Some Results on Interval-Valued Fuzzy Matrices

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Abstract

In this paper, we introduced interval-valued fuzzy matrices (IVFMs) as the generalization of interval-valued fuzzy sets. Some essential unary and binary operations of IVFM and some special types of IVFMs i.e., symmetric, reflexive, transitive and idempotent, constant, etc. are defined here. The idea of convergence, periodicity, determinant and adjoint of IVFMs are also defined. Lot of properties of IVFMs are presented here.

Keywords: Interval-valued fuzzy matrix, adjoint, determinant, constant IVFM.

1. Introduction

In [3], Thomason has introduced the concept of fuzzy matrices. After that a lot of works have been done on fuzzy matrices and its variants [4, 5, 6]. It is well known that the membership value completely depends on the decision maker's, its habit, mentality etc. So, sometimes it happens that the membership value can not be measured as a point, but, it can be measured appropriately as an interval. Sometimes, the measurement becomes impossible due to the rapid variation of the characteristics of the system whose membership values are to be determined. For example, we consider a network $N = (V, E)$ consisting n nodes (cities) and m edges (roads) connecting the cities of a country. If we measure the crowdness of the roads of the network for a particular time duration, it is quite impossible to measure the crowdness as a single value, because, the crowdness in a duration is not fixed, it varies time to time. So, more convenient technique to grade the crowdness is an interval not a point. In this case, the network N is called interval-valued fuzzy network and the corresponding matrix (representing the crowdness) is called interval-valued fuzzy matrix. Let $A_N^{(T)}$ be the interval-valued fuzzy matrix corresponding to the network N, representing the crowdness of N during time interval T. The ij th element a_{ij} of $A_N^{(T)}$ is defined as

$$a_{ij} = \begin{cases} [0, 0], & \text{if } i = j \\ [a_L, a_U], & \text{if } (i, j) \in E, \\ [1, 1], & \text{if } (i, j) \notin E. \end{cases}$$

where a_L and a_U are the lower and upper limits of the crowdness of the road connecting the cities i and j.

For illustration, we consider a net-

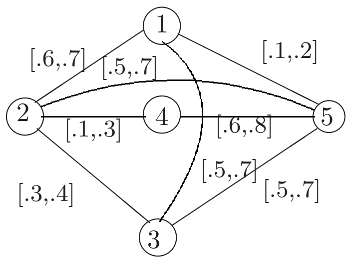

Fig. 1: A network

work shown in Figure 1 containing 5 nodes (cities) and 9 edges (roads). The numbers adjacent to the edges represent the crowdness of the roads connecting the corresponding cities.

The matrix representation of crowdness of the network of Figure 1 during the time interval T is shown below:

$$\begin{bmatrix} [0,0] & [.6,.7] & [.5,.7] & [1,1] & [.1,.2] \\ [.6,.7] & [0,0] & [.3,.4] & [.1,.3] & [.5,.7] \\ [.5,.7] & [.3,.4] & [0,0] & [.8,.9] & [.5,.7] \\ [1,1] & [.1,.3] & [.8,.9] & [0,0] & [.6,.4] \\ [.1,.2] & [.5,.7] & [.5,.7] & [.6,.8] & [0,0] \end{bmatrix}.$$

2. Preliminaries and definitions

In this section, we define some operators between two elements and two matrices.

Let $D[0,1]$ be the set of all subsets of the interval [0,1]. We recall the following operations from [1, 2] for any two elements of $D[0,1]$. Let $x = [x_L, x_U]$ and $y = [y_L, y_U]$, where $0 \le x_L < x_U$ and $0 \le y_L < y_U$. Then

(i) $x+y = [\max\{x_L, y_L\}, \max\{x_U, y_U\}]$

(ii) $x.y = [\min\{x_L, y_L\}, \min\{x_U, y_U\}]$

(iii) $x \oplus y = [x_L + y_L - x_L.y_L, x_U + y_U - x_U.y_U]$

(iv) $x \odot y = [x_L.y_L, x_U.y_U]$

(v) $x@y = \left[\frac{x_L+y_L}{2}, \frac{x_U+y_U}{2}\right]$

(vi) $x^c = [1 - x_U, 1 - x_L]$.

The definition of the interval-valued fuzzy matrix is given below.

Definition 1 Interval-valued fuzzy matrix (IVFM). *An interval-valued fuzzy matrix of order $m \times n$ is defined as, $A = (a_{ij})_{m\times n}$, where $a_{ij} = [a_{ijL}, a_{ijU}]$ is the ijth element of A, represents the membership value. All the elements of an IVFM are intervals and all the intervals are the subintervals of the interval* $[0,1]$.

