Torque criteria for the stable slow-varying 2-DOF Planar Manipulator Robot (PMR) with identical Stokes damping

Deriyan Senjaya[1*], Muhammad Ismail Yunus[2]

[1]Department of Physics, National Tsing Hua University, Hsinchu 300, Taiwan
[2]Department of Mathematics, Institut Teknologi Bandung, Indonesia
deriyan191293@gmail.com, muhammadismailyunus@gmail.com

Abstract. The main goal of this research is to find torque criteria for the stable 2-DOF planar manipulator robot (PMR) with identical Stokes damping in a slow-varying (SV) regime. The analysis starts by finding the 2-DOF PMR with Stokes damping Equation of Motion (EOM) via the Lagrangian mechanics. In the next step, the SV approximation is applied to those EOMs and produce new EOMs with identical Stokes damping. These SV approximation EOM stability criteria are analyzed using the Jacobian formalism. From the Jacobian formalism, this research found that the SV regime stability criteria for identical Stokes damping do not depend on the value of damping constants. Then, from these stability criteria, this research also found six possibilities of stable eigenvalue regions with the example of PMR link masses $M_1 = M_2 = M$ and PMR link lengths $a_1 = a_2 = a$. All six stable eigenvalue regions correspond to the interval of the 2-DOF PMR joint angles. In general, those six stable eigenvalue regions are the same, so this research only picked one of the six stable eigenvalue regions, the fifth region. The fifth region leads to the two scenarios of joint angle intervals, $\{\theta_1^*, \theta_2^*\} = \{(\pi, 2\pi), (-\pi, 0)\}$; $\{(\pi, 2\pi), (0, \pi)\}$. According to those joint angle intervals, the torque criteria for the stable 2-DOF PMR with identical Stokes damping in an SV regime should be
$\{\tau_1', \tau_2'\} = \{(-2Mga, 2Mga), (-0.5Mga, 0.5Mga)\}$ and
$\{\tau_1', \tau_2'\} = \{(-2Mga, 1.5Mga), (-0.5Mga, 0)\}$.

Keywords: First Keyword, Second Keyword, Third Keyword.

1 Introduction

Our world is now in the era of the 4.0 industry revolution. The 4.0 industry revolution insists on the vast development of robotics technology in the industrial sector. It happens because people's need is increasing over time. In the industrial sector, the manipulator robot is the most preferable robot type. The manipulator robot is the most preferable because this robot type can manipulate the object's position and orientation accurately using some links and joints [1]. The popularity of this robot type in industrial applications leads many scientists and engineers to study its dynamics and how to control it. In reality, manipulator robot dynamics are not completely easy to examine. One of the obstructions is the presence of the so-called Stokes damping in the

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of the obstructions is the presence of the so-called Stokes damping in the manipulator's motors. The presence of the Stokes damping makes all of the equations more complex.

According to this fact, this research tried to model and analyze one of the dynamical system parameters, the so-called stability, for the manipulator robot with the Stokes damping. As the first attempt to get the stability analysis of the manipulator robot with the Stokes damping, this research used the simple manipulator robot, the 2-degree of freedom (DOF) manipulator. This research also used the identical damping constant in the two motors. This research employed the Jacobian formalism to find the stability of the 2-DOF PMR produced from the 2-DOF PMR Equation of Motions (EOMs) via the Lagrangian mechanics. The stability criteria for the 2-DOF PMR with identical Stokes damping is easier to understand using the language of physics. The language of physics for the manipulator robot is the motor's torque. The torque is the input for the manipulator robot movement. Therefore, this research also produced the torque criteria for the 2-DOF PMR with identical Stokes damping as the manifestation of the stability criteria. Through this manifestation, the researchers hope that the results are more applicable and give insight to another advanced research in the scope of the dynamics of the PMR with Stokes damping.

2 Kinematics and Dynamics of 2-dof Planar Manipulator Robot (PMR)

The 2-DOF Planar Manipulator Robot (PMR) is a simple example of a manipulator robot that moves in a two-dimensional plane and only has two revolute joints and two links [1]. In this part, the kinematic and dynamic equations are explained first before their implementation in this research.

a. Kinematics of 2-DOF PMR using Denavit-Hartenberg Convention

The kinematic formula of the manipulator robot shows the position and orientation of the robot's end-effector relative to the robot's base in terms of joint parameters (angle or displacement) [2]. There are many methods to obtain the kinematic formula of the manipulator robot. One can use the trigonometry relation, but it becomes complicated for a manipulator robot with a high degree of freedom (DOF). The kinematic formula also enables us to describe the velocity and acceleration of the robot's end-effector relative to the robot's base in terms of joint speed and acceleration. For the open-chain manipulator robot, the general method to calculate the set of kinematic formulas is the Denavit-Hartenberg (D-H) convention. The DH convention employs the successive homogeneous transformation T_i (part of the SE(3) group) for each robot's arm reference frame with specific rules for those reference frame establishments [2]. The equation of T_i is shown in Eq. (1) below.

$$T_i = \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \bar{R}_Z(\theta_i)\bar{P}(a_i, d_i)\bar{R}_X(\alpha_i) \quad (1)$$

where θ_i is the rotation angle about the Z_{i-1} axis of the link i-1 to align the X_{i-1} axis to the X_i axis, α_i is the twist angle about the X_{i-1} axis of the link i-1 to align the Z_{i-1} axis to the Z_i axis, a_i is the perpendicular distance between the Z_{i-1} axis to the Z_i axis, and d_i is the joint offset. The symbols $\bar{R}_Z(\theta_i)$, $\bar{P}(a_i, d_i)$, and $\bar{R}_X(\alpha_i)$ here are the rotation about the Z-axis, the translation in the X and Z direction, and the rotation about the X-axis, respectively, in the form of homogeneous transformation. The kinematic equations obtained from the D-H convention are equal to the total composition transformation of TI as follows (Eq. (2)). Here, the *N* denotes the number of robot links. The $R_{3\times3}$ and $P_{3\times1}$ in Eq. (2) are the kinematic equations (orientation and position, respectively).

$$T = T_1 \circ T_2 \circ T_3 \ldots \circ T_N = \begin{bmatrix} R_{3\times3} & P_{3\times1} \\ 0_{1\times3} & 1 \end{bmatrix} \tag{2}$$

In this paper, the 2-DOF Planar Manipulator Robot (PMR) is the object of this study. Therefore, the kinematic equations for this 2-DOF PMR are obtained in advance by the D-H Convention (via the rules in [2] and using Eq. (1)). This kinematic analysis is a primary point to obtain the dynamical analysis of the 2-DOF PMR in this research. The schematic diagram for the 2-DOF PMR is shown in Fig. (1) below.

