Estimation of unknown function of a class of retarded iterated integral inequalities

Jiting Huang,Wu-Sheng Wang,Yun Ouyang,Yongzhou Qin

School of Mathematics and Statistics, Hechi University

Guangxi,Yizhou 546300,P.R.China

hcxyhjt@163.com,wang4896@126.com, shuxueoyy@126.com, gxqinyongzhou@163.com

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Abstract: It is well known that differential equations and integral equations are important tools to discuss the rule of natural phenomena. Various generalizations of the Gronwall-Bellman inequality are important tools in the study of existence, uniqueness, boundedness, stability, continuous dependence on the initial value and parameters, and other qualitative properties of solutions of differential equations and integral equation. In this paper, we discuss a class of retarded iterated integral inequalities, which includes a nonconstant term outside the integrals. By adopting novel analysis techniques, the upper bound of the embedded unknown function is estimated explicitly. The derived result can be applied in the study of solutions of ordinary differential equations and integral equations.

Introduction

It is well known that differential equations and integral equations are important tools to discuss the rule of natural phenomena. In the study of the existence, uniqueness, boundedness, stability, oscillation and other qualitative properties of solutions of differential equations and integral equations, one often deals with certain integral inequalities. One of the best known and widely used inequalities in the study of nonlinear differential equations is Gronwall- Bellman inequality [1,2], which can be stated as follows: If u and f are non-negative continuous functions on an interval [a, b] satisfying

$$u(t) \le c + \int_a^t f(s)u(s)ds, \quad t \in [a,b], \qquad (1)$$

for some constant $c \ge 0$, then

$$u(t) \le c \exp\left(\int_a^t f(s)u(s)ds\right), \quad t \in [a,b].$$

Pachpatte in [3] investigated the retarded inequality

$$u(t) \le k + \int_a^t g(s)u(s)ds + \int_a^{\alpha(t)} h(s)u(s)ds. \qquad (2)$$

where *k* is a constant. Replacing *k* by a nondecreasing continuous function *f*(*t*) in (1), Rashid in [4] studied the following retarded inequality

$$u(t) \le f(t) + \int_a^t g(s)u(s)ds + \int_a^{\alpha(t)} h(s)u(s)ds. \qquad (3)$$

In 2011, Abdeldaim et al. [5] studied a new integral inequality of Gronwall-Bellman-Pachpatte type

$$u(t) \le u_0 + \int_0^t g(s)u(s)\left[u(s) + \int_0^s h(\tau)\left[u(\tau) + \int_0^\tau g(\xi)u(\xi)d\xi\right]d\tau\right]ds. \qquad (4)$$

In 2014, El-Owaidy, Abdeldaim, and El-Deed [6] investigated several retarded nonlinear integral inequalities

$$u(t) \le f(t) + \int_a^t g(s)u^p(s)ds + \int_a^{\alpha(t)} h(s)u^p(s)ds, \qquad (5)$$

$$u^p(t) \le f^p(t) + \int_a^{\alpha_1(t)} g(s)u(s)ds$$

$$+\int_a^{\alpha_2(t)} h(s)u(s)ds, \quad (6)$$

$$u(t)\le f(t)+\int_a^{\alpha_1(t)} g(s)w_1(u(s))ds$$
$$+\int_a^{\alpha_2(t)} h(s)w_2(u(s))ds, \quad (7)$$

$$u(t)\le f(t)+\int_a^{\alpha_1(t)} g(s)u(s)w_1(\ln u(s))ds$$
$$+\int_a^{\alpha_2(t)} h(s)u(s)w_2(\ln u(s))ds, \quad (8)$$

$$u(t)\le f(t)+\int_a^{\alpha(t)} g(s)u(s)ds$$
$$+\int_a^{\alpha(t)} h(s)u(s)\left[u(s)+\int_a^{\alpha(t)} h(\lambda)u(\lambda)d\lambda\right]ds. \quad (9)$$

During the past few years, some investigators have established a lot of useful and interesting integral inequalities in order to achieve various goals; see [7-9] and the references cited therein.

