Modeling and Quadratic Stability Control of Buck Converter in DCM

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Abstract. Quadratic stability is an important performance for control systems. In this paper, the model of Buck Converter in DCM is built based on the theories of hybrid systems (HS) and switched linear systems (SLS) primarily. Then quadratic stability of SLS and two switching rules based on state feedback and hybrid feedback are introduced. The problem of Buck Converter's quadratic stability is researched afterwards. In the end, the simulation analysis and verification are provided. Both experimental verification and theoretical analysis results indicate that the output of Buck Converter in DCM has an excellent performance via quadratic stability control and switching rules. This analysis method can be also used directly to study the quadratic stability of other switched converters.

Introduction

With the development of modern control theory in recent years, we realize the control systems from linear systems (LS), nonlinear systems (NLS) up to hybrid systems (HS). HS, as a new concept, is defined as a unitive dynamic system interacted by discrete and continuous parts. To study DC-DC converters by HS theory is significant because they are typical HS. For DC-DC Converters, the operation of each mode should be regard as the continuous dynamic subsystems and the turn-on or turn-off of power switch as the discrete dynamic subsystems.

Among some models of HS, such as hybrid automata, hybrid Petri net, switched linear systems (SLS), as an important type of HS, have attracted considerable attention in modeling, analysis and design. SLS is a collection of a series of continuous variable subsystems. In this model of SLS, the switching conditions of subsystems, which can be also regarded as switching rules, are emphasized. In other words, only one subsystem actives at any moment and the system is switched during several different subsystems based on switching signals.

In order to achieve a good performance, switching must be fast enough that may lead to chattering and high switching power loss. Therefore one of the important problems of SLS is the stability, which is frequently defined as quadratic stability. Stability in SLS is more complex since it depends on the switching rules as the stability of all the subsystems.

Looking from the existing literatures[1,2], Lyapunov theories are the dominant approaches used in study of stability. For example, the stability of DC-DC Converters in CCM (Continuous Current Mode) was analyzed in [3]. The aim of this paper is to study the quadratic stability of DC-DC converters via using Lyapunov function based on the model in DCM (Discontinuous Current Mode).

Modeling of Buck Converter in DCM

Buck Converter in DCM. The topology structure of Buck Converter is shown in Fig. 1. Assume all of the components are perfect. Three work modes of Buck converter in DCM are shown in Fig. 2.

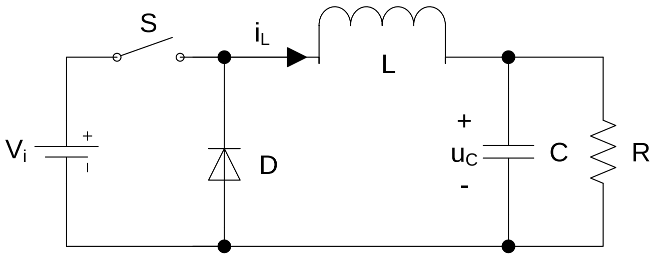
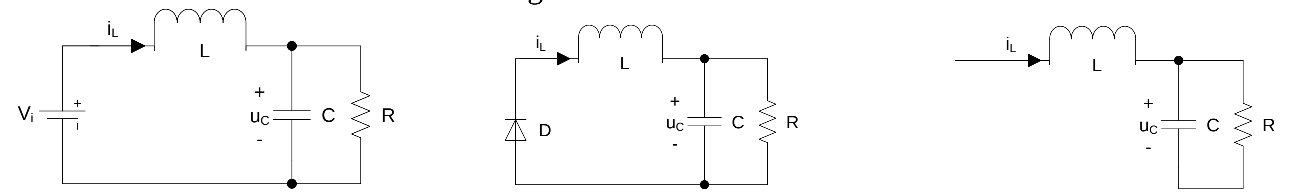


Fig.1 Buck Converter



(a) Switch tube is on (b) Switch tube is off and diode is on (c) Switch tube and diode are all off

Fig.2 Buck Converter in DCM

Model of SLS. Generally speaking, a switched linear system can be described as [4]

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)} x(t) + B_{\sigma(t)} \\ y(t) = C_{\sigma(t)} x(t) + D_{\sigma(t)} \end{cases}. \tag{1}$$

Where $x(t)$ is the state vector of n dimension, $y(t)$ is the output vector of q dimension, the piecewise constant function $\sigma(t):[0,+\infty) \to \eta=\{1,2,\cdots,m\}$ is the switching signal. Moreover, $\sigma(t)=i$ implies that the switched mode (A_i, B_i, C_i, D_i) is active.

