

The Definition and Properties of New Connectedness in Topology Space

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Abstract--Connectedness is an important property in topology space. Based on the definition of connectedness, in this paper, we defined strongly connectedness, that is, the connected spaces were enlarged by concept of the connectedness. It can be found that strongly connected spaces have many interesting properties. Meanwhile, locally strongly connected space and strongly connected component were investigated.

Keywords—connected space; strongly connected space; strongly connected component; locally strongly connectivity

I. INTRODUCTION

Connectedness [1,2,3] is an important aspect of the research in topology space, which has received considerable attention in many fields, with its relative notions used in both mathematic fields such as algebra, geometry, graph theory, ect. And non-mathematic fields such as geographical information system, etc. Meanwhile, it is also involved in the design field of robot kinematics whose technology is now under rapid development.

Although the notion of connectedness is simple, it exerts far-reaching impacts on topology and its application. As to the widely used intermediate value theorem, the key is to guarantee the connectedness of space.

The study on connectedness started when the great mathematician Euler solved the problem of Konigsberg seven bridges. Euler pointed out that “one picture can be drawn with one line only when it is connected and it has no odd vertices or it has two odd vertices”. At that time, the notion of connectedness was not very clear. With the establishment of topology, as one of its invariance, related properties and conclusions appeared successively. Then, the combination of connectedness with countability, separability and compactness enriched the conclusions that had been independent but with many condition constraints. As a result, many practical counterexamples appeared. For example, Roy [4] constructed a countable Hausdorff connected space, Martin [5] also constructed such a space and pointed out that “Does there exist a countable Uryson connected space? The answer is affirmative. Roy, Kallnan [6], Miller [7] all made related constructions. Bing [8] constructed a countable Hausdorff connected space but it was not strongly connected

space. Reference [9] has given connected subset of R^2 but it is not strongly connected space. More discussions about strong connectivity can be found in references [11,12,13,14].

Since American cybernetic expert professor LA. Zadeh put forward the notion of Fuzzy subset in 1965, fuzzy mathematics has made substantial progress, which has opened new ways of thinking for researches in connectedness. There are many scholars who have combined connectedness with fuzzy mathematics and conducted further studies.

Strong connectivity is a special connectedness, but there aren't many in-depth studies on strongly connected space in fuzzy mathematics. This paper mainly conducts related studies on strong connectivity in general topology and discusses the characteristics of locally strong connectivity.

II. CONCEPT OF STRONGLY CONNECTED SPACE

When discussing the concept of connectedness, we often come across the equivalent criterion that a space is connected if and only if any continuous map from it to the discrete space $\{0,1\}$ is constant. It would be interesting to see what concept arises if the discrete space of two points is replaced by some other spaces.

The definition of connectedness, a space is connected if and only if any continuous map f from X to the discrete space $\{0,1\}$ is constant. It is natural to seek a similar definition for strongly connected spaces. The concept of strongly connected space is obtained by replacing the discrete space $\{0,1\}$ by some other space.

Definition 2.1 A space X is strongly connected if and only if it is not a disjoint union of countable many but more than one closed set.

Definition 2.2 A subset Y in a space X is said to be strongly connected if the subset Y with the subspace topology is a strongly connected space.

According to the above definition, Y is a strongly connected space if and only if Y is not a disjoint union of countable many but more than one closed set.

Note the similarity between this definition and that of connectedness.

If X is connected if and only if

$$X \neq E_1 \cup E_2, \quad E_1 = \overline{E_1} \neq \emptyset, \quad E_2 = \overline{E_2} \neq \emptyset, \\ E_1 \cap E_2 = \emptyset.$$

If X is strongly connected spaces if and only if

$$X \neq E_1 \cup E_2 \cup E_3 \cup \dots, \\ E_i = \overline{E_i} \neq \emptyset, E_i \cap E_j = \emptyset, (i \neq j), i, j = 1, 2, 3, \dots.$$

There is another equivalent criterion, which is often taken as the definition. It makes use of the concept of continuous map.

A space is connected if and only if any continuous map f from X to the discrete space $\{0, 1\}$ is constant. Similarly, we can seek the concept of strongly connected spaces.

Definition 2.3 Let Z be a topological space with more than one point. A space X is strongly connected spaces if and only if any continuous map from X to Z is constant.

Proposition 2.1 Definition 2.1 and definition 2.3 are equivalent definitions.

Proof: \Rightarrow If X is strongly connected spaces, let $f : X \rightarrow Z$ is continuous map.

Since $f(X)$ is strongly connected and the only strongly connected subset in Z is one point set. Therefore, f is constant map.

\Leftarrow Suppose X is not strongly connected spaces, namely $X = \cup E_i, i = 1, 2, \dots$.

$$\text{Define } f : X \rightarrow Z, f(x) = i, x \in E_i.$$

Then this f is continuous and not constant. It contradicts the strongly connectedness of X .

Note that in the above definition, Z is restricted to be a space with more than one point. Otherwise, the image of X is always constant and the definition makes no sense.