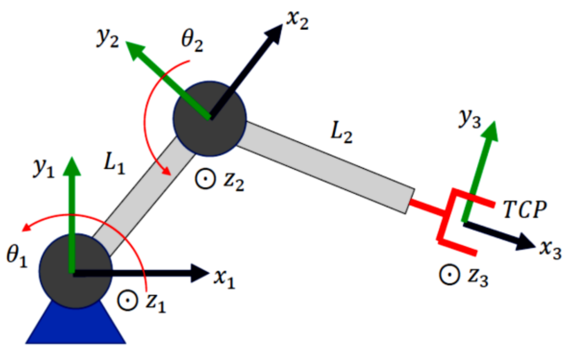


FIGURE 1. The schematic diagram of the 2-DOF PMR: TCP is the abbreviation of the Tool Center Point and the part of the end-effector, L_1 is the length of the first link, and L_2 is the length of the second link.

Based on the D-H convention rules in [2], for this 2-DOF PMR, the D-H parameters (a_i, d_i, θ_i, and α_i) can be tabulated in Table 1. Using Eq. (1) and all of the parameters in Table 1, one can obtain the expression of homogeneous transformation for each link as follows (Eq. (3) and Eq. (4)).

TABLE 1. The tabulation of the D-H parameters for the 2-DOF PMR (θ_{o1} and θ_{o2} are the initial configurations of the joint angle of the first and the second arm)

Link	$\boldsymbol{a_i}$	$\boldsymbol{\theta}$	$\boldsymbol{d_i}$	$\boldsymbol{\alpha_i}$
1	$L_1 = a_1$	$\theta_1 + \theta_{o_1}$	0	0

2	$L_2 = a_2$	$\theta_2 + \theta_{o_2}$	0	0

$$T_1 = \begin{bmatrix} \cos(\theta_1 + \theta_{o_1}) & -\sin(\theta_1 + \theta_{o_1}) & 0 & a_1 \cos(\theta_1 + \theta_{o_1}) \\ \sin(\theta_1 + \theta_{o_1}) & \cos(\theta_1 + \theta_{o_1}) & 0 & a_1 \sin(\theta_1 + \theta_{o_1}) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$

$$T_2 = \begin{bmatrix} \cos(\theta_2 + \theta_{o_2}) & -\sin(\theta_2 + \theta_{o_2}) & 0 & a_2 \cos(\theta_2 + \theta_{o_2}) \\ \sin(\theta_2 + \theta_{o_2}) & \cos(\theta_2 + \theta_{o_2}) & 0 & a_2 \sin(\theta_2 + \theta_{o_2}) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

Using Eq. (2) and the results of Eq. (3) and Eq. (4), one can obtain the $P_{3\times3}$ for the 2-DOF PMR as follows (Eq. (5)). By taking the derivative of all of the elements in Eq. (5) to time, the velocity of the end-effector follows Eq. (6).

$$P_{3\times1} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} a_1 \cos(\theta_1 + \theta_{o_1}) + a_2 \cos(\theta_1 + \theta_2 + \theta_{o_1} + \theta_{o_2}) \\ a_1 \sin(\theta_1 + \theta_{o_1}) + a_2 \sin(\theta_1 + \theta_2 + \theta_{o_1} + \theta_{o_2}) \\ 0 \end{bmatrix} \tag{5}$$

$$\dot{P}_{3\times1} = \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} -a_1\dot{\theta}_1 \sin(\theta_1 + \theta_{o_1}) - a_2(\dot{\theta}_1 + \dot{\theta}_2) \sin(\theta_1 + \theta_2 + \theta_{o_1} + \theta_{o_2} \\ a_1\dot{\theta}_1 \cos(\theta_1 + \theta_{o_1}) + a_2(\dot{\theta}_1 + \dot{\theta}_2) \cos(\theta_1 + \theta_2 + \theta_{o_1} + \theta_{o_2}) \\ 0 \end{bmatrix} \tag{6}$$

b. Dynamics of ideal 2-DOF PMR using Lagrangian Mechanics

Manipulator robot dynamical equations are hard to obtain due to their structural complexities (for example, the number of the degree of freedom). One of the easiest methods to find the manipulator robot's dynamical equations is the Lagrangian mechanics [3]. In physics, Lagrangian mechanics obtain the physical system's equation of motion (EOM) by minimizing the system's action S (Eq. (7)).

$$S = \int L(q_k, \dot{q}_k, t)dt \tag{7}$$

The Lagrangian function $L(q_k, \dot{q}_k, t)$ is a function that describes the total kinetic $T(\dot{q}_k)$ and potential energy $V(q_k)$ of the system as $L(q_k, \dot{q}_k, t) = T(\dot{q}_k) - V(q_k)$. Here, the variables q_k and $\dot{q}_k$ are the generalized coordinates and the generalized speeds. From the functional derivative to Eq. (7) through $q_k \to q_k + \epsilon\delta q_k$, one can write δS as follows (Eq. (9)).

$$\delta S = \int \left[\frac{\partial L}{\partial q_k}\delta q_k + \frac{\partial L}{\partial \dot{q}_k}\delta \dot{q}_k\right] dt = \int \left[\frac{\partial L}{\partial q_k}\delta q_k + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\delta q_k\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\delta q_k\right] dt \tag{8}$$

$$\delta S = \int \left[\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\right]\delta q_k dt + \int \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\delta q_k\right) dt = \int \left[\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\right]\delta q_k dt + \partial\Omega \tag{9}$$

Minimizing the system's action leads to the $\delta S = 0$ (Any physical system should follow the shortest path with minimum energy). So, Eq. (9) turns into Eq. (10) as follows.

$$\int \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k}\right] \delta q_k dt = \int \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\delta q_k\right) dt = \partial\Omega \quad (10)$$

The integrand on the left side of Eq. (10) is the generalized Euler-Lagrange (EL) equation. This EL equation is related to the generalized force F_k (translational forces and torques).

$$F_k = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} \quad (11)$$

Eq. (11) acts as the generalized EOM of the physical system.