In this paper, on the basis of [5,6], we discuss a new retarded nonlinear Volterra- redholm type integral inequality

$$u(t)\le f(t)+\int_a^{\alpha(t)} g(s)u(s)ds + \int_a^{\alpha(t)} g(s)u(s)$$
$$\times\left[u(s)+\int_0^s h(\tau)\left[u(\tau)+\int_0^\tau q(\xi)u(\xi)d\xi\right]d\tau\right]ds, \quad (10)$$

and

$$u^2(t)\le f^2(t)+\int_a^{\alpha(t)} g(s)u(s)ds + \int_a^{\alpha(t)} g(s)u(s)$$
$$\times\left[u(s)+\int_0^s h(\tau)u(\tau)d\tau\right]ds. \quad (11)$$

Results

Throughout this paper, let R+ = [0, +∞); I = [a, +∞); C1(M, S) denotes the class of continuously differentiable functions defined on set M with range in the set S, C(M, S) denotes the class of continuously functions defined on set M with range in the set S, α'(t) denotes the derived function of a function α(t).

Theorem 1. Suppose that $f(t)$, $h(t)$, $q(t) \in C(\boldsymbol{I},\mathbf{R}_+)$, $\alpha\in C^1(\boldsymbol{I},\boldsymbol{I})$ is nondecreasing with $\alpha(t)\le t$ on $\boldsymbol{I}$, $f\in(\mathbf{R}_+,\mathbf{R}_+)$ is a nondecreasing function with $f(t)>0$ for $t>0$. Suppose that

$$1-\int_a^{\alpha(t)} g(s)f(s)\exp\left(\int_a^s [g(\tau)+h(\tau)+q(\tau)]d\tau\right)ds>0. \quad (12)$$

If $u(t)$ satisfies (10), then

$$u(t)\le f(t)\exp\left(\int_a^{\alpha(t)}[g(s)+h(s)+q(s)]ds\right)\Big[1-$$
$$\int_a^{\alpha(t)} g(s)f(s)\exp\left(\int_a^s [g(\tau)+h(\tau)+q(\tau)]d\tau\right)ds\Big]^{-1}. \quad (13)$$

Proof. Since $f(t)$ is a positive and nondecreasing function. From (10) we have

$$\frac{u(t)}{f(t)}\le 1+\int_a^{\alpha(t)} g(s)\frac{u(s)}{f(s)}ds+\int_a^{\alpha(t)} g(s)f(s)\frac{u(s)}{f(s)}\left[\frac{u(s)}{f(s)}+\int_a^s h(\tau)\left[\frac{u(\tau)}{f(\tau)}+\int_a^\tau q(\xi)\frac{u(\xi)}{f(\xi)}d\xi\right]d\tau\right]ds. \quad (14)$$

Let $v(t)=u(t)/f(t)$. We observe that

$$v(t)\le 1+\int_a^{\alpha(t)} g(s)v(s)ds+\int_a^{\alpha(t)} g(s)f(s)v(s)[v(s)+\int_a^s h(\tau)[v(\tau)+\int_a^\tau q(\xi)v(\xi)d\xi]d\tau]ds. \quad (15)$$

Define a function $z(t)$ by the right hand side of the above inequality

$$z(t)=1+\int_a^{\alpha(t)} g(s)v(s)ds+\int_a^{\alpha(t)} g(s)f(s)v(s)[v(s)+\int_a^s h(\tau)[v(\tau)+\int_a^\tau q(\xi)v(\xi)d\xi]d\tau]ds, \quad (16)$$

which is a positive and nondecreasing function on $\boldsymbol{I}$. From (15) and (16), we have

$$v(t)\le z(t),\ v(\alpha(t))\le z(\alpha(t))\le z(t),\ t\in I, \quad (17)$$

$$z(a) = 1. \qquad (18)$$

Differentiating $z(t)$ with respect to t and using (17) we have

$$z'(t) = \alpha'(t)g(\alpha(t))v(\alpha(t)) + \alpha'(t)g(\alpha(t))f(\alpha(t))v(\alpha(t)) \times \left[v(\alpha(t) + \int_a^{\alpha(t)} h(\tau)\left[v(\tau) + \int_a^{\tau} q(\xi)v(\xi)d\xi\right]d\tau \right]$$