In the IVFM, the elements are the membership grade of some attributes, they are not crisp number, so naturally some new operations are needed to handel such matrices. Before applying binary or unary operations between IVFMs we need clear idea about the similar operations between the elements.

Now, we define some operations for any two IVFMs $A = (a_{ij})$ and $B = (b_{ij})$ of order $m \times n$ in the following.

(i) $A + B = (a_{ij} + b_{ij})$

(ii) $A.B = (c_{ij})$ where $c_{ij} = \sum_{k=1}^{n} a_{ik}.b_{kj}$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, p$ and $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{n\times p}$

(iii) $A \oplus B = (a_{ij} \oplus b_{ij})$

(iv) $A \odot B = (a_{ij} \odot b_{ij})$

(v) $A@B = (a_{ij}@b_{ij})$

(vi) $A^c = (a^c_{ij})$ (the complement of A)

(vii) $A' = (a_{ji})$ (the transpose of A)

(viii) $A \le B$ if and only if $a_{ijL} \le b_{ijL}$ and $a_{ijU} \le b_{ijU}$ for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Definition 2 Null IVFM. *An IVFM is said to be a null IVFM if all its elements are zero, i.e., all elements are* $[0,0]$ *and the matrix is denoted by* 0.

Definition 3 Identity IVFM. *An IVFM $A = (a_{ij})$ of order $n\times n$ is called unit IVFM or identity IVFM if all the diagonal entries of A are* $[1,1]$ *and all* *other entries are* $[0,0]$*. It is denoted by* I_n*.*

Definition 4 *An* $n \times n$ *IVFM* A *is said to be*
(i) reflexive iff $A \geq I_n$*,*
(ii) symmetric iff $A = A'$*,*
(iii) transitive iff $A^2 \leq A$*,*
(iv) idempotent iff $A^2 = A$*.*

3. Basic properties

In this section, we present some basic properties of IVFMs. The commutative, associative and distributive laws are valid for IVFMs under the operations addition $(+)$ and multiplication $(\cdot)$.

Property 1 *For any three IVFMs* A*,* B *and* C
(i) $A + B = B + A$
(ii) $A + (B + C) = (A + B) + C$
(iii) $A.(B.C) = (A.B).C$
(iv) $A.(B + C) = A.B + A.C$
(v) $(B + C).A = B.A + C.A$*.*
(vi) $(A + B)' = A' + B'$
(vii) $(A.B)' = B'.A'$*.*

Property 2 *Let* $A = (a_{ij})_{m\times n}$ *be an IVFM then,*
(i) $A + A = A$
(ii) $A + 0 = A$*.*

4. Convergence and periodicity of IVFM

For the first time Thomason [3] studied the convergence of powers of a fuzzy matrix and pointed out that the powers of general fuzzy matrices either converge or oscillate with a finite period. By using this concept, some properties about convergence of powers of IVFM are studied here.

Let $A = (a_{ij})$ be a square IVFM of order n where $a_{ij} = [a_{ijL}, a_{ijU}]$. Thus the powers of A are defined as $A^m = A \cdots A \cdots A$ ('m' in number), and naturally ij th element of A^m is denoted by $a^m_{ij} = [a^m_{ijL}, a^m_{ijU}]$.

Definition 5 Convergence of IVFM. *Let* $A = (a_{ij})$ *be a square IVFM of order* n*, where* $a_{ij} = [a_{ijL}, a_{ijU}]$*. If there exists an integer* m *such that* $A^{m+1} = A^m$ *holds, then the power of an IVFM is said to converge. Generally, an IVFM is said to converge when its powers converge.*

Definition 6 Periodicity of IVFM. *Let* A *be a square IVFM, if there exists two integers* m *and* s *such that,* $A^{m+s} = A^s$ *holds, then* m *is said to be the periodicity of* A *and* s *is the starting point of* A *corresponding to* m*.*

Let $n > s$, be any positive integer then $n - s > 0$ and multiplying A^{n-s} on both sides of $A^{s+m} = A^s$, we get, $A^{m+n} = A^n$ which means that every $n > s$ be also a starting point corresponding to m. Now, it is clear that the periodicity m of an IVFM A indicates a least positive integral value, say, e such that for which every $n \geq e$ is a starting point, while $n < e$ is not a starting point corresponding to m. This e is called an index of A, where $e = \min\{s/A^{s+m} = A^s\}$, m is a given positive integer. We also call d, the least periodicity of A, where $d = \min\{m/A^{s+m} = A^s\}$, for all positive integers s, m. Note that, every periodicity m of an IVFM A is a multiple of the least periodicity d.