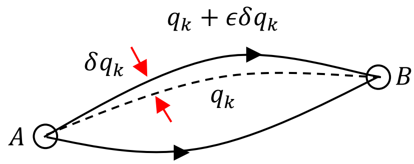


FIGURE 1. Action minimization : This process finds q_k from A to B by variating q_k with the small parameter ϵ of δq_k.

The physical system might have several damping factors. Damping is one of the kinds of friction that acts on the system. Damping induces the reduction of energy in the physical system (energy losses). The manifestation of the damping is the so-called Rayleigh dissipation term (R) in Eq. (12) [4]. Rayleigh dissipation term (R) is a function the generalized velocities $\dot{q}_k$.

$$F_k = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} + \frac{\partial R}{\partial \dot{q}_k} \quad (12)$$

3 Stability Criterion in Dynamical System

2-DOF PMR is one of the dynamical systems. The generalized coordinates for 2-DOF PMR are θ_1 and θ_2 (joint angles). Then, the generalized forces for 2-DOF PMR are τ_1 and τ_2 (torques for first and second motors). All EOM produced from Eq. (11) provides complete dynamical relation between those joint angles and torques in the form of Ordinary Differential Equations (ODE). This ODE can use to see some properties of the 2-DOF PMR, such as stability and controllability. Before finding the torque criteria for stable 2-DOF PMR, this section describes a general mathematical method to deduce the stability criterion from the ODE first.

a. Mathematical Method to Analyze Stability Via ODE

In mathematics, there are two kinds of ODE of the Dynamical System. Those kinds of ODE are autonomous and non-autonomous [5, 6]. This part emphasizes the autonomous-type ODE only. The Autonomous-type ODE is the ODE that does not explicitly depend on the independent variable [6]. If the variable is time *t*, this ODE is sometimes called a time-invariant (TI) ODE. So, the autonomous-type ODE describes in the following form (see Eq. (13) below).

$$\dot{x} = f(x) \quad f: D \rightarrow \mathbb{R}^n \tag{13}$$

Here *D* is an open and connected subset of $\mathbb{R}^n$, and *f* is a local Lipschitz map from *D* into $\mathbb{R}^n$.

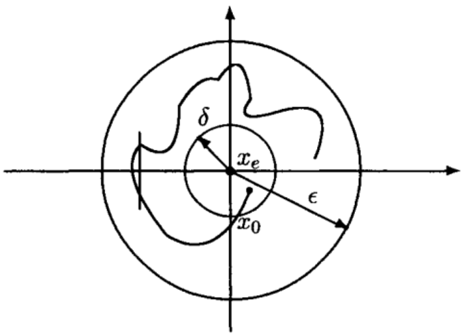


FIGURE 2. Illustration of Stable Equilibrium Point of Autonomous ODE [6]

Let the autonomous ODE has an equilibrium point at $x = x_e = x^*$, so $f(x_e) = 0$. The equilibrium point $x_e = x^*$ of the autonomous ODE is said to be stable if for each $\varepsilon > 0, \exists\delta = \delta(\varepsilon) > 0$ [6]

$$\|x(0) - x_e\| < \delta \rightarrow \|x(t) - x_e\| < \varepsilon \quad \forall t \geq t_o \tag{14}$$

The illustration of the equilibrium point of the autonomous ODE describes in Fig. 2. If the condition in Eq. (13) does not occur, the equilibrium point of the autonomous ODE is unstable. Eq. (14) can precisely describe in the following way. Let $\lambda = \lambda(t)$ be a small perturbation to the x_e, so $x' = x_e + \lambda(t)$. Then,

$$\dot{x}' = f(x') \rightarrow \frac{dx_e}{dt} + \frac{d\lambda}{dt} = \dot{\lambda} = f(x_e + \lambda) \tag{15}$$

Taking the Taylor series to $f(x_e + \lambda)$, then Eq. (15) can be written in Eq. (16).

$$\dot{\lambda} = f(x_e) + \lambda \left[\frac{df}{dx}\right]_{x=x_e} + O(\lambda^2) \tag{16}$$

Because $f(x_e) = 0$ (equilibrium point of the autonomous ODE), then Eq. (16) can be written in Eq. (17).

$$\dot{\lambda} = \lambda \left[\frac{df}{dx}\right]_{x=x_e} + O(\lambda^2) = \lambda f'(x_e) + O(\lambda^2) \tag{17}$$

The perturbation $\lambda(t)$ is small enough compared to the x_e. So, the dominant term in the series in Eq. (17) is only the first term. This argument leads to the neglect of higher terms. Then, Eq. (17) can be written in Eq. (18) and Eq.(19) as follows.

$$\begin{aligned} \dot{\lambda} \approx \lambda f'(x_e) \rightarrow \int \frac{d\lambda}{\lambda} \\ = \int f'(x_e)dt \rightarrow \ln \lambda(t) = f'(x_e)t + C \rightarrow \lambda(t) = e^C e^{tf'(x_e)} \end{aligned} \tag{18}$$

$$\lambda(t) = \lambda_o e^{tf'(x_e)} \qquad \lambda_o = \lambda(0), t \geq 0 \tag{19}$$

If $f'(x_e) > 0$, λ increases higher for $t \rightarrow \infty$. The increase in λ indicates an unstable equilibrium point. Meanwhile, if $f'(x_e) < 0$, λ decreases for $t \rightarrow \infty$ and is a stable equilibrium point.

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, \dots, x_n) \\ \dot{x}_2 &= f_2(x_1, x_2, \dots, x_n) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, x_2, \dots, x_n) \end{aligned} \tag{20}$$

The analysis in Eq. (18) and Eq. (19) works well for the single autonomous ODE. How about the autonomous-type ODE system like in Eq. (20)? For the autonomous-type ODE system, the Jacobian formalism is more convenient to figure out the stability problem. The first step is the same as the one ODE, that is, finding the equilibrium point. Then, to obtain the Jacobian of the ODE system, write Eq. (20) into the matrix equation as Eq. (21).