$$\le \alpha'(t)g(\alpha(t))z(t)\{1 + f(\alpha(t))[z(t)$$

$$+ \int_a^{\alpha(t)} h(\tau)\left[v(\tau) + \int_a^{\tau} q(\xi)v(\xi)d\xi\right]d\tau]\}$$

$$= \alpha'(t)g(\alpha(t))z(t)[1 + f(\alpha(t))w_1(t)], \qquad (19)$$

where

$$w_1(t) = z(t) + \int_a^{\alpha(t)} h(\tau)\left[z(\tau) + \int_a^{\tau} q(\xi)z(\xi)d\xi\right]d\tau, \qquad (20)$$

which is a positive and nondecreasing function on ***I***. From (18) and (20), we have

$$z(t) \le w_1(t),\ z(\alpha(t)) \le w_1(\alpha(t)) \le w_1(t),\ t \in I, \qquad (21)$$

$$w_1(a) = z(a) = 1. \qquad (22)$$

Differentiating $w_1(t)$ with respect to t and using (19) and (21), we have

$$w_1'(t) = z'(t) + \alpha'(t)h(\alpha(t))\left[z(\alpha(t)) + \int_a^{\alpha(t)} q(\xi)z(\xi)d\xi\right]$$

$$\le \alpha'(t)g(\alpha(t))w_1(t)[1 + f(\alpha(t))w_1(t)]$$

$$+ \alpha'(t)h(\alpha(t))\left[w_1(t) + \int_a^{\alpha(t)} q(\xi)w_1(\xi)d\xi\right]$$

$$\le \alpha'(t)g(\alpha(t))w_1(t)[1 + f(\alpha(t))w_1(t)]$$

$$+ \alpha'(t)h(\alpha(t))w_2(t), \qquad (23)$$

where

$$w_2(t) = w_1(t) + \int_a^{\alpha(t)} q(\xi)w_1(\xi)d\xi, \qquad (24)$$

which is a positive and nondecreasing function on ***I***. From (22) and (24) we have

$$w_1(t) \le w_2(t),\ w_1(\alpha(t)) \le w_2(\alpha(t)) \le w_2(t),\ t \in I, \qquad (25)$$

$$w_2(a) = z(a) = 1. \qquad (26)$$

Differentiating $w_2(t)$ with respect to t and using (23) and (25), we have

$$w_2'(t) = w_1'(t) + \alpha'(t)q(\alpha(t))w_1(\alpha(t))$$

$$\le \alpha'(t)g(\alpha(t))w_2(t)[1 + f(\alpha(t))w_2(t)]$$

$$+ \alpha'(t)h(\alpha(t))w_2(t) + \alpha'(t)q(\alpha(t))w_2(t)$$

$$\le \alpha'(t)g(\alpha(t))f(\alpha(t))w_2^2(t) + \alpha'(t)[g(\alpha(t))$$

$$+ h(\alpha(t)) + q(\alpha(t))]w_2(t). \qquad (27)$$

Since $w_2(t) > 0$, from (27) we have

$$-[(w_2(t))^{-1}]' \le \alpha'(t)g(\alpha(t))f(\alpha(t)) + \alpha'(t)[g(\alpha(t))$$

$$+ h(\alpha(t)) + q(\alpha(t))](w_2(t))^{-1}. \qquad (28)$$

Let $x(t) = (w_2(t))^{-1}$, then $x(a) = (w_2(a))^{-1} = 1$ from (28) we get

$$x'(t) \ge -\alpha'(t)[g(\alpha(t)) + h(\alpha(t))$$

$$+ q(\alpha(t))]x(t) - \alpha'(t)g(\alpha(t))f(\alpha(t)). \qquad (29)$$

Consider ordinary differential equation

$$y'(t) = -\alpha'(t)[g(\alpha(t)) + h(\alpha(t)) + q(\alpha(t))]y(t)$$

$$-\alpha'(t)g(\alpha(t))f(\alpha(t)),\ y(a) = 1. \qquad (30)$$

Using method of variation of constant, we obtain that the solution of Eq. (30) is

$$y(t) = \exp\left(-\int_a^t \alpha'(s)[g(\alpha(s)) + h(\alpha(s)) + q(\alpha(s))]ds\right) \times \{1 - \int_a^t \alpha'(s)g(\alpha(s))f(\alpha(s))\exp[\int_a^t (\alpha'(\tau))(g(\alpha(\tau))$$

