

On the Equivalence of Limit Distributions of a Sum and of a Maximum Sum of $\tilde{\rho}$ -mixing Random Variables

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Abstract— $\{X_n\}$ is a sequence of independent random variables defined on a fixed probability space (Ω, \mathcal{F}, P) , Kruglov and Bo (1996) proved the equivalence of the limit distributions of an appropriately centered and normalized sum and the maximum sum of random variables $\{X_n\}$ which have finite variances. M. Sreehari (2010) obtained the similar results with the assumption on finiteness of $E|X_n|^p$, where $1 < p < 2$. In this paper, let $\{X_n\}$ is a sequence of $\tilde{\rho}$ -mixing random variables defined on a fixed probability space (Ω, \mathcal{F}, P) , by use of the Yang Shanchao's moment inequality(1998): $E\left|\sum_{i=a+1}^{a+n} X_i\right|^p \leq C \sum_{i=1}^n E|X_i|^p, \forall n \geq 1, a \geq 0$, We prove the equivalence of the limit distributions of an appropriately centered and normalized sum and the maximum sum of random variables $\{X_n\}$ with the assumption on finiteness of $E|X_n|^p$, where $1 < p < 2$. The result is an extension of a result of Sreehari (2010).

Keywords- $\tilde{\rho}$ -mixing; Limit distribution of maximum sum; Moment inequality; Maximal inequalities. MSC: 60G50, 60E07.

I. INTRODUCTION

Let $\{X_n\}$ be a random variable sequence defined on a fixed probability space (Ω, \mathcal{F}, P) . Write $\mathcal{F}_s = \sigma(X_i, i \in S \subset N)$. Given σ -algebras \mathcal{B}, \mathcal{R} in \mathcal{F} , let

$$\rho(\mathcal{B}, \mathcal{R}) = \sup_{X \in L_2(\mathcal{B}), Y \in L_2(\mathcal{R})} \frac{|EXY - EXEY|}{(\text{Var}X \cdot \text{Var}Y)^{1/2}}.$$

Following Bradley (1990), for $k \geq 0$, define the $\tilde{\rho}$ -mixing coefficients by

$$\beta(k) = \sup\{\rho(\mathcal{F}_S, \mathcal{F}_T) : \text{finite subsets } S,$$

$$T \subset N, \text{ such that } \text{dist}(S, T) \geq k\}, k \geq 0.$$

Definition 1.1 A sequence of random variables $\{X_n\}$ is said to be a $\tilde{\rho}$ -mixing sequence if there exists $k \in N$ such that $\beta(k) < 1$.

$\tilde{\rho}$ -mixing random variables were introduced by Bradley (1992) and many applications have been found. $\tilde{\rho}$ -mixing is similar to ρ -mixing, but they are in some ways quite different. Many authors have studied this concept and provided interesting results and applications. See, for example, Bradley (1992,1993) and Miller (1994) for limit properties under the condition $\tilde{\rho}(k) \rightarrow 0$, Bryc and Smolenski (1993) for moment inequalities and almost sure convergence, Peligrad and Gut (1999) for almost sure results, Utev and Peligrad (1999) for moment inequalities, Meng Y. J., Lin Z.Y. for strong laws of large numbers, Kuczmaszewska(2008) for Chung-Teicher type SLLN, and so forth.

For independent random variables, Kruglov and Bo (1996) obtained necessary and sufficient conditions for weak convergence with a mixing property in Renyi's sense of distribution functions (dfs) of normalized monotone sequences of random variables with random index. In particular, they investigated the convergence in distribution of S_{N_n} and $\max_{1 \leq k \leq N_n} S_k$ where $S_k = \sum_{j=1}^k X_j$; $\{X_n\}$ is a sequence of independent random variables with finite variances but a common mean and N_n is a sequence of positive integer valued random variables independent of $\{X_n\}$ such that $\frac{N_n}{n}$ converges in probability to a random variable ν . In a subsequent paper, Kruglov (1999) showed that the method of proof given in Kruglov and Bo (1996) is applicable to prove the following:

Theorem 1.1 (Kruglov, 1999).