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \frac{\partial \dot{x}_1}{\partial x_2} & \cdots & \frac{\partial \dot{x}_1}{\partial x_n} \\ \frac{\partial \dot{x}_2}{\partial x_1} & \frac{\partial \dot{x}_2}{\partial x_2} & \cdots & \frac{\partial \dot{x}_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \dot{x}_n}{\partial x_1} & \frac{\partial \dot{x}_n}{\partial x_2} & \cdots & \frac{\partial \dot{x}_n}{\partial x_n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = J(x_1, x_2, \dots, x_n) \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{21}$$

The stability of the ODE system via Jacobian formalism obtains from the eigenvalue ϵ of the Jacobian. To find the eigenvalue of the Jacobian, solve the secular equation first (Eq. (22)).

$$\det\left|J(x_1, x_2, \dots, x_n) - \epsilon \hat{I}\right| = 0 \tag{22}$$

where $\hat{I}$ is the identity matrix. When the $Re\big(\epsilon(x_1^*, x_2^*, \dots, x_n^*)\big) < 0\ \forall \epsilon(x_1^*, x_2^*, \dots, x_n^*)$, the ODE system is stable. The ODE system is unstable if there is one of the $\epsilon(x_1^*, x_2^*, \dots, x_n^*)$ with $Re\big(\epsilon(x_1^*, x_2^*, \dots, x_n^*)\big) > 0$.

4 Results

This section starts from the formulation of EOM for the 2-DOF PMR with Stokes-type damping. After that, the explanation will continue to the implementation of the SV approximation and identical Stokes damping to those EOMs. The final part describes the stable torque criteria analysis for the 2-DOF PMR with identical Stokes damping in the SV regime in detail via the Jacobian formalism.

a. Equation of Motion for the 2-DOF PMR with Stokes-Type Damping

The EOM for the 2-DOF PMR with Stokes-type damping starts from writing the kinetic energy of each link. Let M_1 and M_2 be the link masses, a_1 and a_2 be the link lengths, I_1 and I_2 be the link moment of inertia, and then the kinetic energy for each link is written in Eq. (23) and Eq. (24).

$$T_1 = \frac{1}{2} M_1 a_{c_1}^2 \dot{\theta}_1^2 + \frac{1}{2} I_1 \dot{\theta}_1^2 \tag{23}$$

$$T_2 = \frac{1}{2} M_2 a_1^2 \dot{\theta}_1^2 + M_2 a_1 a_{c_2} \cos\theta_2 \left[\dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2\right] + \frac{1}{2} M_2 a_{c_2}^2 \dot{\theta}_1^2 + M_2 a_{c_2}^2 \dot{\theta}_1 \dot{\theta}_2 + P \tag{24}$$

The $P = \frac{1}{2} M_2 a_{c_2}^2 \dot{\theta}_2^2 + \frac{1}{2} I_2 \dot{\theta}_1^2 + I_2 \dot{\theta}_1 \dot{\theta}_2 + \frac{1}{2} I_2 \dot{\theta}_2^2$, and a_{c_1} and a_{c_2} are $\frac{1}{2} a_1$ and $\frac{1}{2} a_2$.

This 2-DOF PMR only experienced the gravitational potential energy. Eq. (25) and Eq. (26) show the gravitational potential energy for each link in the 2-DOF PMR.

$$V_1 = M_1 g a_{c_1} \sin\theta_1 \tag{25}$$

$$V_2 = M_2 g a_1 \sin\theta_1 + M_2 g a_{c_2} \sin(\theta_1 + \theta_2) \tag{26}$$

The Lagrangian L in general is as $L = T - V$. The total Lagrangian for each link will lead to the 2-DOF PMR Lagrangian. To introduce the Stokes-type damping, define the two Rayleigh dissipation terms as a function of the motor's angular velocity $\dot{\theta}_1$ and $\dot{\theta}_2$ as $R_1 = \alpha \dot{\theta}_1^2$ and $R_2 = \beta \dot{\theta}_2^2$. The R_1 and R_2 are chosen as the Rayleigh dissipation because the derivation of R_1 and R_2 to the $\dot{\theta}_1$ and $\dot{\theta}_2$ are analogous to the damping due to the viscosity of a fluid with damping coefficients α and β [7]. Therefore, the Lagrangian for each link has R_1 and R_2 terms as Eq. (27) and Eq. (28).

$$L_1 = T_1 - V_1 + R_1 = T_1 - V_1 + \alpha \dot{\theta}_1^2 \tag{27}$$

$$L_2 = T_2 - V_2 + R_2 = T_1 - V_1 + \beta\dot{\theta}_1^2 \tag{28}$$

According to the EL equation in Eq. (12), one can obtain the full EOMs for each 2-DOF PMR robot link with Stokes-type damping (Eq. (29)) in the matrix form.

$$\vec{\tau} = \begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = M_{2\times2}(\vec{q})\ddot{\vec{q}} + C_{2\times2}(\vec{q},\dot{\vec{q}})\dot{\vec{q}} + G_{2\times1}(\vec{q}) \tag{29}$$

Where *M*, *C*, and *G* are inertia, Coriolis, and gravity matrices of the 2-DOF PMR with Stokes-type damping (see Eq. (30), Eq. (31), and Eq. (32)).

$$M_{2\times2}(\vec{q}) = \begin{bmatrix} M_1 a_{c_1}^2 + M_2 a_1^2 + 2M_2 a_1 a_{c_2}\cos\theta_2 + M_2 a_{c_2}^2 + I_1 + I_2 & M_2 a_1 a_{c_2}\cos\theta_2 + M_2 a_{c_2}^2 + I_2 \\ M_2 a_1 a_{c_2}\cos\theta_2 + M_2 a_{c_2}^2 + I_2 & M_2 a_{c_2}^2 + I_2 \end{bmatrix} \tag{30}$$

$$C_{2\times2}(\vec{q},\dot{\vec{q}}) = \begin{bmatrix} -2M_2 a_1 a_{c_2}\dot{\theta}_2\sin\theta_2 + 2\alpha & -M_2 a_1 a_{c_2}\dot{\theta}_2\sin\theta_2 \\ M_2 a_1 a_{c_2}\dot{\theta}_1\sin\theta_2 & 2\beta \end{bmatrix} \tag{31}$$

$$G_{2\times1}(\vec{q}) = \begin{bmatrix} M_1 g a_{c_1}\cos\theta_1 + M_2 g a_1\cos\theta_1 + M_2 g a_{c_2}\cos(\theta_1+\theta_2) \\ M_2 g a_{c_2}\cos(\theta_1+\theta_2) \end{bmatrix} \tag{32}$$

b. Slow-Varying Equation of Motion for the 2-DOF PMR with Stokes-Type Damping

The slow-varying approach (SV) is the approximation method that assumes $\dot{\theta}_1$ and $\dot{\theta}_2$ are small enough. So, the second derivative of θ_1 and θ_2 to the time (acceleration) is approaching zero ($\ddot{\theta}_1 = \ddot{\theta}_2 \approx 0$). The SV approach reduces the full EOMs of the 2-DOF PMR with Stokes-type damping in Eq. (29) to become Eq. (33) and Eq. (34).