SLS Model of Buck Converter in DCM. Supposing all of the components are ideal and the state vector is $x(t) = [\, i_L \;\; u_C \,]^T$,the output vector is $y(t) = u_C$. Consequently, the parameters matrixes of Buck Converter in DCM are expressed as follows

(a) Switch tube is on: $A_1 = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix}$, $B_1 = \begin{bmatrix} \frac{V_{in}}{L} \\ 0 \end{bmatrix}$, $C_1 = [0 \quad 1]$, $D_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(b) Switch tube is off and diode is on: $A_2 = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix}$, $B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $C_2 = [0 \quad 1]$, $D_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(c) Switch tube and diode are all off: $A_3 = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{RC} \end{bmatrix}$, $B_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $C_3 = [0 \quad 1]$, $D_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

So it can be seen that Buck Converter in DCM has three switched modes, (A_1, B_1, C_1, D_1), (A_2, B_2, C_2, D_2) and (A_3, B_3, C_3, D_3),which belong to three subsystems respectively, Σ_1, Σ_2 and Σ_3.

Quadratic Stability of SLS

Propaedeutics. As for the quadratic stability of system (1), the problem is to find a switching signal such that a given point $\bar{\boldsymbol{x}}$ is a stable equilibrium. And $\bar{\boldsymbol{x}}$ may be called a switched equilibrium due to the regulation can be achieved only through switching, even if all the subsystems are asymptotically stable.

Since any other equilibrium point can be shifted to the origin via a change of variable $\tilde{x} = x - \bar{x}$,we can assume the switched equilibrium is the origin $\bar{x} = 0$ without loss of generality. Quadratic stability for SLS can be defined as follows.

Definition: if and only if there exist a matrix $P = P^T > 0$ and a constant $\varepsilon > 0$ such that for the quadratic function $V(x) = x^T P x$, we have $\dot{V}(x(t)) \le -\varepsilon x^T x$ along all system trajectories, the switched equilibrium $\bar{\boldsymbol{x}} = 0$ can be said quadratically stable.

The derivative $\dot{V}(x(t))$ [5] is defined as follows

(a) when the subsystem $\sum_i$ is active

$\dot{V}(x(t)) = \dot{V}_i(x(t)) = x^T(A_i^T P + PA_i)x + 2b_i^T Px$,

(b) at the switching point between subsystem $\sum_i$ and subsystem $\sum_j$

$\dot{V}(x(t)) = \sup_{\gamma\in[0,1]}\{\gamma\dot{V}_i(x(t)) + (1-\gamma)\dot{V}_j(x(t))\}$.

If we denote $A_{eq} = \sum_{i=1}^m \alpha_i A_i$ and $b_{eq} = \sum_{i=1}^m \alpha_i b_i$,where $0<\alpha_i<1$ and $\sum_{i=1}^m \alpha_i = 1$, a convex combination of the subsystems is defined as follows

$$\textstyle\sum_{eq} : \dot{x} = A_{eq}x + b_{eq} . \tag{2}$$

Quadratic Stability.

Theorem: As for the system (l), the point $\bar{\boldsymbol{x}} = 0$ is a quadratic stable switched equilibrium if there exist $\alpha_i \in (0,1)$, i =1,...,m such that

$$\sum_{i=1}^m \alpha_i = 1 , \tag{3}$$

$$A_{eq} = \sum_{i=1}^m \alpha_i A_i \text{ is Hurwitz,} \tag{4}$$

$$b_{eq} = \sum_{i=1}^m \alpha_i b_i = 0 . \tag{5}$$

Since the convex combination A_{eq} in Eq. 4 is stable, there exist two positive definite symmetric matrices P and Q such that

$$A_{eq}^T P + PA_{eq} = -Q . \tag{6}$$

According to Eq. 4, Eq. 6 can be rewritten as follows

$$\sum_{i=1}^m \alpha_i x^T(A_i^T P + PA_i)x = -x^T Qx . \tag{7}$$

We can also get the null term

$$2(\alpha_1 b_1 + \alpha_2 b_2 + ... + \alpha_m b_m)^T Px = 0 \tag{8}$$

from Eq. 5.

