A Note on $B_{\pi'}$-Characters

X.J. Liu, J.K. Hai
College of Mathematics, Qingdao University
Qingdao, China

***Abstract*—Let π be a set of primes.Isaacs established the π-theory of characters,which generalizes the theory of Brauer module characters. Based on Isaacs's work, the concept of principal indecomposable $B_{\pi'}$-characters of G is introduced, where π' denotes the complement set of π.It is shown that some characteristics of the principal indecomposable $B_{\pi'}$-characters of G.**

Keywords-* π *-separable group;* π *-regular class functions;* B_{π} *-character

I. INTRODUCTION

In literature[1],Isaacs introduced the concept of B_{π}-characters of a π-separable group G,where π denotes a set of primes. B_{π}-character of G are also called π-Brauer characters. Denote by $B_{\pi}(G)$ the set of all π-Brauer characters of G.Let χ be a character of G and denote by χ^* the restriction of χ to the set of all π-elements of G. In literature [1], Isaacs proved that the set of χ^* forms a basis of the space of π-class functions of G.In addition, he also proved that the number of π-characters of G equals the number of conjugacy classes of π-elements of G.

We shall establish some elementary facts on p-modular representations for a fixed prime number p. Let R be a complete discrete valuation ring with quotient field K of characteristic 0. We assume that the residue field $F=R/(\pi)$ has characteristic p,where (π) denotes the unique maximal ideal of R.With this assumption we refer to the triple (K,R,F) as a p-modular system We fix a valuation ν of K such that $\nu(\pi)=1$,and denote by α^* the image of $\alpha\in R$ under the natural map $R\to F=R/(\pi)$.

For $\alpha\in RG$, α^* denote the image of α under the natural map

$$RG\to RG/\pi(RG)=FG\left(\sum_{x\in G}\alpha_x x\to\sum_{x\in G}\alpha_x^* x\right)$$

Write $P_i=e_iRG$,and so $P_i^*=e_i^*FG=P_i/\pi P_i$ .Then $\{P_i,1\le i\le l\}$ and $\{P_i^*,1\le i\le l\}$ give complete sets of representatives for nonisomorphic principal indecomposable RG- and FG-modules, respectively. Let η_i be the R-character defined by $P_i=e_iRG$, which is called a principal indecomposable character of G.Thus the Brauer character defined by P_i^* coincides with $\eta_i|_{G_P}$ .refer to the literature [2], [5] or [8].

Motivated by these results due to Isaacs, we want to know if other properties of Brauer characters can be generalized? In this note, by using the decomposition matrices in the π-theory of characters, The concept of principal indecomposable $B_{\pi'}$-characters of G is introduced, where π' denotes the complement set of π.

It is shown that some characteristics of the principal indecomposable $B_{\pi'}$-characters of G.Refer to the literature [3], [6]or [7] .

Throughout this paper, groups considered are finite π-separable. Irr(G) denotes the set of all irreducible ordinary characters of G; $G_{\pi'}$ denotes the set of all π-regular elements of G; χ^* denotes the restriction of χ to $G_{\pi'}$. Let H be a subgroup of G. Let χ and φ be characters of G and H respectively. Then we write χ_H the restriction of χ to H and φ^G the lift of φ to G. Unless stated otherwise, other notation and terminologies used are standard, refer to the literature [1], [4] or [9].

II. PRELIMINARIES

First we recall some basis concepts and present some lemmas which will be used in the sequel.

Definition2.1 ([4,Definition2.1]) Write $\aleph_{\pi}(G)$ for the set of χ in $Irr(G)$ such that $\chi(1)$ is a number and for every $M\lhd\lhd G$ and every irreducible constituent ϕ of χ_M, $o(\phi)$ is a π-number.

Remark (a) If G is a π-group, then we have $\aleph_{\pi}(G)=Irr(G)$.

(b) If G is a π'-group, then we have $\aleph_{\pi}(G)=\{1_G\}$

(c) Every automorphism of G leaves $\aleph_{\pi}(G)$ invariant.

Definition2.2 ([1,Definition5.1]) Let G be a π-separable group.We write $B_{\pi}(G)$ to denote the set of $\chi \in Irr(G)$ such that some(and therefore each) nucleus character of χ is a π-special.

Note that if $\chi \in Irr(G)$ is π-special,then it is its own nucleus character and so lies in $B_{\pi}(G)$.

Definition2.3 Let $\chi \in Irr(G)$. Then χ^* is said to be irreducible if χ^* can not be expressed as $\chi^* = \mu^* + \nu^*$, where μ, ν are two class functions of G.

$I^{\pi}(G)$ denotes the set of all irreducible χ^*.