$$+ h(\alpha(\tau)) + q(\alpha(\tau)))d\tau]ds\}$$

$$= \exp\left(-\int_a^{\alpha(t)} [g(s) + h(s) + q(s)]ds\right) \cdot \{1 -$$

$$\int_a^t \alpha'(s) g(\alpha(s)) f(\alpha(s)) \exp[\int_a^{\alpha(s)} (g(\tau)+h(\tau)+q(\tau))d\tau]ds\}$$

$$= \exp\left(-\int_a^{\alpha(t)} [g(s)+h(s)+q(s)]ds\right)\cdot\{1-$$

$$\int_a^{\alpha(t)} [g(s) f(s) \exp[\int_a^s (g(\tau)+h(\tau)+q(\tau))d\tau]ds\}. \quad (31)$$

From (29), (30) and (31), we have

$$(w_2(t))^{-1} = x(t) \ge y(t) = \exp(-\int_a^{\alpha(t)} [g(s)+h(s)$$

$$+q(s)]ds)\{1-\int_a^{\alpha(t)} g(s) f(s) \exp[\int_a^s (g(\tau)$$

$$+h(\tau)+q(\tau))d\tau]ds\}, \quad (32)$$

that is

$$w_2(t) \le \exp\left(\int_a^{\alpha(t)} [g(s)+h(s)+q(s)]ds\right)\cdot\{1-$$

$$\int_a^{\alpha(t)} g(s) f(s) \exp[\int_a^s (g(\tau)+h(\tau)+q(\tau))d\tau]ds\}^{-1}. \quad (33)$$

From (17), (21) and (25), we get

$$u(t)/f(t) = v(t) \le z(t) \le w_1(t) \le w_2(t). \quad (34)$$

Therefore

$$u(t) \le f(t) \exp\left(\int_a^{\alpha(t)} [g(s)+h(s)+q(s)]ds\right)\cdot[1-$$

$$\int_a^{\alpha(t)} g(s) f(s) \exp(\int_a^s [g(\tau)+h(\tau)+q(\tau)]d\tau)ds]^{-1}. \quad (35)$$

We get the required estimation (13). The proof is complete.

Theorem 2. Suppose that $h(t) \in C(\boldsymbol{I},\mathbf{R}_+)$, $\alpha \in C^1(\boldsymbol{I}, \boldsymbol{I})$ is nondecreasing with $\alpha(t) \le t$ and $\alpha(a) \le a$ on $\boldsymbol{I}$, $f\in(\mathbf{R}_+,\mathbf{R}_+)$ is a nondecreasing function with $f(t) > 1$ for $t > 0$. If $u(t)$ satisfies (11), then

$$u(t) \le \exp\left(\int_a^{\alpha(t)} [g(s)/2+h(s)]ds\right)\{f(a)$$

$$+\int_a^{\alpha(t)} [f(s) f'(s)+\alpha'(s) g(\alpha(s))/2]$$

$$\times \exp[-\int_a^s (g(\tau)/2+h(\tau))d\tau]\}\cdot \quad (36)$$

Proof. Let $z_1^2(t)$ denote the right hand side in (11), i.e.

$$z_1^2(t) = f^2(t) + \int_a^{\alpha(t)} g(s)u(s)ds$$

$$+\int_a^{\alpha(t)} g(s)u(s)[u(s)+\int_a^s h(\tau)u(\tau)d\tau]ds. \quad (37)$$

Then $z_1(t)$ is a positive and nondecreasing function on $\boldsymbol{I}$. We have

$$u(t) \le z_1(t), u(\alpha(t)) \le z_1(\alpha(t)) \le z_1(t), t \in I, \quad (38)$$

$$z_1(a) = f(a). \quad (39)$$

Differentiating $z_1(t)$ with respect to t and using (38), we have

$$2z_1(t)z_1'(t) \le 2f(t)f'(t)+\alpha'(t)g(\alpha(t))z_1(t)$$

$$+\alpha'(t)g(\alpha(t))z_1(t)[z_1(t)+\int_a^{\alpha(t)} h(\tau)z_1(\tau)d\tau].$$