Property 3 *The powers of an IVFM* A *either converge to* A^p *for a finite* p*, or oscillate with a finite period.*

Proof. In the powers of A there may occur a new interval which is not in A originally. But, if we check lower limits and upper limits of these intervals separately, then it will be clear that these numbers are not a new at all. These exist already in other intervals in A. Because, n is finite and max-min operations are deterministic, so it can not introduce lower limits and upper limits which are not in A. Thus if A does not converge in its powers, then it must oscillates with finite period. □

Property 4 *Let A, B and C be three IVFMs such that $A \leq B$ then $A.C \leq B.C$.*

Proof. We consider three IVFMs $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{m\times n}$ and $C = (c_{ij})_{n\times p}$ such that $A \leq B$. Let $A.C = D = (d_{ij}) = ([d_{ijL}, d_{ijU}])_{m\times p}$, where $d_{ijL} = \sum_{k=1}^{n} a_{ikL}.c_{kjL}$; $d_{ijU} = \sum_{k=1}^{n} a_{ikU}.c_{kjU}$ and $B.C = E = (e_{ij}) = ([e_{ijL}, e_{ijU}])_{m\times p}$, where $e_{ijL} = \sum_{k=1}^{n} b_{ikL}.c_{kjL}$; $e_{ijU} = \sum_{k=1}^{n} b_{ikU}.c_{kjU}$. Since $A \leq B$ i.e., $a_{ijL} \leq b_{ijL}$ and $a_{ijU} \leq b_{ijU}$. Again, $c_{ij} = [c_{ijL}, c_{ijU}]$ is a subset of $[0, 1]$, therefore for any $c_{ij} = [c_{ijL}, c_{ijU}]$ we have,

$d_{ijL} = \sum_{k=1}^{n} a_{ikL}.c_{kjL} \leq \sum_{k=1}^{n} b_{ikL}.c_{kjL} = e_{ijL}$ and

$d_{ijU} = \sum_{k=1}^{n} a_{ikU}.c_{kjU} \leq \sum_{k=1}^{n} b_{ikU}.c_{kjU} = e_{ijU}$.

Hence, $A.C \leq B.C$ □

Property 5 *If either $A^p \leq A^q$ or $A^p \geq A^q$ holds for $p < q$, then A converges.*

Proof. If $A^p \leq A^q$ holds then $a^{(p)}_{ijL} \leq a^{(q)}_{ijL}$ and $a^{(p)}_{ijU} \leq a^{(q)}_{ijU}$. By property 4.5.3, we have

$$a^{(p)}_{ijL} \leq a^{(q)}_{ijL} \leq a^{(q+1)}_{ijL} \leq \cdots$$

and $a^{(p)}_{ijU} \leq a^{(q)}_{ijU} \leq a^{(q+1)}_{ijU} \leq \cdots$.

Since, $a^{(p)}_{ijL} \leq a^{(q)}_{ijL}$, $a^{(p)}_{ijU} \leq a^{(q)}_{ijU}$ and max-min operation is deterministic, therefore a finite number of distinct lower intervals and upper intervals in the corresponding interval (position) occur in the powers of A so that,

$a^{(p)}_{ijL} \leq a^{(q)}_{ijL} \leq a^{(q+1)}_{ijL} \leq \cdots \leq a^{(s)}_{ijL} = a^{(s+1)}_{ijL} = \cdots$ and

$a^{(p)}_{ijU} \leq a^{(q)}_{ijU} \leq a^{(q+1)}_{ijU} \leq \cdots \leq a^{(t)}_{ijU} = a^{(t+1)}_{ijU} = \cdots$

for some finite s and t, where $s \geq$ or $\leq t$. Simply, a finite number of distinct IVFMs occur in the powers of A.

Hence, A converges. □

Similarly, it can be shown when $A^p > A^q$.

Property 6 *A be an IVFM of order $n \times n$, for all $i, j \leq n$, if there is $k \leq n$ such that $a_{ijL} \leq a_{ikL}.a_{kjL}$ and $a_{ijU} \leq a_{ikU}.a_{kjU}$, then A converges to A^p, for some $p \leq n - 1$.*

5. Interval-valued fuzzy determinant

Definition 7 Interval-valued fuzzy determinant. *The interval-valued fuzzy determinant (IVFD) of an IVFM A of order $n \times n$ is denoted by $\mid A \mid$ or $det(A)$ and we define it as*

$$\mid A \mid = \sum_{\sigma\in S_n} a_{1\sigma(1)}a_{2\sigma(2)}...a_{n\sigma(n)} = \sum_{\sigma\in S_n} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where $a_{i\sigma(i)} = [a_{i\sigma(i)_L}, a_{i\sigma(i)_U}]$ and S_n denotes the symmetric group of all permutations of the indices $\{1, 2, ..., n\}$.