III. PROPOSITION OF STRONGLY CONNECTED SPACE

Proposition 3.1 A continuous image of strongly connected space is connected space.

proof: Let X be any strongly connected spaces, let $f : X \rightarrow Z$ be a continuous surjection.

Suppose $f(X)$ is not strongly connected spaces, by definition,

$$f(X) = E_1 \cup E_2 \cup E_3 \cup \dots,$$

$$E_i = \overline{E_i} \neq \emptyset, E_i \cap E_j = \emptyset, (i \neq j), i, j = 1, 2,$$

Since f is continuous, and the inverse image of closed sets are still closed, i.e., $f^{-1}(E_i)$ is closed, then

$$X = f^{-1}(E_1) \cup f^{-1}(E_2) \cup f^{-1}(E_3) \cup \dots.$$

It contradicts the strongly connectedness of X , therefore $f(X)$ is strongly connected spaces.

Proposition 3.2 A strongly connected space is connected spaces.

proof: Since Z has at least two points, there exists a continuous injection i such that $i : \{0, 1\} \rightarrow Z$, for any continuous map

$$f : X \rightarrow \{0, 1\}$$

Then $i \circ f$ also a continuous function.

Now X is connected, by definition, f is constant, therefore X is connected.

Proposition 3.3 If $\{X_\alpha\}$ is a collection of strongly connected subspaces of a space X such that $\bigcap_\alpha X_\alpha \neq \emptyset$, then $\bigcup_\alpha X_\alpha$ is strongly connected spaces.

proof: Let $f : \bigcup_\alpha X_\alpha \rightarrow Z$ be any continuous map and $i : X_\alpha \rightarrow \bigcup_\alpha X_\alpha$ be the inclusion map.

Since X_α is strongly connected,

$$f \circ i : X_\alpha \rightarrow Z$$

is continuous and constant, and $\bigcap_\alpha X_\alpha \neq \emptyset$, so there exist a p such that $p \in X_\alpha$ for all α . Then therefore f is constant.

Proposition 3.4

Let $A \subseteq X, B \subseteq X$ and $A \subseteq B \subseteq \overline{A}$, A is strongly connected spaces, then B is strongly connected spaces.

Proof: Suppose B is not strongly connected spaces, and

$$B = \bigcup E_i, E_i = \overline{E_i} \neq \emptyset,$$

$$E_i \cap E_j = \emptyset, (i \neq j), i, j = 1, 2, \dots.$$

Since A, B is strongly connected, so $A \subseteq E_i$ or $A \subseteq E_j$ ($i \neq j$). we can suppose $A \subseteq E_i$ and $A \subseteq E_i \subset B$, then

$$\overline{A}^B \subseteq \overline{E_i}^B, \overline{A}^B = \overline{A} \cap B = B \supseteq \overline{E_i}^B.$$

Therefore $B = \overline{E_i}$. It contradicts with the given conditions, so B is strongly connected spaces.

Proposition 3.5 The topological product of an arbitrary family of strongly connected spaces is strongly connected.

proof: First, suppose X, Y is strongly connected spaces, we prove $X \times Y$ is strongly connected spaces. Let

$$(a, b) \in X \times Y, (c, d) \in X \times Y,$$

Then $X \times \{b\}$ and $\{c\} \times Y$ is strongly connected,

$$X \times \{b\} \cap \{c\} \times Y = (c, b).$$

By Proposition 2.3, $X \times \{b\} \cup \{c\} \times Y$ is strongly connected.

$$X \times Y = \bigcup \{(X \times \{b\}) \cup (\{c\} \times Y)\}$$

$$\bigcap \{(X \times \{b\}) \cup (\{c\} \times Y)\} = (a, b).$$

Then $X \times Y$ is strongly connected spaces.

Next, suppose an arbitrary family $\{X_s\}_{s \in S}$ is strongly connected spaces, there exist $\alpha_s \in X_s$ for all $s \in S$.

Suppose $R = \{T : T \subset S\}$, T is finite,

$$C_T = \prod_{s \in S} A_s, A_s = \begin{cases} \{\alpha_s\} & s \notin T \\ X_s & s \in T \end{cases} \text{ for all } T \in R$$

By above conclusion: $\{C_T\}_{T \in R}$ is strongly connected spaces.

Since $\bigcap_{T \in R} C_T \neq \emptyset$ and $C = \bigcup_{T \in R} C_T$ is strongly connected spaces.

We prove C is dense set in $\prod_{s \in S} X_s$, i.e., $\overline{C} = \prod_{s \in S} X_s$.

In fact, let $U = \prod U$ be open neighbourhood of x and x be a given point in X . $U \cap C \neq \emptyset$, then $U_\alpha \in X_\alpha$ is open set and $U_\alpha = X_\alpha$ for $s \notin T$.