Let $\{X_n\}$ be a sequence of independent random variables. Suppose $0 < a \leq EX_n \leq b < \infty$ and $0 < \sigma^2 \leq \text{var}(X_n) \leq \Delta^2 < \infty$, $n = 1, 2, \Lambda$. Then

$$\lim_{n \rightarrow \infty} P \left\{ \frac{S_n - ES_n}{\sqrt{\text{var}(S_n)}} \leq x \right\} = F(x), \quad x \in R$$

with some distribution function F if and only if

$$\lim_{n \rightarrow \infty} P \left\{ \frac{\max_{1 \leq k \leq n} S_k - ES_n}{\sqrt{\text{var}(S_n)}} \leq x \right\} = F(x), \quad x \in R.$$

Let $\{B_n\}$ be a sequence of scaling constants satisfying the conditions:

$$0 < B_n \leq B_{n+1} \rightarrow \infty, B_n = n^{1/\alpha} L(n) \quad (1.1)$$

for some α ; $1 < \alpha \leq 2$, and $L(x)$ be a slowly varying function. Suppose, further, that for some p , $1 < p < \alpha$; $E(X_n^p)$ exists and satisfies

$$0 < a \leq EX_n < E|X_n|^p \leq b < \infty, \quad n = 1, 2, \Lambda \quad (1.2)$$

M. Sreehari (2010) obtained the similar results with the assumption on finiteness of $E|X_n|^p$, where $1 < p < 2$:

Theorem 1.2 (Sreehari, 2010) Under the assumptions (1.1) and (1.2),

$$\lim_{n \rightarrow \infty} P \{S_n - E(S_n) \leq xB_n\} = F(x), \quad x \in C(F)$$

with some distribution function F if and only if

$$\lim_{n \rightarrow \infty} P \left\{ \max_{1 \leq k \leq n} S_k - E(S_n) \leq xB_n \right\} = F(x), \quad x \in C(F).$$

Holds, where $C(F)$ is the set of continuity points of F .

The aim of this work is to relax the assumption on independence of the random variables $\{X_n\}$ and assume $\{X_n\}$ is $\tilde{\rho}$ -mixing random variables. Our proof is essentially the same as that of Kruglov (1999) and we use lemma 2.2 in place of Kolmogorov's inequality.

2. THE MAIN RESULT

We assume that all the random variables that we consider are nondegenerate and defined on a common probability space (Ω, \mathcal{F}, P) . Let $\{X_n\}$ be a sequence of $\tilde{\rho}$ -mixing random variables, Let $S_0 = 0, S_n = \sum_{i=1}^n X_i$.

We now state and prove the main theorem, that is, theorem 1.2 is also true if $\{X_n\}$ is a sequence of $\tilde{\rho}$ -mixing random variables.

Theorem 2.1 Let $\{X_n\}$ be a sequence of $\tilde{\rho}$ -mixing random variables. Under the assumptions (1.1) and (1.2), and $\tilde{\rho}(1) < 1$,

$$\lim_{n \rightarrow \infty} P \{S_n - ES_n \leq xB_n\} = F(x), \quad x \in R \quad (2.1)$$

with some distribution function F , if and only if

$$\lim_{n \rightarrow \infty} P \left\{ \max_{1 \leq k \leq n} S_k - ES_n \leq xB_n \right\} = F(x), \quad x \in R. \quad (2.2)$$

In order to prove the theorem, we need the following lemmas.

Lemma 2.1 (Yang Shanchao, 1998) Let $\{X_n\}$ be a sequence of $\tilde{\rho}$ -mixing random variables. Suppose $\tilde{\rho}(1) < 1$, $EX_i = 0, E|X_i|^p < \infty, 1 < p \leq 2$. Then there exists a positive constant C depending only on p and $\tilde{\rho}$ such that

$$E \left| \sum_{i=a+1}^{a+n} X_i \right|^p \leq C \sum_{i=1}^n E|X_i|^p, \quad \forall n \geq 1, a \geq 0.$$

Lemma 2.2 Let $\{X_n\}$ be a sequence of $\tilde{\rho}$ -mixing random variables. Suppose $\tilde{\rho}(1) < 1, EX_i = 0,$

$E|X_i|^p < \infty, 1 < p \leq 2$. Then there exists a positive constant C depending only on p and $\tilde{\rho}$ such that

$$E \max_{1 \leq k \leq n} \left| \sum_{i=1}^k X_i \right|^p \leq C (\log 4n)^p \sum_{i=1}^n E|X_i|^p$$

Proof of Lemma 2.2. The first step consists in proving the inequality for $n = 2^k$, k being an arbitrary positive integer. To avoid obscuring the basic idea we present the proof for $k = 6$ only. Let $X_{r,s} = \sum_{i=r+1}^s X_i$ for $0 \leq r < s \leq 2^6$. Consider the following collections of $X_{r,s}$:

$$\{X_{0,64}\};$$

$$\{X_{0,32}, X_{32,64}\};$$

$$\{X_{0,16}, X_{16,32}, X_{32,48}, X_{48,64}\};$$

$$\{X_{0,8}, X_{8,16}, X_{16,24}, X_{24,32}, X_{32,40}, X_{40,48}, X_{48,56}, X_{56,64}\};$$

$$\{X_{0,4}, X_{4,8}, X_{8,12}, X_{12,16}, X_{16,20}, X_{20,24}, X_{24,28}, X_{28,32}, X_{32,36}, X_{36,40}, X_{40,44}, X_{44,48}, X_{48,52}, X_{52,56}, X_{56,60}, X_{60,64}\};$$