$$\tau_1 = M_1 g a_{c_1}\cos\theta_1 + M_2 g a_1\cos\theta_1 + M_2 g a_{c_2}\cos(\theta_1+\theta_2) + 2\alpha\dot{\theta}_1 \tag{33}$$

$$\tau_2 = M_2 g a_{c_2}\cos(\theta_1+\theta_2) + 2\beta\dot{\theta}_2 \tag{34}$$

Let $\tilde{\alpha} = \frac{M_1 g a_{c_1}}{2\alpha}$, $\tilde{\beta} = \frac{M_2 g a_1}{2\alpha}$, $\tilde{\gamma} = \frac{M_2 g a_{c_2}}{2\alpha}$, $\tilde{\delta} = \frac{\tau_1}{2\alpha}$, $\tilde{\varepsilon} = \frac{M_2 g a_{c_2}}{2\beta}$, and $\tilde{f} = \frac{\tau_2}{2\beta}$, and re-write Eq. (33) and Eq. (34) in the form of $\dot{\theta}_1$ and $\dot{\theta}_2$, one can obtain Eq. (35) and Eq. (36) as follows.

$$\dot{\theta}_1 = \tilde{\delta} - (\tilde{\alpha} + \tilde{\beta})\cos\theta_1 - \tilde{\gamma}\cos(\theta_1+\theta_2) = F_1(\theta_1,\theta_2) \tag{35}$$

$$\dot{\theta}_2 = \tilde{f} - \tilde{\varepsilon}\cos(\theta_1+\theta_2) = F_2(\theta_1,\theta_2) \tag{36}$$

c. The Implementation of Stability Criterion to the Slow-Varying Equation of Motion

To implement the stability criterion into the SV approach EOMs (Eq. (35) and Eq. (36)), find the equilibrium point for this system first as follows.

$$\dot{\theta}_1(\theta_1^*,\theta_2^*) = 0 \to \tilde{\delta} = (\tilde{\alpha}+\tilde{\beta})\cos\theta_1^* + \tilde{\gamma}\cos(\theta_1^*+\theta_2^*) \tag{37}$$

$$\dot{\theta}_2(\theta_1^*,\theta_2^*) = 0 \to \tilde{f} = \tilde{\varepsilon}\cos(\theta_1^*+\theta_2^*) \tag{38}$$

Using Eq. (37) and Eq. (38), one can obtain the equilibrium joint angles for each arm as Eq. (39) and Eq. (40).

$$\theta_1^* = \arccos\left[\frac{\tilde{\varepsilon}\tilde{\delta}-\tilde{\gamma}\tilde{f}}{\tilde{\varepsilon}(\tilde{\alpha}+\tilde{\beta})}\right] \tag{39}$$

$$\theta_2^* = \arccos\left[\frac{\tilde{f}}{\tilde{\varepsilon}}\right] - \arccos\left[\frac{\tilde{\varepsilon}\tilde{\delta}-\tilde{\gamma}\tilde{f}}{\tilde{\varepsilon}(\tilde{\alpha}+\tilde{\beta})}\right] \tag{40}$$

After determining the equilibrium joint angles for each arm, the next step is forming the Jacobian matrix (Eq. (41)). The formation of the Jacobian matrix involves the first derivative of Eq. (35) and Eq. (36) to the joint angles for each arm as Eq. (21).

$$J = \begin{bmatrix} \frac{\partial\dot{\theta}_1}{\partial\theta_1} & \frac{\partial\dot{\theta}_1}{\partial\theta_2} \\ \frac{\partial\dot{\theta}_2}{\partial\theta_1} & \frac{\partial\dot{\theta}_2}{\partial\theta_2} \end{bmatrix} = \begin{bmatrix} (\tilde{\alpha}+\tilde{\beta})\sin\theta_1 + \tilde{\gamma}\sin(\theta_1+\theta_2) & \tilde{\gamma}\sin(\theta_1+\theta_2) \\ \tilde{\varepsilon}\sin(\theta_1+\theta_2) & \tilde{\varepsilon}\sin(\theta_1+\theta_2) \end{bmatrix} \tag{41}$$

After determining the Jacobian matrix, obtain the eigenvalue (let the eigenvalue of the Jacobian matrix be ϕ) of the Jacobian matrix by using the secular equation (Eq. (22)). One can obtain that ϕ as a function of the joint angles for each arm as Eq. (42) below.

$$\phi_{1,2}(\theta_1,\theta_2) = \frac{1}{2}\left[(\tilde{\alpha}+\tilde{\beta})\sin\theta_1 + (\tilde{\gamma}+\tilde{\varepsilon})\sin(\theta_1+\theta_2)\right] \pm \frac{1}{2}Q \tag{42}$$

where

$$Q = \sqrt{\left[(\tilde{\alpha}+\tilde{\beta})\sin\theta_1 + (\tilde{\gamma}+\tilde{\varepsilon})\sin(\theta_1+\theta_2)\right]^2 - 4\tilde{\varepsilon}(\tilde{\alpha}+\tilde{\beta})\sin\theta_1\sin(\theta_1+\theta_2)}\,.$$

According to the theory of stability criterion for autonomous ODE (see *Mathematical Methods to Analyze Stability via ODE* section), the autonomous ODE is stable if all $Re(\phi)$ of the Jacobian matrix's eigenvalue for all equilibrium points are negative $Re\left(\phi_{1,2}(\theta_1^*,\theta_2^*)\right) < 0, \forall\phi_{1,2}(\theta_1^*,\theta_2^*)$. By this theory, one can obtain the stability criterion for the 2-DOF PMR with Stokes-damping in the SV approximation, like in Eq. (43).

