Exact Travelling Wave Solutions of Nonlinear Wave equations Using Tanh-function Method

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Abstract. This paper is concerned with the exact traveling wave solutions of nonlinear wave equations. Using the tanh function method, we can obtain the accurate expression of the solutions. Further, according to the images of the solutions, we can get the variation depends on the velocity v.

Introduction

It is well known that nonlinear phenomena are very important in a variety of scientific fields, especially in fluid mechanics, solid state physics, plasma physics, plasma waves, capillary-gravity waves, and chemical physics. Most of these phenomena are described by nonlinear partial differential equations. Analytical solutions of this problems are usually not available, especially when the nonlinear terms are involved. Therefore, finding its travelling solutions is of practical importance.

The methods of looking for exact traveling wave solutions of nonlinear evolution equations, has been tremendous development in recent decades, such as inverse scattering method [2], HI Rota's bilinear technique [5], the Painlve expansion method [13]. In the early nineties of last century, Huibin and Kelin [7] proposed a new method. The main idea of this method is taking hyperbolic tangent function of the power series as possible traveling wave solutions of the nonlinear evolution equations. Then they substituted the power series directly to KdV equation, and obtained the coefficients of the power series. However this method involved very complicated algebra computation. In order to reduce the complex algebra computation, Malfiety [9-11] proposed the tanh-function method. Since all the derivatives of hyperbolic tangent can be expressed by the hyperbolic tangent in itself, this simple translation makes the method can be applied to more nonlinear evolution equations. Fan et al. [3] proposed the extended hyperbolic tangent method, which replace the tanh-function by the solutions of Riccati equation. In [1, 4, 14, 16, and 17], using the tanh function method, they got the exact form of traveling wave solutions of various types of evolution equations. In recent years, the G′/G function method [15], the auxiliary function method [6] is based on tanh-function method. This shows that the hyperbolic tangent function method is very effective and direct method when looking for the exact traveling wave solutions of nonlinear evolution equations.

The Tanh-function Method

Let's consider the nonlinear partial differential equations

$$N\left(u, u_t, u_x, u_{xx}, u_{xxx}, \cdots\cdots\right) = 0 \tag{2.1}$$

Where $u(x, t)$ is the real function on R^2 ? At first, we assume the traveling wave solutions of (2.1) are the form of

$$u(x, t) = U(\omega) = U(c(x - \upsilon t)), \tag{2.2}$$

With the velocity v, and the constant c. Submitted (2.2) into (2.1), we can get the ODEs About ω

$$N(U, U', U'', U''', ...) = 0. \tag{2.3}$$

Second, we assume the possibly traveling wave solutions can be written

$$u(\mathrm{x,t}) = \mathrm{U}(\omega) = \mathrm{H(Y)} = \sum_{i=0}^{K} a_i Y^i, \tag{2.4}$$

Where $Y = \tanh(\omega) = \dfrac{e^{\omega} - e^{-\omega}}{e^{\omega} + e^{-\omega}}$ the highest order K will be determined late. Then we can;

Get;

$$\frac{dY}{d\omega} = 1 - Y^2, \frac{dU}{d\omega} = (1-\mathrm{Y}^2)\mathrm{H}', \frac{d^2U}{d\omega^2} = (1-\mathrm{Y}^2)(-2Y.\mathrm{H}' + (1-\mathrm{Y}^2)\mathrm{H}''),$$

$$\frac{d^3U}{d\omega^3} = (1-\mathrm{Y}^2)\big((6Y^2-2)\mathrm{H}' + 6Y(1-\mathrm{Y}^2)\mathrm{H}'' + (1-Y^2)^2\mathrm{H}'''\big),$$

Submitted above equations into (2.3), we can get the ODEs with Y

$$N(\mathrm{Y,H,H',H'',H''',...}) = 0 \tag{2.5}$$

Where $H' = \dfrac{dH}{dY}$ To determine the parameter K, we usually balance the nonlinear term and the highest order derivative term in equation (2.5). Then, we submitted (2.4) (with the determined K) into (2.5), and get the polynomial equation with Y. Collecting all the coefficients of power of Y, and letting the coefficients of each power of Y to be vanished, we can determined all the coefficient $a_1, a_2, \cdots\cdots, a_K$. According to (2.4), we can get the traveling wave solutions of (2.1).

The Nonlinear Wave Equations

Consider the nonlinear wave equations;

$$u_{tt} = u_{xx} + u(1-u^2). \tag{3.1}$$

Submitted (2.2) into (3.1), we can get;

$$c^2 v^2 U'' = c^2 U'' + U(1-\mathrm{U}^2). \tag{3.2}$$

According to (2.5), we have;

$$c^2 v^2 U''(-2\,\mathrm{YH}' + (1-\mathrm{Y}^2)\mathrm{H}'') = c^2(1-\mathrm{Y}^2)(-2Y\mathrm{H}' + (1-\mathrm{Y}^2)\mathrm{H}'') + \mathrm{H}(1-\mathrm{H}^2).$$

Balancing $Y^4 H''$ with H^3, yields $4 + K - 2 = 3K$ i.e. $K = 1$. Thus $H(Y) = a_0 + a_1 Y$ $H' = a_1, H'' = 0$..