Now, a new equation is obtained as follows

$$\begin{aligned} &\alpha_1(x^T(A_1^T P + PA_1)x + 2b_1^T Px) + \alpha_2(x^T(A_2^T P + PA_2)x + 2b_2^T Px) + ... \\ &+ \alpha_m(x^T(A_m^T P + PA_m)x + 2b_m^T Px) = -x^T Qx \le -\varepsilon x^T x \end{aligned} . \tag{9}$$

Where $0<\varepsilon\le \lambda_{\min}$ and $\lambda_{\min}$ is the smallest positive real eigenvalue of Q. Then Eq. 9 is equivalent to Eq. 10 as follows

$$\begin{aligned} &\alpha_1(x^T(A_1^T P + PA_1)x + 2b_1^T Px + \varepsilon x^T x) + \alpha_2(x^T(A_2^T P + PA_2)x + 2b_2^T Px + \varepsilon x^T x) + ... \\ &+ \alpha_m(x^T(A_m^T P + PA_m)x + 2b_m^T Px + \varepsilon x^T x) \le 0 \end{aligned} . \tag{10}$$

Consequently we can conclude that there exists a figure i, which may lead to the inequality $x^T(A_i^T P + PA_i)x + 2b_i^T Px + \varepsilon x^T x \le 0$ satisfied for every nonzero x. This inequality is equivalent to $x^T(A_i^T P + PA_i)x + 2b_i^T Px \le -\varepsilon x^T x$.

Now, m regions are defined as follows

$\Omega_i = \{x | x^T (A_i^T P + PA_i) x + 2b_i^T Px \le -\varepsilon x^T x\}, i \in \{1,2,...,m\}$.

These m regions are closed to overlap and cover $\Re^n \backslash \{0\}$. Quadratic stability is assured when the subsystem $\sum_i$ is active in region Ω_i using the Lyapunov function $V(x) = x^T Px$. And Eq.11 is satisfied within the region Ω_i

$$\dot{V}(x(t)) = \dot{V}_i(x(t)) = x^T (A_i^T P + PA_i) x + 2b_i^T Px \le -\varepsilon x^T x. \tag{11}$$

While at the switching points (which are interior to the region $\Omega_i \cap \Omega_j$)

$$\dot{V}(x(t)) = \sup_{\gamma \in [0,1]} \{\gamma \dot{V}_i(x(t)) + (1-\gamma)\dot{V}_j(x(t))\} \le \max_{i=1,2,...,m} \{\dot{V}_i(x(t))\} \le -\varepsilon x^T x. \tag{12}$$

Switching Rules

State Feedback Switching Rule. In this rule, it takes the form of a state-feedback and the subsystem with the highest rate of decrease of $V(x)$ is activated

$$\sigma(x) = \arg\min\{\dot{V}_i(x)\}. \tag{13}$$

Therefore, the activation region of the i-th subsystem is defined as

$$\Psi_i = \{x | \dot{V}_i(x) < \dot{V}_j(x), \forall j \ne i\}. \tag{14}$$

In the strategy of state feedback, the highest decrease rate $\dot{V}(x(t))$ can be assured but the occurrence of sliding modes may not be avoided. The sliding modes may appear in some region of the state space, even when the state is far away from the switched equilibrium.

Hybrid Feedback Switching Rule. The occurrence of sliding modes can be avoided in hybrid feedback switching rule via hysteresis. The procedures of strategy of hybrid feedback switching rule is listed as follows

(*a :initialization*) at time t = 0 activate the subsystem $\sum_{i_0}$ with $i_0 = \arg\min\{(\dot{V}_i(x_0))\}$.

(*b: switching off rule*) if subsystem $\sum_i$ is active and $x^T (A_i^T P + PA_i) x + 2b_i^T Px > -\varepsilon x^T x$,the system will switch to subsystem $\sum_j$ with $j = \arg\min\{(\dot{V}_j(x))\}$.

(*c: equilibrium neighbourhood rule*) if $\|x\| \le \rho_{off}$ switching will be stopped until $\|x\| \ge \rho_{on}$ ($\rho_{off} < \rho_{on}$) .

The admissible region of activation (where subsystem $\sum_i$ can be active) can be defined as follows

$$\Phi_i = \{x | \dot{V}_i(x) + \varepsilon x^T x < 0\}. \tag{15}$$

Thus this strategy assures that the interval between two consecutive switching is always bounded away from zero.