(40)

From (37), we have $z_1(t)>f(t)>1$, from (40) we have

$$z_1'(t) \le f(t)f'(t)+\alpha'(t)g(\alpha(t))/2$$

$$+\alpha'(t)g(\alpha(t))[z_1(t)+\int_a^{\alpha(t)} h(\tau)z_1(\tau)d\tau]/2$$

$$= f(t)f'(t)+\alpha'(t)g(\alpha(t))/2+\alpha'(t)g(\alpha(t))z_2(t)/2. \quad (41)$$

where

$$z_2(t) = z_1(t) + \int_a^{\alpha(t)} h(\tau)z_1(\tau)d\tau. \quad (42)$$

From (39) and (42), we have

$$z_1(t) \le z_2(t),\ z_1(\alpha(t)) \le z_2(\alpha(t)) \le z_2(t),\ t \in I, \quad (43)$$

$$z_2(a) = z_1(a) = f(a). \quad (44)$$

Differentiating $z_2(t)$ with respect to t and using (41) and (43), we have

$$\begin{aligned} z_2'(t) &\le z_1'(t) + \alpha'(t)h(\alpha(t))z_1(\alpha(t)) \\ &\le f(t)f'(t) + \alpha'(t)g(\alpha(t))/2 \\ &\quad + \alpha'(t)g(\alpha(t))z_2(t)/2 + \alpha'(t)h(\alpha(t))z_2(t). \quad (45) \end{aligned}$$

Consider ordinary differential equation

$$\begin{aligned} y'(t) &= f(t)f'(t) + \alpha'(t)g(\alpha(t))/2 \\ &\quad + \alpha'(t)[g(\alpha(t))/2 + h(\alpha(t))]z_2(t),\ y(a) = a. \quad (46) \end{aligned}$$

Using method of variation of constant, we obtain that the solution of Eq. (46) is

$$\begin{aligned} y(t) &= \exp\left(\int_a^{\alpha(t)} [g(s)/2 + h(s)]ds\right)\{f(a) \\ &\quad + \int_a^{\alpha(t)} [f(s)f'(s) + \alpha'(s)g(\alpha(s))/2] \\ &\quad \times \exp[-\int_a^s (g(\tau)/2 + h(\tau))d\tau]\} \cdot \quad (47) \end{aligned}$$

From (38), (43), (45), (46) and (47), we have

$$\begin{aligned} u(t) &\le z_1(t) \le z_2(t) \le y(t) \\ &= \exp\left(\int_a^{\alpha(t)} [g(s)/2 + h(s)]ds\right)\{f(a) \\ &\quad + \int_a^{\alpha(t)} [f(s)f'(s) + \alpha'(s)g(\alpha(s))/2] \\ &\quad \times \exp[-\int_a^s (g(\tau)/2 + h(\tau))d\tau]\} \cdot \end{aligned}$$

We get the required estimation (36). The proof is complete.

Conclusions

In this paper, we discuss a class of retarded iterated integral inequalities

$$\begin{aligned} u(t) &\le f(t) + \int_a^{\alpha(t)} g(s)f(s)ds + \int_a^{\alpha(t)} g(s)[u(s) \\ &\quad + \int_a^s h(\tau)[u(\tau) + \int_a^\tau q(\xi)u(\xi)]d\tau]ds. \end{aligned}$$

which includes a nonconstant term $f(t)$ outside the integrals. Under the condition

$$1 - \int_a^{\alpha(t)} g(s)f(s)\exp(\int_a^s [g(\tau) + h(\tau) + q(\tau)]d\tau)ds > 0,$$

by adopting novel analysis techniques, such as: change of variable, amplification method, differential and integration, we obtain the upper bounds of the embedded unknown function $u(t)$

$$\begin{aligned} u(t) &\le f(t)\exp(\int_a^{\alpha(t)} [g(s) + h(s) + q(s)]ds)[1 \\ &\quad - \int_a^{\alpha(t)} g(s)f(s)\exp(\int_a^s [g(\tau) + h(\tau) + q(\tau)]d\tau)ds]^{-1}. \end{aligned}$$

The derived result can be applied in the study of solutions of ordinary differential equations and integral equations.

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