied. Let $(u_1,\cdots,u_s)$ be a positive and radially symmetric solution of (8). Then we have the following asymptotic estimates for $r>0$:

$$u_i(r)\leq (N(N-2))^{\alpha_i}r^{-2\alpha_i},\quad i=1,2,\cdots,s. \tag{10}$$

Proof of Theorem 1 Let $(u_1,u_2,\cdots,u_s)$ be a nontrivial positive and radially symmetric solution of (8). Next, we divide the proof into two cases.

Case i Without loss of generality, we assume $\delta_1>0$. By (9), $(r^{N-2}u_1)'\geq 0$. Then we know that the function $r^{N-2}u_1$ is non-decreasing on $(0,\infty)$ and there exists a constant $c_0>0$ such that

$$u_1(r)\geq c_0r^{-(N-2)},\quad for\quad r>r_0>0 \tag{11}$$

Thus, (11) and (10) lead to a contradiction for r sufficiently large since $\delta_1>0$.

Case ii Suppose that $\delta_1=0$ (the case $\delta_i=0, i=2,\cdots,s$, being similar). At first, by (10), we have

$$r^{N-2}u_1\leq (N(N-2))^{\alpha_1}r^{(N-2)-2\alpha_1}=(N(N-2))^{\alpha_1}. \tag{12}$$

Secondly, according to (8), we obtain

$$r^{N-1}(-u_1'(r))\geq r^{N-1}(-u_1'(r))-r_0^{N-1}(-u_1'(r_0))=\int_{r_0}^{r}\sigma^{N-1}u_2^{p_1}d\sigma. \tag{13}$$

At the same time, it follows from (9) that

$$u_2^{p_1}\geq (N(N-2))^{-\sum_{j=1}^{s-1}p_1\cdots p_j}\times r^{2\sum_{j=1}^{s-1}p_1\cdots p_j}\times u_1^{p_1p_2\cdots p_s}.$$

Together with (11), we obtain from (13) that

$$r^{N-1}(-u_1')\geq c_0^{p_1\cdots p_s}(N(N-2))^{-\sum_{j=1}^{s-1}p_1\cdots p_j}\left(\ln\frac{r}{r_0}\right),$$

which will lead to a contradiction for r sufficiently large. Thus, the proof of Theorem 1 is complete.

III BLOW-UP ESTIMATES FOR PARABOLIC SYSTEM

Motivated by [1] and [2], we use the nonexistence result of the elliptic system obtained in section 2 to establish the blow-up estimate (2) for the reaction-diffusion system (1), (3) and (4). With the ideas in [10], we have the following lemma which is a comparison relationship between u_i and u_{i+1}.

Lemma 4 If the exponents $p_1,p_2,\cdots,p_s$ of (1) satisfy

$$p_i\left(1+\sum_{j=1}^{s-2}p_{i+1}\cdots p_{i+j}\right)\geq(\leq)p_{i+1}\left(1+\sum_{j=2}^{s-1}p_{i+2}\cdots p_{i+j}\right), i=1,\cdots,s. \tag{14}$$

Then the solution $(u_1,u_2,\cdots,u_s)$ of (1), (3) and (4) satisfies either

$$u_{i+1}(x,t)\leq K_iu_i(x,t)^{\alpha_{i+1}/\alpha_i},\ \alpha_{s+1}:=\alpha_1,\ u_{s+1}:=u_1, i=1,2,\cdots,s. \tag{15}$$

or

$$u_{i+1}(x,t)\geq L_iu_i(x,t)^{\alpha_{i+1}/\alpha_i},\ \alpha_{s+1}:=\alpha_1,\ u_{s+1}:=u_1, i=1,2,\cdots,s. \tag{16}$$

for all $(x,t)\in Q_T\setminus Q_\eta$, with some positive constants $K_1,\cdots,K_s,L_1,\cdots,L_s$ and $\eta\in(0,T)$.

Theorem 2 Let $(u_1,u_2,\cdots,u_s)$ be a classical solution of (1), (3) and (4), defined on $\Omega\times(0,T)$, with $(0,T)$ maximal time interval of existence, and $T<+\infty$. Assume that the following hypotheses are satisfied:

(i) $u_1(\cdot,t),\cdots,u_s(\cdot,t)$ are radially, decreasing, and symmetric functions of $r=|x|$;

(ii) $u_i(x,t), u_{it}(x,t)\geq 0$ in $Q_T=\Omega\times(0,T), i=1,2,\cdots s$;

(iii) $u_{1t}(x,t),\cdots,u_{st}(x,t)$ achieve the maximum at $x=0$, for every $t\in(0,T)$;

(iv) $\delta_i>0$ or $\delta_1=\cdots=\delta_s=0$;

(v) $p_i\left(1+\sum_{j=1}^{s-2}p_{i+1}\cdots p_{i+j}\right)\geq p_{i+1}\left(1+\sum_{j=2}^{s-1}p_{i+2}\cdots p_{i+j}\right), i=1,\cdots,s$

Then there are constants $C_1,\cdots,C_s>0$ and $t_1\in(0,T)$ such that

$$u_i(x,t)\leq u_i(0,t)\leq C_i(T-t)^{-\alpha_i},\ i=1,2,\cdots,s \tag{17}$$

for all $(x,t)\in Q_T \setminus Q_{t_1}$.