Submitting it into above equation, we have

$$-2a_1 c^2 v^2 Y + 2a_1 c^2 v^2 Y^3 = -2a_1 c^2 Y + 2a_1 c^2 Y^2 + (a_0 + a_1\,\mathrm{Y})(1 - (a_0 + a_1\,\mathrm{Y})^2).$$

Collecting the coefficients of each power of Y, we get the algebraic for a_0, a_1, c, v:

$$\begin{cases} a_0 - a_0^3 = 0, \\ 2a_1 c^2 v^2 - 2a_1 c^2 - 3a_0^2 a_1 + a_1 = 0, \\ -3a_0 a_1^2 = 0, \\ -2a_1 c^2 v^2 + 2a_1 c^2 - a_1^3 = 0. \end{cases} \tag{3.2}$$

Then, the solutions of (3.2) are

$$a_1 = -1, a_0 = 0, v = -\frac{\sqrt{-1+2c^2}}{\sqrt{2}c}; a_1 = -1, a_0 = 0, v = \frac{\sqrt{-1+2c^2}}{\sqrt{2}c};$$

$$a_1 = 1, a_0 = 0, v = -\frac{\sqrt{-1+2c^2}}{\sqrt{2}c}; a_1 = -1, a_0 = 0, v = \frac{\sqrt{-1+2c^2}}{\sqrt{2}c},$$

With $c \neq 0$. Recall that

$$u(\mathrm{t,x}) = \mathrm{U}(\omega) = \mathrm{U}(c(\mathrm{x} - vt)) = a_0 + a_1 Y(\omega) = a_0 + a_1 \tanh(c(\mathrm{x} - vt)),$$

Hence we obtain the traveling wave solutions of (3.1)

$$u_1(\mathrm{x,t}) = -\tanh[c(\mathrm{x} + \frac{\sqrt{-1+2c^2}}{\sqrt{2}c}t)]; \mathrm{u}_2(\mathrm{x,t}) = -\tanh[c(\mathrm{x} - \frac{\sqrt{-1+2c^2}}{\sqrt{2}c}t)];$$

$$u_3(\mathrm{x},\mathrm{t}) = \tanh[c(\mathrm{x} + \frac{\sqrt{-1+2c^2}}{\sqrt{2}c}\mathrm{t})]; u_4(\mathrm{x},\mathrm{t}) = \tanh[c(\mathrm{x} - \frac{\sqrt{-1+2c^2}}{\sqrt{2}c}\mathrm{t})];$$

Since the velocity $v = -\dfrac{\sqrt{-1+2c^2}}{\sqrt{2}c}$, the constant c must satisfies $c \geq \sqrt{\dfrac{1}{2}}$. We plot the

Images of (3) with specially values $c = \sqrt{\dfrac{1.000001}{2}}, c = 1.5, c = 100000$, which the correspond

Velocity v is close to 0, constant and 1.

$$u(\mathrm{x},\mathrm{t}) = -\coth[c(\mathrm{x} + \frac{\sqrt{-1+2c^2}}{\sqrt{2}c}\mathrm{t})]; u(\mathrm{x},\mathrm{t}) = -\coth[c(\mathrm{x} - \frac{\sqrt{-1+2c^2}}{\sqrt{2}c}\mathrm{t})];$$

$$u(\mathrm{x},\mathrm{t}) = \coth[c(\mathrm{x} + \frac{\sqrt{-1+2c^2}}{\sqrt{2}c}\mathrm{t})]; u(\mathrm{x},\mathrm{t}) = \coth[c(\mathrm{x} - \frac{\sqrt{-1+2c^2}}{\sqrt{2}c}\mathrm{t})];$$
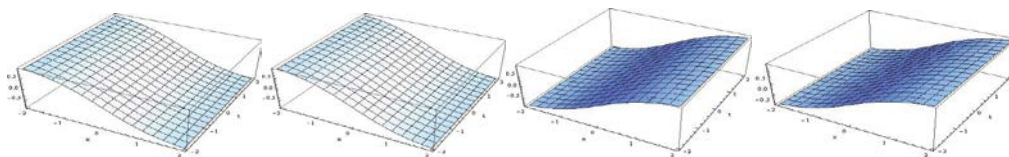
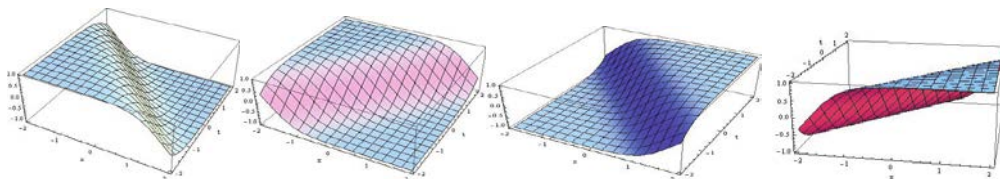Fig. 1. The travelling wave solutions of (3), when $c = \sqrt{1.000001/2}, (x,t) \in [-2,2] \times [-2,2]$

Fig. 2. The travelling wave solutions of (3), when $c = 1.5, (\mathrm{x},\mathrm{t}) \in [-2,2] \times [-2,2]$.
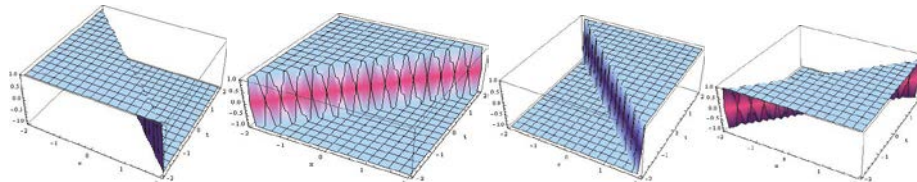
Fig. 3. The travelling wave solutions of (3), when $c = 100000, (\mathrm{x},\mathrm{t}) \in [-2,2] \times [-2,2]$.

Conclusion

In this paper, we have applied the tanh-function method to construct a series of traveling wave solutions for nonlinear wave equations. These traveling waves' solutions are expressed in terms of hyperbolic tangent or hyperbolic cotangent functions depending on different parameters. The tanh-function method is direct, concise and effective, which can be applied to many other nonlinear evolution equations.

Acknowledgments

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