The addition and multiplication between two elements a_{ij} and b_{ij} of $D[0, 1]$ are defined in the following.

$a_{ij} + b_{ij} = [a_{ijL}, a_{ijU}] + [b_{ijL}, b_{ijU}]$
$= [\max\{a_{ijL}, b_{ijL}\}, \max\{a_{ijU}, b_{ijU}\}]$
$a_{ij}.b_{ij} = [a_{ijL}, a_{ijU}].[b_{ijL}, b_{ijU}]$
$= [\min\{a_{ijL}, b_{ijL}\}, \min\{a_{ijU}, b_{ijU}\}]$.

Property 7 *Let $A = (a_{ij})_{n\times n}$ be an IVFM.*
(i) If a row (column) be multiplied by a scalar $k \in (0,1]$, then $\mid A \mid$is multiplied by k.
(ii) If all elements of a row (column) of A are [0,0], then $|A| = [0,0]$.
(iii) If A is triangular, then
$|A| = \prod_{i=1}^{n} [a_{iiL}, a_{iiU}]$.

Property 8 *If any two rows (or columns) of a square IVFM are interchanged then the determinant of that IVFM remains unchange.*

Property 9 *If A is a square IVFM of order $n \times n$ then, $|A| = |A^{'}|$.*

6. Adjoint of an IVFM

The adjoint of an IVFM A is defined as in classical matrix and it is denoted by $adj(A)$.

Property 10 *For any two $n \times n$ IVFMs A and B,*
(i) if $A \leq B$ then $adj(A) \leq adj(B)$,
(ii) $adj(A) + adj(B) \leq adj(A+B)$,
(iii) $adj(A^{'}) = (adj(A))^{'}$,
(iv) if A contains a zero row then $(adjA).A = [0,0]$.

Proof. (i) Let $A = (a_{ij})$ and $B = (b_{ij})$ be two IVFMs of order $n \times n$, where $a_{ij} = [a_{ijL}, a_{ijU}]$ and $b_{ij} = [b_{ijL}, b_{ijU}]$. Let $adj(A) = C = (c_{ij})$ and $adj(B) = D = (d_{ij})$.
Then, $c_{ij} = \sum_{\sigma_{n_j n_i}} \prod_{t \in n_j} [a_{t\sigma(t)L}, a_{t\sigma(t)U}]$ and $d_{ij} = \sum_{\sigma_{n_j n_i}} \prod_{t \in n_j} [b_{t\sigma(t)L}, b_{t\sigma(t)U}]$.

Now it is clear that $c_{ij} \leq d_{ij}$ because $a_{t\sigma(t)L} \leq b_{t\sigma(t)L}$ and $a_{t\sigma(t)U} \leq b_{t\sigma(t)U}$ for every $t \neq j$, $\sigma(t) \neq \sigma(j)$. Therefore $C \leq D$, i.e., $adj(A) \leq adj(B)$.
Proofs of (ii) and (iii) are trivial.
(iv) Suppose $A = (a_{ij})$ be a square IVFM of order $n \times n$, where $a_{ij} = [a_{ijL}, a_{ijU}]$. Let $B = adj(A)$, then $b_{ij} = |A_{ji}|$ and let $C = (adj(A)).A = B.A$, where $c_{ij} = \sum_{k=1}^{n} b_{ik}.a_{kj} = \sum_{k=1}^{n} |A_{ki}|.a_{kj}$ if the ith row of A is zero, i.e., $a_{ij} = [a_{ijL}, a_{ijU}] = [0,0]$ for all j then A_{ki} contains a zero row where $k \neq i$ and therefore $|A_{ki}| = [0,0]$ (by the property 4.6.1(ii)) for every $k \neq i$. Again $[a_{ijL}, a_{ijU}] = [0,0]$ for all j, $j = 1, 2, \ldots, n$ when $k = i$. Then $c_{ij} = \sum_{k=1}^{n} |A_{ki}|.a_{kj} = [0,0]$.
Hence, $C = (adj(A)).A = [0,0]$. □

Property 11 *Let A be an IVFM of order $n \times n$, then*
(i) $A.(adj(A)) \geq |A|.I_n$,
(ii) $(adj(A)).A \geq |A|.I_n$.