$$\{X_{0,2}, X_{2,4}, X_{4,6}, X_{6,8}, X_{8,10}, X_{10,12}, X_{12,14}, X_{14,16}, X_{16,18}, X_{18,20}, X_{20,22}, X_{22,24}, X_{24,26}, X_{26,28}, X_{28,30}, X_{30,32}, X_{32,34}, X_{34,36}, X_{36,38}, X_{38,40}, X_{40,42}, X_{42,44}, X_{44,46}, X_{46,48}, X_{48,50}, X_{50,52}, X_{52,54}, X_{54,56}, X_{56,58}, X_{58,60}, X_{60,62}, X_{62,64}\};$$

$$\{X_{0,1}, X_{1,2}, L, X_{62,63}, X_{63,64}\}.$$

There are $k+1=7$ collections. Choose $1 \leq i \leq 2^6$. We expand S_i , choosing the terms of this expansion from the collections above and choosing the minimal possible number of terms in the expansion. It's clear that at most one term is required from each collection. As an example,

$$X_{0,63} = X_{0,32} + X_{32,48} + X_{48,56} + X_{56,60} + X_{60,62} + X_{62,63}.$$

Thus each expansion then has at most $k+1=7$ terms in it. Represent the expansion of S_i by

$$S_i = \sum_{j=1}^h X_{i_{j-1}, i_j} \quad (h \leq 7),$$

$$|S_i| \leq \sum_{j=1}^h |X_{i_{j-1}, i_j}|.$$

Obviously

$$\begin{aligned} \sum_{j=1}^h |X_{i_{j-1}, i_j}| &\leq |X_{0,64}| + (|X_{0,32}| + |X_{32,64}|) \\ &+ (|X_{0,16}| + |X_{16,32}| + |X_{32,48}| + |X_{48,64}|) \\ &+ \Lambda \\ &+ (|X_{0,1}| + |X_{1,2}| + L + |X_{63,64}|); \end{aligned}$$

It is easy to see

$$\begin{aligned} |X_{0,64}| &\leq |X_{0,1}| + |X_{1,2}| + L + |X_{63,64}| \\ |X_{0,32}| + |X_{32,64}| &\leq |X_{0,1}| + |X_{1,2}| + L + |X_{63,64}| \\ \Lambda, \end{aligned}$$

$$\begin{aligned} |X_{0,2}| + |X_{2,4}| + L + |X_{62,64}| \\ \leq |X_{0,1}| + |X_{1,2}| + L + |X_{63,64}| \end{aligned}$$

So

$$\begin{aligned} |S_i| &\leq \sum_{j=1}^h |X_{i_{j-1}, i_j}| \\ &\leq 7(|X_{0,1}| + |X_{1,2}| + L + |X_{63,64}|) \\ |S_i|^p &\leq 7^p (|X_{0,1}| + |X_{1,2}| + L + |X_{63,64}|)^p \end{aligned}$$

$$\max_{1 \leq i \leq 2^6} |S_i|^p \leq 7^p (|X_{0,1}| + |X_{1,2}| + L + |X_{63,64}|)^p$$

By use of lemma 2.1,

$$\begin{aligned} E \max_{1 \leq i \leq 2^6} |S_i|^p &\leq 7^p E (|X_{0,1}| + |X_{1,2}| + L + |X_{63,64}|)^p \\ &\leq 7^p C \sum_{i=1}^{2^6} E |X_i|^p \end{aligned} \quad (2.3)$$

The above argument generalized to arbitrary $k \geq 1$ yields

$$E \max_{1 \leq i \leq 2^k} |S_i|^p \leq C(k+1)^p \sum_{i=1}^{2^k} E |X_i|^p.$$

Given an n such that $n \neq 2^k$ for every $k \geq 1$, choose k such that $2^{k-1} < n < 2^k$ and redefine $X_i = 0$ if $n < i \leq 2^k$. By (2.3),

$$E \max_{i \leq n} |S_i|^p \leq C(k+1)^p \sum_{i=1}^{2^k} E |X_i|^p.$$

Since $2^{k-1} < n$ implies $k+1 \leq \frac{\log 4n}{\log 2}$,

$$E \max_{1 \leq k \leq n} \left| \sum_{i=1}^k X_i \right|^p \leq C (\log 4n)^p \sum_{i=1}^n E |X_i|^p.$$

The proof of Lemma 2.2 is complete.