Quadratic Stability of Buck Converter

Tab. 1 Parameters Matrixes of Buck converter after Coordinate Transformating

$\sum_1$:w=1, v=1 (Switch tube is on)	$\sum_2$:w=1, v=0 (Switch tube is off and diode is on)	$\sum_3$:w=0, v=0 (Switch tube and diode are all off)
$A_1 = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix}$	$A_2 = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix}$	$A_3 = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{RC} \end{bmatrix}$
$B_1 = \begin{bmatrix} \frac{V_{in}}{L} - \frac{1}{L}\bar{x}_2 \\ \frac{1}{C}\bar{x}_1 - \frac{1}{RC}\bar{x}_2 \end{bmatrix}$	$B_2 = \begin{bmatrix} -\frac{\bar{x}_2}{L} \\ \frac{1}{C}\bar{x}_1 - \frac{1}{RC}\bar{x}_2 \end{bmatrix}$	$B_3 = \begin{bmatrix} 0 \\ -\frac{1}{RC}\bar{x}_2 \end{bmatrix}$

Coordinate Transformation of Buck Converter. Assume the anticipant output $\bar{x} = [\bar{x}_1 \quad \bar{x}_2]^T$ as the switched equilibrium, we use $\tilde{x} = x - \bar{x}$ for conversion of coordinates to transfer switched equilibrium to original point. The state equations after coordinate transformation are given as follows

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{L} w \\ \frac{1}{C} w & -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} \frac{V_{in}}{L} v - \frac{1}{L} w\bar{x}_2 \\ \frac{1}{C} w\bar{x}_1 - \frac{1}{RC} \bar{x}_2 \end{bmatrix}. \tag{16}$$

The parameters matrixes of three subsystems, $\sum_1$, $\sum_2$ and $\sum_3$ after coordinate transformation are presented in Tab.1.

Switching Control of Buck Converter. As for Buck Converter in DCM, there exist three coefficients α_1 ,α_2 and α_3 in the convex combination of Eq.2 .According to Eq.3, we have α_3 =1- (α_1 +α_2). Then, the convex combination can be written as follows

$$A_{eq} = \begin{bmatrix} 0 & -\frac{\alpha_1 + \alpha_2}{L} \\ \frac{\alpha_1 + \alpha_2}{C} & -\frac{1}{RC} \end{bmatrix}, \quad b_{eq} = \begin{bmatrix} \frac{\alpha_1 V_{in} - (\alpha_1 + \alpha_2)\bar{x}_2}{L} \\ \frac{(\alpha_1 + \alpha_2)R\bar{x}_1 - \bar{x}_2}{RC} \end{bmatrix}.$$

Moreover, there exists a relationship between α_1 , α_2 and switched equilibrium $\bar{x}$ as follows based on Eq.5.

$$\alpha_1 = \frac{\bar{x}_2^2}{R\bar{x}_1 V_{in}} \quad , \alpha_2 = \frac{\bar{x}_2 V_{in} - \bar{x}_2^2}{R\bar{x}_1 V_{in}} \quad .$$

Then a Lyapunov function appropriately must be found. Since Lyapunov functions can always be chosen as the total energy of the system in a port control Hamilton system, such as in a mechanical or an electrical system, we assume $P = \begin{bmatrix} L & 0 \\ 0 & C \end{bmatrix}$, so the Lyapunov function is

$$V(\tilde{x}) = \tilde{x}^T P \tilde{x} = \frac{1}{2} L\tilde{x}_1^2 + \frac{1}{2} C\tilde{x}_2^2 = \frac{1}{2} Li_L^2 + \frac{1}{2} Cu_C^2 .$$

Obviously $V(\tilde{x})$ is an energy function. Therefore we can also get

$$A_{eq}^T P + PA_{eq} = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{2}{R} \end{bmatrix} = -Q .$$

Where Q is semi-definite positive. Q can be chosen as a semi-definite matrix because $\dot{V}(\tilde{x})$ is not identically vanishing along all system trajectories for DC-DC converters. Let $\lambda_{\min} = 2/R$ is the smallest positive real eigenvalue of Q and $0<\varepsilon \le \lambda_{\min}$. Thus the regions of subsystems can be divided according to state feedback switching rule or hybrid feedback switching rule. Then we may use this approach to regulate the output of Buck converter to the desired switched equilibrium.