Remark 1 The conditions (i)-(iii) in Theorem 2 are reasonable if we impose appropriate assumptions on the initial data $u_{i0}(x)$, for example, positivity, radial symmetry, and a suitable decreasing property with

$$\Delta u_{i0} + u_{(i+1)0}^{p_i} \geq 0,\ u_{(s+1)0} += u_{10},\ i=1,2,\cdots,s$$

Proof of Theorem 2 Let $(\sigma_1,\sigma_2,\cdots,\sigma_s)$ be a solution of linear system (6) with constant $\varepsilon_0=-2$. Define for $t\in(0,T)$,

$$\rho_1(t)=u_1(0,t)^{1/\sigma_1}, \rho_2(t)=u_2(0,t)^{1/\sigma_2},\cdots,\rho_s(t)=u_s(0,t)^{1/\sigma_s}$$

and $\rho(t)=\rho_1(t)+\rho_2(t)+\cdots+\rho_s(t)$. By putting

$$w_i(r,t)=\frac{u_i(r/\rho(t),t)}{\rho(t)^{\sigma_i}},\ i=1,2,\cdots,s$$

and taking into account that $u_i(\cdot,t)$ achieve its maximum at $r=0$ since the assumption (i), we easily see that w_i is bounded. On the other hand, it follows from the condition (iii) of Theorem 2 and Lemma 1 that

$$0\leq \Delta w_i(r,t)+w_{i+1}^{p_i}(r,t)\leq \frac{u_{it}(0,t)}{\rho(t)^{\sigma_i+2}},\ i=1,2,\cdots,s. \quad (18)$$

By using the assumptions (i), (ii) and (18), we get

$$0\leq w_{i,rr}+\frac{N-1}{r}w_{i.r}+w_{i+1}^{p_i}\leq \sum_{j=1}^{s}\frac{u_{jt}(0,t)}{\rho(t)^{\sigma_j+2}},\ i=1,2,\cdots,s \quad (19)$$

for any $t\in(0,T)$ and $r\in(0[,R\rho(t))$. Multiplying the sum of (19) by $(w_1+\cdots+w_s)_r$ we have

$$\frac{1}{2}\frac{d}{dr}\left(\sum_{i=1}^{s} w_i\right)_r^2+\frac{N-1}{r}\left(\sum_{i=1}^{s} w_i\right)_r^2+\left(\sum_{i=1}^{s} w_{i+1}^{p_i}\right)\left(\sum_{i=1}^{s} w_i\right)\leq 0$$

which implies that

$$\frac{1}{2}\frac{d}{dr}\left(\sum_{i=1}^{s} w_i\right)_r^2+\left(\sum_{i=1}^{s} w_{i+1}^{p_i}\right)\left(\sum_{i=1}^{s} w_i\right)\leq 0 \quad (20)$$

Integrating (20) on $(0,r)$ and together with $w_{i,r}\leq 0, 0\leq w_i\leq 1$, we obtain

$$\frac{1}{2}\left(\sum_{i=1}^{s} w_i\right)_r^2\leq \sum_{i=1}^{s}\frac{1}{1+p_i}+s \quad (21)$$

for any $t\in(0,T)$ and $r\in[0,R\rho(t))$. The inequality (21) implies the equicontinuity of $w_i, i=1,2,\cdots,s$.

Next, we claim that

$$\liminf_{t\to T}\left(\frac{u_{1t}(0,t)}{\rho(t)^{\sigma_1+2}}+\frac{u_{2t}(0,t)}{\rho(t)^{\sigma_2+2}}+\cdots+\frac{u_{st}(0,t)}{\rho(t)^{\sigma_s+2}}\right)>0 \quad (22)$$

Otherwise, there exists a sequence $\{t_m\}\subseteq(0,T)$ with $t_m\to T$ such that

$$\liminf_{t\to T}\left(\frac{u_{1t}(0,t)}{\rho(t)^{\sigma_1+2}}+\frac{u_{2t}(0,t)}{\rho(t)^{\sigma_2+2}}+\cdots+\frac{u_{st}(0,t)}{\rho(t)^{\sigma_s+2}}\right)=0$$

By using the Ascoli-Alzela theorem, there exists a sequence (still denoted by $\{t_m\}$) such that

$$w_i(\cdot,t_m)\to \overline{w}_i(\cdot),\quad m\to\infty\ (i=1,2,\cdots,s)$$

uniformly on compact subsets of $[0,+\infty)$. Now in the sense of distributions

$$\overline{w}_{i,rr}+\frac{N-1}{r}\overline{w}_{i,r}+\overline{w}_{i+1}^{p_i}=0,\ i=1,2,\cdots,s \quad (23)$$

The absolute continuity of w_i implies that $\overline{w}_i\in C^2(0,+\infty)$ and, by local existence and uniqueness of the initial value problem for (23) and using the arguments in [1,2], we conclude that $\overline{w}_i(r)>0$ on $(0,+\infty)$ with $\overline{w}_i'(0)=0, i=1,2,\cdots,s$.