Proof. (i) Let $C = A.(adj(A))$.
The ith row of A is $[a_{i1L}, a_{i1U}][a_{i2L}, a_{i2U}] \ldots [a_{inL}, a_{inU}]$. Then by definition of $adj(A)$, the jth column of $adj(A)$ is given by $[A_{j1L}, A_{j1U}][A_{j2L}, A_{j2U}] \ldots [A_{jnL}, A_{jnU}]$, where $[A_{ijL}, A_{ijU}]$ is the cofactor of the element a_{ij} in A.

Therefore, $c_{ij} = \sum_{k=1}^{n} [a_{ikL}, a_{ikU}][A_{jkL}, A_{jkU}] \geq [0,0]$ and hence $c_{ii} = \sum_{k=1}^{n} [a_{ikL}, a_{ikU}][A_{ikL}, A_{ikU}]$ which is equal to $|A|$.
Therefore, $C = A.(adjA) \geq |A|.I_n$.
(ii) Proof is similar to (i). □

Property 12 *For an IVFM $A = (a_{ij})$ of order $n \times n$,*
(i) $|A| = |adjA|$,
(ii) $A.(adjA)$ is transitive.

Property 13 *Let $A = (a_{ij})$ be an $n \times n$ reflexive IVFM. Then,*
(i) $adj(A^2) = (adj(A))^2 = adj(A)$,
(ii) $adj(adj(A)) = adj(A)$,

(iii) $adj(A) \geq A$,
(iv) $A.(adj(A)) = (adj(A)).A = adj(A)$.

7. Constant IVFM

Definition 8 Constant IVFM. *An $n \times n$ IVFM A is said to be constant if all its rows are equal to each other, i.e., if $a_{ikL} = a_{jkL}$ and $a_{ikU} = a_{jkU}$ for all i, j, k.*

Property 14 *If $A = (a_{ij})$ is a constant IVFM of order $n \times n$ and $B \geq I_n$ is an IVFM of the same order, then $A.B$ and $B.A$ are constant IVFMs.*

Proof. Since $A = (a_{ij})$ is a constant IVFM of order $n \times n$ where $a_{ij} = [a_{ijL}, a_{ijU}]$ then $a_{ik} = [a_{ikL}, a_{ikU}]$ are the same for all $i, i, k = 1, 2, \ldots, n$. Let $A.B = \sum_{k=1}^{n} a_{ik}.b_{kj} = D = (d_{ij})$ and $d_{ij} = [d_{ijL}, d_{ijU}]$. Then $d_{ijL} = \sum_{k=1}^{n} a_{ikL}.b_{kjL}$ and $d_{ijU} = \sum_{k=1}^{n} a_{ikU}.b_{kjU}$. Here, $a_{ikL} = a_{jkL}$ and $a_{ikU} = a_{jkU}$ for all i, j, k. This implies that d_{ijL} and d_{ijU} are both independent of i, $i \in \{1, 2, \ldots, n\}$ i.e., d_{ij} is independent of i. Therefore, $A.B$ is constant.

Let $B.A = E = (e_{ij})$, *where* $e_{ij} = [e_{ijL}, e_{ijU}]$. Then, $e_{ijL} = \sum_{k=1}^{n} b_{ikL}.a_{kjL}$ and $e_{ijU} = \sum_{k=1}^{n} b_{ikU}.a_{kjU}$. Now, $e_{ijL} = \max_k\{\min\{b_{ikL}, a_{kjL}\}\}$, $B \geq I_n$ implies $b_{ii} = [1, 1]$, for all $i, i \in \{1, 2, \ldots, n\}$. Since a_{kjL} is independent of k, denoted by a_{jL} and $e_{ijL} = \max_k\{\min\{a_{ikL}, a_{kjL}\}\} = \min\{a_{jL}, \max_k\{b_{ikL}\}\} = a_{jL}$, as, $\max_k\{b_{ikL}\} = 1$. Similarly, $e_{ijU} = a_{ijU}$. Therefore, $B.A$ is constant. It should be noted that if $B \not\geq I_n$ then $B.A$ is not constant. □

Property 15 *If $A = (a_{ij})$ is a constant IVFM where $a_{ij} = [a_{ijL}, a_{ijU}]$ then,*
(i) $A^2 = A$, i.e., every constant IVFM is idempotent,
(ii) $(adj(A))'$ is constant,
(iii) $A.(adj(A))$ is constant,
(iv) $|A| = [c_L, c_U]$, where $c_L = \min_{ij}\{a_{ijL}\}$ and $c_U = \min_{ij}\{a_{ijU}\}$.

8. Conclusion

In this article the IVFM is defined. The convergence, periodicity, determinant of IVFM along with basic properties are studied here. The concept of constant IVFM is introduced in this article also. An application of IVFM is also cited. At present, we are trying to find out the eigenvalues and eigenvectors of IVFM and other results.

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