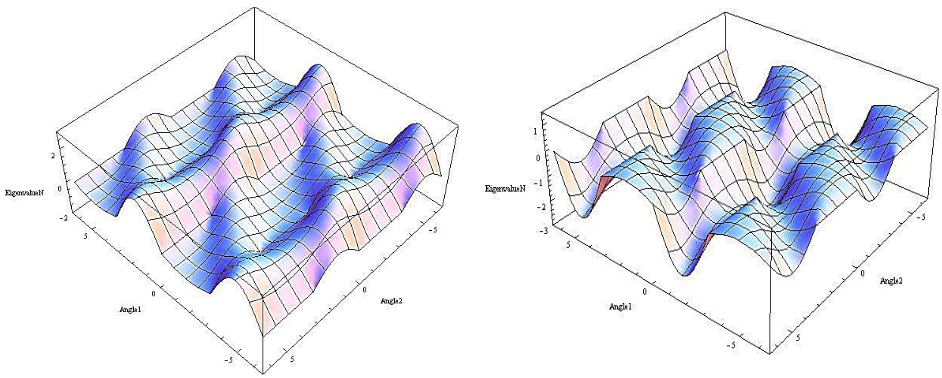
$$Re\left(\phi_{1,2}(\theta_1^*, \theta_2^*)\right) = Re\left(\frac{1}{2}\left[\left(\tilde{\alpha} + \tilde{\beta}\right)\sin\theta_1^* + (\tilde{\gamma} + \tilde{\varepsilon})\sin(\theta_1^* + \theta_2^*)\right] \pm \frac{1}{2}Q(\theta_1^*, \theta_2^*)\right) < 0 \quad (43)$$

This research takes an example of 2-DOF PMR with identical Stokes damping ($\alpha = \beta = C$) where its masses and link lengths are equal ($M1 = M2 = M$), ($a_1 = a_2 = a$). For this example, Eq. (43) becomes Eq. (45) below.

$$Re\left(\phi_{1,2}(\theta_1^*, \theta_2^*)\right) = \frac{Mga}{2C} Re\left(\frac{3}{4}\sin\theta_1^* + \frac{1}{2}\sin(\theta_1^* + \theta_2^*) \pm \sqrt{\frac{9}{16}\sin^2\theta_1^* + \frac{1}{4}\sin^2(\theta_1^* + \theta_2^*)}\right) < 0 \quad (44)$$

$$Re\left(\frac{3}{4}\sin\theta_1^* + \frac{1}{2}\sin(\theta_1^* + \theta_2^*) \pm \sqrt{\frac{9}{16}\sin^2\theta_1^* + \frac{1}{4}\sin^2(\theta_1^* + \theta_2^*)}\right) < 0 \quad (45)$$

Eq. (45) shows the stability criteria for the 2-DOF PMR with identical Stokes damping does not depend on the value of the damping constants. The left-hand side (LHS) of the Eq. (45) has two variables, θ_1^* and θ_2^*. So, the LHS of the Eq. (45) forms the so-called manifold (surface). Fig.3 below shows the Eq. (45) as manifold.



(a) (b)

FIGURE 3. Manifold (surface) of the Stability Criteria for the 2-DOF PMR with identical Stokes damping. The part (a) is the LHS of the Eq. (44) by taking (+) sign and part (b) is the

LHS of the Eq. (44) by taking (-) sign. The interval of θ_1^* and θ_2^* are the same as $\theta_1^* = [-2\pi, 2\pi]$ and $\theta_2^* = [-2\pi, 2\pi]$ respectively

To see the region in the manifold for Fig.3(a) and Fig.3(b) that fulfill Eq. (45), plot those manifolds in $\frac{2\phi(\theta_1^*,\theta_2^*)}{Mga} < 0$ intervals. Then, by combining those two manifolds in $\frac{2\phi(\theta_1^*,\theta_2^*)}{Mga} < 0$ intervals into one plot (Fig.4), there are six stable eigenvalue regions. Table 2 shows those six stable eigenvalue regions.

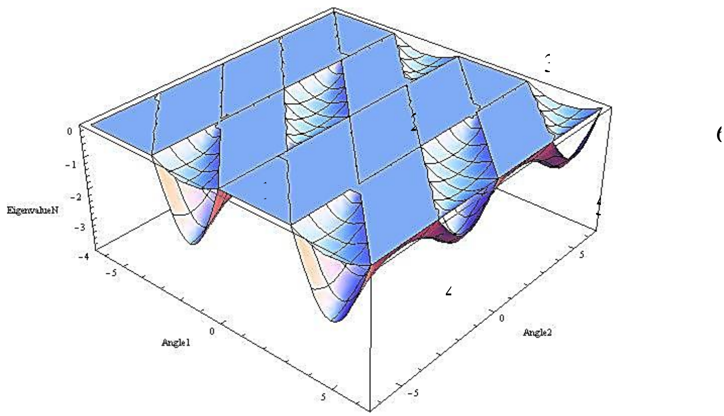
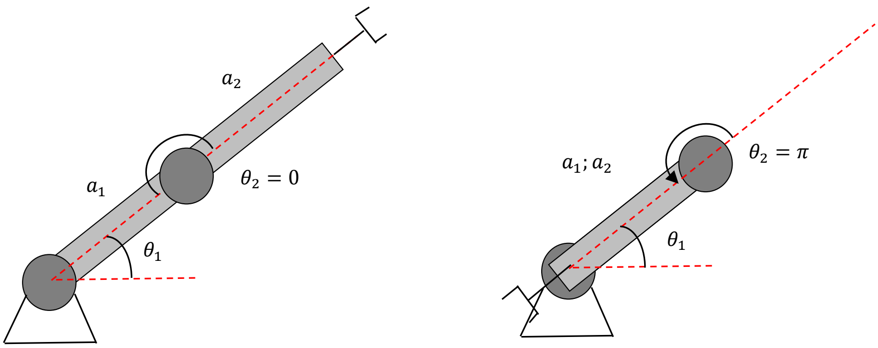


FIGURE 4. The Diagram of the Six Stable Eigenvalue Regions that fulfill Eq. (44)

From the result in Table 2, those six stable eigenvalue regions of the 2-DOF PMR with identical Stokes damping follow the form of 2-DOF PMR singularity. Singularity in a manipulator robot is a configuration in which the manipulator robot end-effector is not accessible in specific directions [8]. Therefore, the singularity configuration in the manipulator robot induces instability in the controller (see Fig.5 for the singularity configuration in the 2-DOF manipulator robot from the kinematics).