Now, in case $N=1$, we easily reach a contradiction by noting that, since (23) hold, then, $\overline{w}_1,\cdots,\overline{w}_s$ must be strictly concave on $(0,+\infty)$, which is impossible.

If $N=2$, we proceed as follows: from (23), we infer that $r\overline{w}_1',\cdots,r\overline{w}_s'$ are strictly decreasing on $(0,+\infty)$. Hence there exist constants $M>0$ and $r_0>0$ such that $r\overline{w}_i'<-M, r\in(r_0,+\infty)$. This gives a contradiction when $r\to+\infty$.

If $N\geq 3$, we know from Theorem 1 that the elliptic system (23) has no positive solutions. Therefore, we conclude that (22) is true for all $N\geq 1$, that is, there exists a positive constant c such that

$$\liminf_{t\to T}\left(\frac{u_{1t}(0,t)}{\rho(t)^{\sigma_1+2}}+\frac{u_{2t}(0,t)}{\rho(t)^{\sigma_2+2}}+\cdots+\frac{u_{st}(0,t)}{\rho(t)^{\sigma_s+2}}\right)=c>0 \quad (24)$$

It follows from (24) that there exists $t_1\in(0,T)$ such that $c\leq \frac{u_{1t}(0,t)}{\rho(t)^{\sigma_1+2}}+\cdots+\frac{u_{st}(0,t)}{\rho(t)^{\sigma_s+2}}\leq \frac{u_{1t}(0,t)}{u_1(0,t)^{(\sigma_1+2)/\sigma_1}}+\cdots+\frac{u_{st}(0,t)}{u_s(0,t)^{(\sigma_s+2)/\sigma_s}}$ holds for all $t\in(t_1,T)$. Integrating above inequality on $(t,\tau)\subseteq(t_1,T)$ and then letting $\tau\to T$, we have

$$c(T-t)\leq \alpha_1 u_1(0,t)^{-1/\alpha_1}+\cdots+\alpha_s u_s(0,t)^{-1/\alpha_s} \quad (25)$$

Now, from Lemma 4, it follows that

$$u_i(0,t)^{-1/\alpha_i}\leq \mu_i u_s^{-1/\alpha_s},\ i=1,2,\cdots,s-1 \quad (26)$$

for some suitable constants $\mu_i>0$. Thus, (25) and (26) imply that

$$u_s(x,t) \le u_s(0,t) \le C_s(T-t)^{-\alpha_s} \quad (27)$$

for all $(x,t) \in Q_T \setminus Q_{t_1}$. In order to obtain the analogous estimates on $u_1, \cdots, u_{s-1}$, we observe that from the symmetry assumptions it follows that $\Delta u_i \le 0, i = s-1, s-2, \cdots, 2, 1$. Hence, from (1) and (27), we find that $u_{(s-1)t}(0,t) \le u_s(0,t)^{p_{s-1}} \le C(T-t)^{-p_{s-1}\alpha_s} = C(T-t)^{-1-\alpha_{s-1}}$.

Integrating the last inequality on $(0,t)$, it follows that

$$u_{s-1}(x,t) \le u_{s-1}(0,t) \le C_{s-1}(T-t)^{-\alpha_{s-1}}$$

for all $(x,t) \in Q_T \setminus Q_{t_1}$. Similarly, we can obtain the another estimates. Then the proof is complete.

Finally, we give lower bounds for the blow-up rates.

Theorem 3 Assume the conditions in Theorem 2 hold, then for $t \in (\eta, T)$, there are positive constant c_1 such that

$$u_i(0,t) \ge c_i(T-t)^{-\alpha_i}, \ i = 1,2,\cdots,s \quad (28)$$

Proof. For $t \in (0,T)$, since $\Delta u_i \le 0$ at $x = 0$ by the assumptions of Theorem 2, we see that $u_{it}(0,t) \le u_{i+1}(0,t)^{p_i}$. Together with Lemma 4, we obtain

$$u_{it}(0,t) \le u_{i+1}(0,t)^{p_i} \le K_i^{p_i} u_i^{(p_i\alpha_i+1)/\alpha_i} = K_i^{p_i} u_i^{(1+\alpha_1)/\alpha_i} \quad (29)$$

Integrating (29) over $(t,\tau) \subseteq (\eta, T)$ and then letting $\tau \to T$, we get the inequalities (28) hold for $t \in (\eta, T)$. The proof is complete.

Remark 2 Combining Theorems 2 and 3, we conclude that the blow-up rates of radial positive solutions of the reaction-diffusion system (1), (3) and (4) under the conditions of the theorems are $u_i(0,t) = O((T-t)^{-\alpha_i}), \ i = 1,2,\cdots,s$, as t tends to T.

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