Simulation of Buck Converter. In this section, simulation results of Buck converter based on hybrid feedback switching rule are presented in Fig. 3. Fig. 4 demonstrates the simulation results when Buck Converter is open-loop as a compared object. The parameters of Buck Converter are given as follows: V_{in}=24V, f=20KHz, R=10Ω, L=400μH, C=750 μF.

The load resistor R has two saltations during the process of simulation, one is from 10Ω to 20Ω when t=0.2s, the other is from 20Ω to 10Ω when t=0.4s. According to Fig. 4,we can find that the output voltage reach its steady state with some overshoots and it spends more time to stabilize compared with the converter in Fig .3 when the load resistor changed. So the hybrid feedback switching controller appropriates good performance of transient and steady state dynamic responses.

Conclusion

In this paper, the model of Buck Converter in DCM, which is based on the concept and theory of switched linear systems, is built first and foremost. Then, the quadratic stability and switching rules of it are studied. Afterwards, simulation results are presented. This approach can also be extended to other converters and port controlled Hamilton linear switched system. And the research results are beneficial to the development of nonlinear control strategy and the practical applications of power electronic systems.

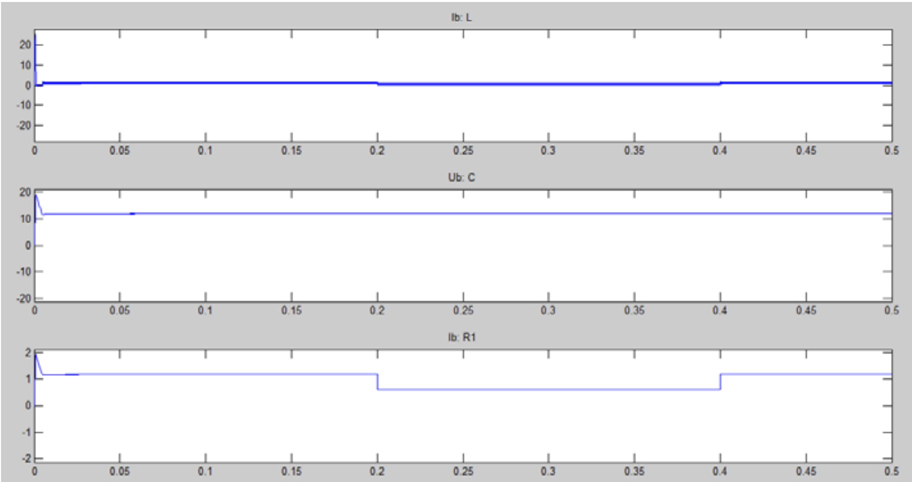


Fig.3 Simulation Results of Buck Converter on Hybrid Feedback switching control rule

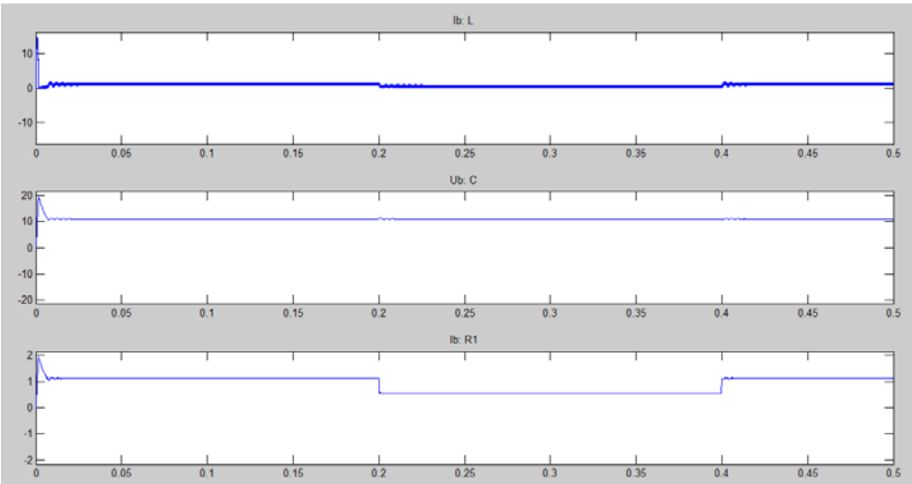


Fig.4 Simulation Results of Buck Converter in Open-loop Control

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