We suppose $c \in X$ such that $c = b$ for $s \in T$. Since $c \in X_s, c \in C$ and $c \in U$. Thus

$$c = x \in U \text{ for } s \in T \text{ and } c = b \in X \text{ for } s \notin T$$

Therefore $U \cap C \neq \emptyset$, i.e. $\overline{C} = \prod_{s \in S} X_s$. By Proposition

3.4, $\prod_{s \in S} X_s$ is strongly connected spaces.

IV. PROPOSITION OF STRONGLY CONNECTED SPACE COMPONENT

Definition 4.1 A strongly connected component C_x is defined as the largest strongly connected set in X which contains the point. See [15, 16]

By definition, let E be strongly connected set in X and C_x be strongly connected component, then $E \subset C_x$, i.e., any strongly connected set in X is included in a strongly connected component.

Proposition 4.1 Any strongly connected set A in X is included in the uniquely strongly connected component.

proof: Suppose A is strongly connected set A in X ,

$R = \{F \subset X \mid F \text{ is strongly connected set, } F \cap A \neq \emptyset\}$, $Y = \bigcup_{F \in R} F$

Then $A \subset Y$. By Proposition 3.3, Y is strongly connected.

Suppose $B \supset Y$ is strongly connected set in Y , then $B \cap A = A \neq \emptyset$, and $B \subset R$, $B \subset Y$, therefore Y is strongly connected component.

Suppose Y' is strongly connected component in A , then $Y' \in R$ and $Y' \subset Y$,

Since Y' is largest strongly connected set, thus $Y' = Y$.

Proposition 4.2 Each strongly connected component in X is closed.

proof: Let A be strongly connected component, i.e., A is largest strongly connected set in X containing p . By Proposition 2.4, \overline{A} is strongly connected, and $\overline{A} \supseteq A$, then $\overline{A} = A$. Therefore strongly connected component A in X is closed.

V. PROPOSITION OF LOCALLY STRONGLY CONNECTED SPACE

Definition 5.1 A space is locally strongly connected at a point x , if every neighbourhood U of x contains a strongly connected neighbourhood V of x , such that $x \in V \subset U$.

X is said to be locally strongly connected if it is locally strongly connected at each of its points.

Proposition 5.1 Strongly connected component of locally strongly connected spaces is open.

proof: Suppose X is locally strongly connected spaces, its topological base \mathfrak{B} such that $\forall B_i \in \mathfrak{B}, B_i$ is strongly connected.

Let A be strongly connected component in X . There exist $B \in \mathfrak{B}$ for $x \in A$, such that $x \in B$. By definition of

strongly connected component, $B \subseteq A$, then A is the union of open set, therefore it is open.

Proposition 5.2 Let X be locally strongly connected spaces, and $A \subset X$ is open set in X , then A is locally strongly connected.

proof: Let X be locally strongly connected spaces, its topological base β such that $\forall B_i \in \beta, B_i$ is strongly connected.

Suppose A is open subspace, then

$$\beta^* = \{B \in \beta \mid B \subset A\}$$

is strongly connected open set, then A is locally strongly connected spaces.

Proposition 5.3 Let X be locally strongly connected spaces, X is connected if and only if X is strongly connected.

proof: \Leftarrow : Suppose X is strongly connected, by Proposition 3.2, X is connected.

\Rightarrow : Suppose X is connected and locally strongly connected, A is the strongly connected component of X , by Proposition 4.2, A is open in X . By Proposition 4.3, A is closed in X .

Since X is connected, $A = X$. Therefore X is strongly connected.

Proposition 5.4 $\{X_\alpha\}_{\alpha \in D}$ are the family spaces and $X_\alpha \cap X_\beta = \emptyset, (\alpha \neq \beta)$, then $\bigoplus_{\alpha \in D} X_\alpha$ is locally strongly connected spaces if and only if $\forall \alpha \in D, X_\alpha$ is locally strongly connected spaces.

proof: \Leftarrow : Let $X = \bigoplus_{\alpha \in D} X_\alpha$. By the given conditions, $\forall \alpha \in D, X_\alpha$ is locally strongly connected spaces. $\forall x \in X$, there exist $x \in X_{\alpha_1}$ for $\alpha_1 \in D$.

Suppose U containing x is open in X , then $U \cap X_{\alpha_1}$ is open in X_{α_1} , and X_{α_1} is locally strongly connected spaces, thus there exist strongly connected open set V in X_{α_1} , such that $x \in V \subset U \cap X_{\alpha_1} \subset U$.

Since X_{α_1} is open set in, then V is strongly connected open set in X , therefore $\bigoplus_{\alpha \in D} X_\alpha$ are locally strongly connected spaces.

\Rightarrow : Suppose $\bigoplus_{\alpha \in D} X_\alpha$ is locally strongly connected spaces, $\forall \alpha \in D, x \in X_\alpha$, and U containing x is open set of X_α , then $x \in X$, and U containing x is open set of X .

Since X is locally strongly connected spaces, then there exist strongly connected open set V in X , such that $x \in V \subset U$, and X_α is open and close subspace, therefore V is locally strongly connected open set in X , i.e., X_α is locally strongly connected spaces.

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