Lemma2.4 ([1,Corollary9.1]) Let G be a π-separable group.

Then $\{\chi^* | \chi \in B_{\pi'}(G)\}$ are linearly independent.

Lemma2.5([1,Corollary9.2]) Let G be a finite π-separable group.

Then $|B_{\pi'}(G)|$ equals the number of π-regular classes of G.

Lemma2.6 ([1,Corollary10.1]) Let G be a finite π-separable group,

$\xi \in Irr(G)$ and $\eta \in B_{\pi'}(G)$ .Then there exists a nonnegative integral "decomposition number" $d_{\xi\eta}$ such that $\xi^* = \sum_{\eta \in B_{\pi'}(G)} d_{\xi\eta}\eta$ for any $\xi \in Irr(G)$.

Lemma2.7 ([1,Theorem7.1]) Let G be a π-separable group, $N \trianglelefteq G$ and G/N be a π'- group .If $\psi \in Irr(G)$ and $\chi \in Irr(G|\psi)$,then $\psi \in B_{\pi'}(N)$ if only if $\chi \in B_{\pi'}(G)$.

Lemma2.8 ([1,Corollary10.2]) Let G be a finite π -separable group.Then the map $B_{\pi'}(G) \to I_{\pi'}(G)(\psi \mapsto \psi^*)$ is a bijection.In particular, $I_{\pi'}(G)$ is a basis of the space of π'-class function of G.

Lemma2.9 ([1,Corollary10.3]) Let G be a p-solvable group. Then the map

$B_{p'}(G) \to IBr(G)(\psi \mapsto \psi^*)$ is a bijection.

Lemma2.10 ([1,Corollary6.3]) Let G be a π-separable group, $N \trianglelefteq G$ and G/N be a π- group .If $\psi \in B_{\pi'}(N)$ is a G-invariant, then there exists a unique

$\chi \in B_{\pi'}(G)$ such that $\chi_N = \psi$.

Lemma2.11([1,Corollary6.4]) Let G be a π-separable group and

$N \le K \le G$,where N is normal is G such G/N is a π-group .If $\eta \in B_{\pi'}(K)$,then there exists an irreducible constituent χ of η^G such that χ belong to $B_{\pi'}(G)$.

Lemma2.12([1,Corollary6.6]) Let G be a π-separable group and

$N \le K \le G$,where N is normal is G such G/N is a π-group .If $\chi \in B_{\pi'}(G)$,then every irreducible constituent of χ_K belong to $B_{\pi'}(K)$.

Lemma2.13([1,Corollary6.5]) Let G be a π-separable group, $N \trianglelefteq G$ and G/N be a π- group .If $\chi \in B_{\pi'}(G)$,then every irreducible constituent of χ_N belong to $B_{\pi'}(N)$.

Lemma2.14([8,Theorem6.8]) Let $D = (d_{ij})$ and $C = (c_{ij})$ be the decomposition and Cartan matrices of G,respectively.Then

(i) $\eta_i = \sum_{j=1}^{k} d_{ji}\chi_j \qquad (1 \le i \le l)$.

(ii) $C={}^tDD$.

Lemma2.15([8,Theorem6.9])

(i) For $y \in G_{p'}$,we have $\sum_{i=1}^{l} \overline{\eta_i(x)}\varphi_i(y) = \delta_{x^G, y^G} |C_G(x)|$.

(ii) If $x \in G - G_{P'}$,then $\eta_i(x) = 0$ $(1 \le i \le l)$.-

III. MAIN THEOREM

Set $Irr(G) = \{\chi_1, \chi_2, \cdots, \chi_k\}$ $B_{\pi'}(G) = \{\varphi_1, \varphi_2, \cdots, \varphi_l\}$.

By Lemma2.6[1]

$(\chi_1^*, \chi_2^*, \cdots, \chi_k^*)^t = (d_{ij})_{k \times l}(\varphi_1, \varphi_2, \cdots \varphi_l)^t$.

write $D = (d_{ij})_{k \times l}$, $C = D^tD$ .Then D and C are said to be the decomposition matrix and the Cartan matrix of G,respectively.

Let $(\eta_1, \eta_2, \cdots, \eta_l)^t = D(\chi_1, \chi_2, \cdots, \chi_k)^t$ .Then $\eta_i (1 \le j \le l)$ are said to be the principal indecomposable $B_{\pi'}$-characters of G. It is easy to verify that the relation of the principal indecomposable $B_{\pi'}$-characters and the $B_{\pi'}$-characters of G is as follows:

$$\left(\eta_1^*,\quad \eta_2^*,\cdots,\quad \eta_l^*\right)^t = C(\varphi_1,\varphi_2,\cdots,\varphi_l)^t .$$

In particular, if G is a π'-group,then $\chi_i = \eta_i = \varphi_i$,where $1 \le i \le k$.