TABLE 2. The list of the Six Stable Eigenvalue Regions that fulfill Eq. (44)

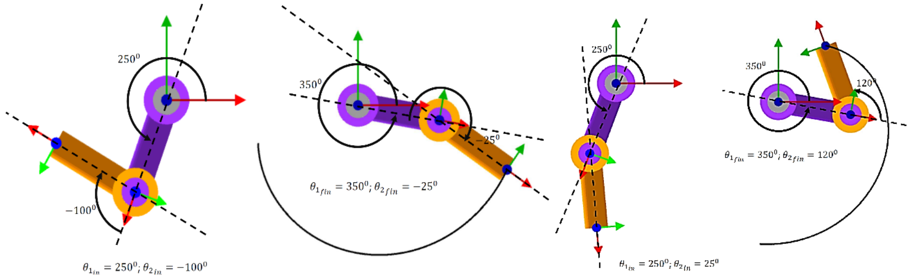
Region	Interval of θ_1^*	Interval of θ_2^*
1	$(-\pi, 0)$	$(-2\pi, -\pi)$
2	$(-\pi, 0)$	$(-\pi, 0) \wedge (0, \pi)$
3	$(-\pi, 0)$	$(\pi, 2\pi)$
4	$(\pi, 2\pi)$	$(-2\pi, -\pi)$
5	$(\pi, 2\pi)$	$(-\pi, 0) \wedge (0, \pi)$
6	$(\pi, 2\pi)$	$(\pi, 2\pi)$



(a) (b)

FIGURE 5. Singularity configuration of the 2-DOF PMR

All of the six stable eigenvalue regions have the same TCP trajectories. The fifth region is enough to see the behaviour of the stability. The fifth region is enough because there are two scenarios of stabilities. Here are the examples of those scenarios (Fig.6). Fig.6 was produced from the program of *RoboAnalyzer 7.5* [9].



(a) (b)

FIGURE 6. Two scenarios of stabilities in the fifth region. The part (a) shows $\theta_1^* \rightarrow (\pi, 2\pi)$ with the example of $\theta_1^* = (250^0, 350^0)$ and $\theta_2^* \rightarrow (-\pi, 0)$ with the example of $\theta_2^* = (-100^0, -25^0)$. The part (b) shows $\theta_1^* \rightarrow (\pi, 2\pi)$ with the example of $\theta_1^* = (250^0, 350^0)$ and $\theta_2^* \rightarrow (0, \pi)$ with the example of $\theta_2^* = (25^0, 120^0)$

In the next section, the stability criteria for the 2-DOF PMR with identical Stokes damping manifested in the torque criteria as a physical language. Torque is easier to access by scientists and engineers because the torque becomes an input for 2-DOF PMR manipulation.

d. Torque Criteria for the Stable Slow-Varying 2-DOF PMR with identical Stokes Damping

This section focuses on the stability criteria for the 2-DOF PMR with identical Stokes damping in the SV regime in the form of torque criteria. From the first scenario in the fifth region, one can obtain Eq. (46) where $\Delta\tau = \tau_1' - \tau_2'$.

$$\theta_1^* = \arccos\left[\frac{2\Delta\tau}{3Mga}\right] \to \frac{2\Delta\tau}{3Mga} = \cos\theta_1^* \to \Delta\tau = \frac{3}{2}Mga\cos\theta_1^* \tag{46}$$

Because the θ_1^* is in the interval of $(\pi, 2\pi)$, there is an allowed $\Delta\tau$ for the stable 2-DOF PMR with identical Stokes damping in the SV regime on the first scenario as the following equation (Eq. (47)). The minus sign represents the rotational direction of the motors.

$$-\frac{3}{2}Mga < \Delta\tau < \frac{3}{2}Mga \tag{47}$$

By taking the $\cos(\theta_1^* + \theta_2^*) = \frac{2\tau_2'}{Mga}$ and $\theta_2^* = (-\pi, 0)$, one can determine the intervals of the stable 2-DOF PMR with identical Stokes damping in the SV regime on the first scenario torque criteria for each motor as the following equations (Eq. (48) and Eq. (49)).

$$-2Mga < \tau_1' < 2Mga \tag{48}$$

$$-\frac{1}{2}Mga < \tau_2' < \frac{1}{2}Mga \tag{49}$$

For the second scenario, the $\Delta\tau$ is the same as the first scenario because θ_1^* is in the interval of $(\pi, 2\pi)$. But the θ_2^* interval is different. This difference leads to another stable torque criteria as follows.

$$-2Mga < \tau_1' < \frac{3}{2}Mga \tag{50}$$

$$-\frac{1}{2}Mga < \tau_2' < 0 \tag{51}$$

To see the crude validity of Eq. (48) to Eq. (51), take the example of no-damping 2-DOF PMR with a mass of 1 kg each and a length of 0.2 meters each with free movement. The non-damping case is suitable for crude approximation to the SV regime and identical Stokes damping case. Because in the SV regime, the damping has little effect on its dynamics. Based on Eq. (48) to Eq. (51), the first and second scenarios have the following torque criteria.