Proof of Theorem 2.1. Suppose α is such that $1 < p < \alpha \leq p^2 \leq 2$. Choose a γ satisfying

$$1 - \frac{p}{\alpha} < \gamma < 1 - \frac{1}{p}.$$

Set $r_n = [n - n^{1-\gamma}]$, where $[x]$ denotes the largest integer not exceeding x . Denote $T_n = S_n - ES_n$. For any real $\varepsilon > 0$ and x , the inequalities

$$\begin{aligned} P\{S_n \leq ES_n + xB_n\} &\geq P\left\{\max_{1 \leq k \leq n} S_k \leq ES_n + xB_n\right\} \\ &= P\left\{\max_{1 \leq k \leq r_n} S_k, \max_{r_n \leq k \leq n} S_k \leq ES_n + xB_n\right\} \\ &\geq P\left\{\max_{r_n < k \leq n} S_k \leq ES_n + xB_n\right\} \\ &\quad - P\left\{\max_{1 \leq k \leq r_n} S_k > ES_n + xB_n\right\} \\ &\geq P\left\{S_n \leq ES_n + (x - \varepsilon)B_n, \max_{r_n < k \leq n} S_k - S_n \leq xB_n\right\} \\ &\quad - P\left\{\max_{1 \leq k \leq r_n} S_k > ES_n + xB_n\right\} \end{aligned}$$

$$\begin{aligned}
&\geq P\{S_n \leq ES_n + (x - \varepsilon)B_n\} \\
&\quad - P\left\{\max_{r_n < k \leq n} S_k - S_n > xB_n\right\} \\
&\quad - P\left\{\max_{1 \leq k \leq r_n} S_k > ES_n + xB_n\right\} \\
&\geq P\{S_n \leq ES_n + (x - \varepsilon)B_n\} \\
&\quad - P\left\{\max_{r_n < k \leq n} (T_k - T_n) > \varepsilon B_n\right\} \\
&\quad - P\left\{\max_{1 \leq k \leq r_n} T_k > ES_n - ES_{r_n} + xB_n\right\}
\end{aligned}$$

Hold. So the equivalence of (2.1) and (2.2) follows if we establish

$$P\left\{\max_{1 \leq k \leq r_n} T_k > ES_n - ES_{r_n} + xB_n\right\} \rightarrow 0 \quad \forall x \in R \quad (2.4)$$

$$P\left\{\max_{r_n < k \leq n} (T_k - T_n) > xB_n\right\} \rightarrow 0 \quad \forall \varepsilon > 0 \quad (2.5)$$

Since $1 - \gamma > 1/\lambda$, we note that

$$ES_n - ES_{r_n} + xB_n > an^{1-\gamma} + xB_n > 0$$

for all n large enough. For these n , by lemma 2.2 we have

$$\begin{aligned}
&P\left\{\max_{1 \leq k \leq r_n} T_k > ES_n - ES_{r_n} + xB_n\right\} \\
&\leq P\left\{\max_{1 \leq k \leq r_n} |T_k| > E(S_n - S_{r_n}) + xB_n\right\} \\
&\leq \frac{C(\log 4r_n)^p}{\left[E(S_n - S_{r_n}) + xB_n\right]^p} \sum_{i=1}^{r_n} E|X_i - EX_i|^p
\end{aligned}$$

Since

$$\begin{aligned}
&\left[E(S_n - S_{r_n}) + xB_n\right]^p \\
&> \left[(n - r_n)a + xB_n\right]^p \\
&> \left[n^{1-\gamma}a + xn^{1/\alpha}L(n)\right]^p \\
&\approx n^{(1-\gamma)p}
\end{aligned}$$

and $(1 - \gamma)p > 1$, (2.4) holds.

Now we prove (2.5). We note that

$$\begin{aligned}
&P\left\{\max_{r_n < k \leq n} (T_k - T_n) > \varepsilon B_n\right\} \\
&\leq P\left\{\max_{r_n < k \leq n} |T_k - T_n| > \varepsilon B_n\right\}
\end{aligned}$$

$$\begin{aligned}
&\leq P\{\max\{|V_{r_n+2} + L + V_n|, \\
&\quad |V_{r_n+3} + L + V_n|, L, |V_n|, 0\} > \varepsilon B_n\}
\end{aligned}$$

Where $V_i = X_i - EX_i$. as $n \rightarrow \infty$, by Lemma 2.2, we have

$$\begin{aligned}
&P\left\{\max_{r_n < k \leq n} (T_k - T_n) > \varepsilon B_n\right\} \\
&\leq \frac{C(\log 4(n - r_n - 2))^p}{(xB_n)^p} \sum_{i=0}^{n-r_n-2} E|X_{n-i} - EX_{n-i}|^p \\
&\leq \frac{C(\log n)^p (n - r_n)}{(B_n)^p} \\
&\approx \frac{(\log n)^p n^{1-\gamma}}{\left[n^{1/\alpha}L(n)\right]^p}.
\end{aligned}$$

Since $1 - \gamma < p/\alpha$, (2.5) holds. The proof is complete.

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