Theorem3.1 Let G be a finite π-separable group and notation be as above. Then the following statements hold:

(1) $\sum_{i=1}^{l} \overline{\eta_i(x)}\varphi_i(y) = \delta_{x^G y^G}|C_G(x)|$ for any $y \in G_{\pi'}$;

(2) $\eta_i(x) = 0$ for any $x \in G - G_{\pi'}$,where $1 \le i \le l$.

Proof (1) For any $y \in G_{\pi'}$,we have $\chi_i(y) = \sum_{j=1}^{l} d_{ij}\varphi_j(y)$. By the second orthogonal relation of characters, one gets that

$$\begin{aligned}\delta_{x^G y^G}|C_G(x)| &= \sum_{i=1}^{k} \overline{\chi_i(x)}\chi_i(y)\\ &= \sum_{i=1}^{k} \overline{\chi_i(x)} \sum_{j=1}^{l} d_{ij}\varphi_j(y)\\ &= \sum_{i=i}^{k} \sum_{j=1}^{l} d_{ij}\overline{\chi_i(x)}\varphi_j(y)\\ &= \sum_{j=1}^{k} d_{ij} \sum_{i=1}^{l} \overline{\chi_j(x)}\varphi_i(y)\\ &= \sum_{i=1}^{l} d_{ji}\left(\sum_{j=1}^{k} \overline{\chi_j(x)}\right)\varphi_i(y)\\ &= \sum_{i=1}^{l} \overline{\eta_i(x)}\varphi_i(y)\end{aligned}$$

Consequently, we have $\sum_{i=1}^{l} \overline{\eta_i(x)}\varphi_i(y) = \delta_{x^G y^G}|C_G(x)|$.

(2) For any $x \in G - G_{\pi'}$ and $y \in G_{\pi'}$,we have $\sum_{i=1}^{l} \overline{\eta_i(x)}\varphi_i(y) = 0$

As y is arbitrary,we obtain that $\sum_{i=1}^{l} \overline{\eta_i(x)}\varphi_i = 0$ .Note that $\{\varphi_i | 1 \le i \le l\}$ is linearly independent, so the previous equality yields that $\eta_i(x) = 0$ for each $1 \le i \le l$ This completes the proof of Theorem 3.1.

Let G be a π-separable group and let φ and η be class functions defined on $G_{\pi'}$ .Write $(\varphi,\eta)' = \frac{1}{|G|}\sum_{x \in G_{\pi'}} \varphi(x)\overline{\eta(x)}$.

Theorem3.2 Let G be a π-separable group and notation be as above. Then

$$(\eta_i,\varphi_j)' = \delta_{ij} .$$

Proof By lemma 2.5 we may assume that

$$Cl(G_{\pi'}) = \{C_1, C_2, \cdots C_l\},$$

where $x_i \in C_i (1 \le i \le l)$

Write $\Phi = (\varphi_i(x_j))_{l \times l}$, $Y = (\eta_i(x_j))_{l \times l}$ and

$$S = \begin{pmatrix} |C_G(x_1)| & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & |C_G(x_l)| \end{pmatrix}$$

By Theorem $3.1(1)$, we have $\overline{Y}^t \Phi = S$ and thus $\overline{Y}^t (\Phi S^{-1}) = I$, $i,e.,$

$$(\Phi S^{-1})^t \overline{Y} = I .$$

Therefore, $\sum_{\nu}^{l} \varphi_i(x_\nu)\frac{1}{|C_G(x_\nu)|}\overline{\eta_i(x_\nu)} = \delta_{ij}$, $i,e.,$

$$\frac{1}{|G|}\sum_{\nu=1}^{t} \varphi_i(x_\nu)\frac{|G|}{|C_G(x_\nu)|}\overline{\eta_i(x_\nu)} = \delta_{ij} ,$$

On the other hand ,

$$\begin{aligned}&\frac{1}{|G|}\sum_{\nu=1}^{t} \varphi_i(x_\nu)\frac{|G|}{|C_G(x_\nu)|}\overline{\eta_i(x_\nu)}\\ &= \frac{1}{|G|}\sum_{\nu=1}^{l} |Cl(x_\nu)|\varphi_i(x_\nu)\overline{\eta_i(x_\nu)}\\ &= \frac{1}{|G|}\sum_{x \in G_{\pi'}} \varphi_i(x)\overline{\eta_j(x)}\\ &= (\varphi_i,\eta_j)' = \delta_{ij} .\end{aligned}$$

Consequently, $(\varphi_i,\eta_j)' = \delta_{ij}$.

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