$$-4 < \tau_1' < 4 \tag{52}$$

$$-1 < \tau_2' < 1 \tag{53}$$

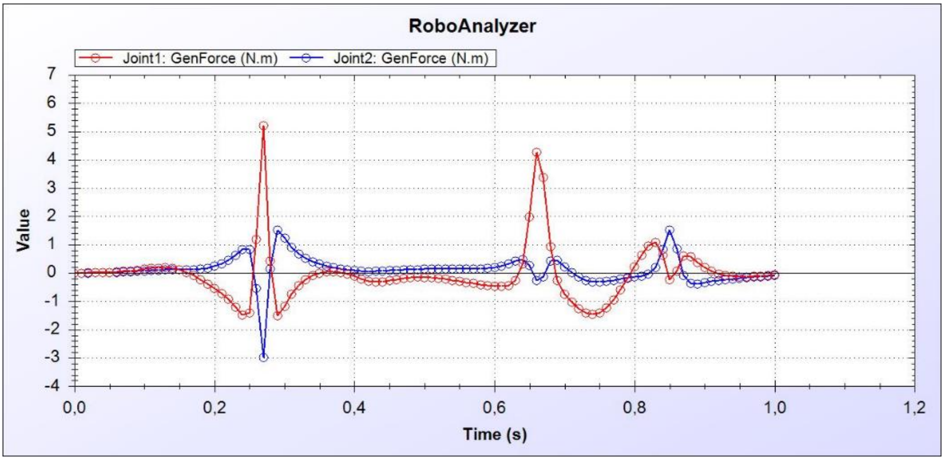
$$-4 < \tau_1' < 3 \tag{54}$$

$$-1 < \tau_2' < 0 \tag{55}$$

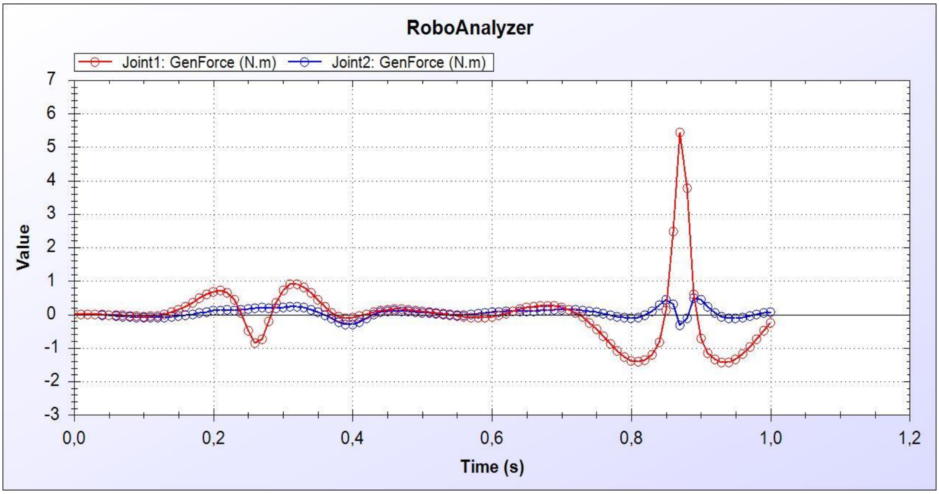
Fig.7 shows the dynamical analysis for the no-damping 2-DOF PMR with a mass of 1 kg each and a length of 0.2 meters each with free movement using the *Robo Analyzer 7.5* [9]. Fig.7(a) shows the minimum torque for the first and second joints is -1.6 Nm and -3.0 Nm for the first scenario. The maximum torque for the first and second joints is 5.2 Nm and 1.4 Nm for the first scenario. Meanwhile, Fig.7(b) shows the minimum torque for the first and second joints is -1.4 Nm and -0.4 Nm. For the second scenario, the maximum torque for the first and second joints is 5.4 Nm and 0.6 Nm. Table 3 compares this SV regime analysis to the free movement no-damping case analysis.

TABLE 3. Comparison of this SV regime analysis and the Free Movement No-Damping Case

Scenario	SV Regime Analysis	Free Movement No-Damping
First	$-4 < \tau_1' < 4$	$-1.6 < \tau_1' < 5.2$
	$-1 < \tau_2' < 1$	$-3.0 < \tau_2' < 1.4$
Second	$-4 < \tau_1' < 3$	$-1.4 < \tau_1' < 5.4$
	$-1 < \tau_2' < 0$	$-0.4 < \tau_2' < 0.6$



(a)



(b)

FIGURE 7. The Dynamical Analysis of the No-Damping 2-DOF PMR with Free Movement. The part (a) is the dynamical analysis for the first scenario and part (b) is the dynamical analysis for the second scenario. The red and blue colors belong to the first and second joints

From Table 3, the trend of the torque criteria for the stable SV 2-DOF PMR with identical Stokes damping looks similar to the no-damping free movement 2-DOF PMR but smaller. The SV 2-DOF PMR is smaller than the no-damping free movement 2-DOF PMR because, in SV 2-DOF PMR, the torque in each joint acts as an input. These inputs torque still have a role controlling the 2-DOF PMR. Meanwhile, for no-damping free movement case, the torque in each joint is the effect of rotational movement due to the gravitational field. The dynamical analysis in the no-damping 2-DOF PMR also supports the SV regime analysis. All maximum torque for the no-damping case in both scenarios is outside the torque criteria for the SV regime analysis and has sharp peaks. It indicates that the no-damping case is unstable at these points. This fact supports the torque criteria for stable SV regime 2-DOF PMR.

5 Conclusions

This research gives a mathematical analysis of the stable torque criteria for SV 2-DOF PMR with identical Stokes damping using Lagrangian Mechanics and Jacobian Formalism. This analysis shows there are two major conclusions. The first conclusion is the stability criteria of the SV regime with identical Stokes damping do not depend on the value of damping constants. The second conclusion shows that the stable torque

criteria for this 2-DOF PMR are X and Y for the first and second scenarios. The dynamical analysis of the 2-DOF PMR with no damping and free movement ($M_1 = M_2 = 1\ kg, A_1 = A_2 = 0.2\ m$) supports the validity of the
$\{\tau_1', \tau_2'\} = \{(-2Mga, 2Mga), (-0.5Mga, 0.5Mga)\}$ and
$\{\tau_1', \tau_2'\} = \{(-2Mga, 1.5Mga), (-0.5Mga, 0)\}$. Because the maximum torque for both scenarios of the 2-DOF PMR with no damping and free movement is outside of the range of
$\{\tau_1', \tau_2'\} = \{(-2Mga, 2Mga), (-0.5Mga, 0.5Mga)\}$ and
$\{\tau_1', \tau_2'\} = \{(-2Mga, 1.5Mga), (-0.5Mga, 0)\}$. Moreover, the 2-DOF PMR with no damping and free movement maximum torque has sharp peaks. The sharp peaks are manifestations of instability outside the range of
$\{\tau_1', \tau_2'\} = \{(-2Mga, 2Mga), (-0.5Mga, 0.5Mga)\}$ and
$\{\tau_1', \tau_2'\} = \{(-2Mga, 1.5Mga), (-0.5Mga, 0)\}$. This research result is the primary step for further analysis to obtain stable control of the robot operations in the damping environment.

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