Forward Kinematic and Inverse Kinematic Analysis of A 3-DOF RRR Manipulator

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Abstract. The increasing need for precise and multifunctional robotic systems in industries such as assembly, welding, and painting shows the importance of kinematic analysis. This paper aims to study and explain the kinematic properties of a three-degree-of-freedom (3-DOF) manipulator, which encompasses a revolving disk like the foundation and two revolving joints, also called the rotating joints. By analyzing forward and inverse kinematics, this paper aims to understand better and control the nature of movement shown by this robot arm. Forward kinematics entails calculating the values of the location and orientation of the end effector in connection with particular joint parameters. On the other hand, inverse kinematics aims to find the specific joint parameters to achieve a specific end-effector position. This paper uses a mathematical model and computational algorithms to solve the kinematic equations, allowing the manipulator to move precisely inside its domain. By comparing detailed models of transformation matrices used, the robot arm's working movement is entirely predicted and regulated. The resulting conclusion drawn from this critical analysis is that the proposed solutions lead to a significant leap in the theoretical understanding of the movement of robots and have efficient implications for a precise working automation context. Therefore, this basis creates more paths for developing advanced robot control algorithms.

Keywords: Forward Kinematics, Inverse Kinematics, Three-degree-of-freedom, Motion Control

1 Introduction

In the changing industrial environment, the automation sector, specifically robotic arms, plays a critical role in facilitating production efficiency and accuracy. The increasing use of robotics in assembly, welding, painting, and other complex processes raises the need for enhanced apparatus and control systems. Therefore, to make robotic arms more precise and flexible, kinematic analysis should be addressed. The assessment of how such mechanical devices are designed and operated provides general ideas regarding the prerequisites and typical elements of the current robotic arms most suitable for the modern

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manufacturing system's requirements. One of the central manipulators, which illustrate the simplicity of the structure, its wide use, and manipulability, is the three-degree-of-freedom. The risk of the complexity of other types of manipulators with a greater number of degrees is also avoided, and, thus, the 3-DOF apparatus needs to become a focus of the recent studies' estimations. Kinematics, or the branch of the mechanics in this field, is important as far as making the proper calculations of where or how a robot should move is concerned. As far as manipulators are limited to a limited number of DOF, it is possible to choose different angles of the joints and give the position of the effector and, conversely, find the position of the effector if such angles of the joints at which the manipulator should reinforce is given.

In recent years, there has been a trend of notable contributions following the rapid technological advancements in the field of robot kinematics. On one hand, studies such as that of Huang et al. have presented the design of control systems for pneumatic parallel mechanisms, which proposed an alternative to a classic PID controller in the form of synchronized and differential motion control systems. Overall, it can be said that the novel approach is superior as it allows for more precise control of the robot's motion [1]. On the other hand, Liu et al. have put significant efforts into the kinematic calibration of heavy-duty manipulators which was proposed as achieved through the use of optimization algorithms, ensuring the necessary precision that is essential for any robot. In the marine sector, Liu et al. presented the adaptive control systems for maritime propulsion auxiliary sails, and the proposed solutions can be rated as an overall success as it utilized an advanced method of precise control based on the adaptive sliding mode control [2]. In addition, Sheng et al. have developed decoupling strategies for the case of stable 3-DoF systems, which has also proved to be an efficient way of optimizing the control [3]. Finally, there are many possible applications with plenty of demanding mathematical challenges, and the above selected articles demonstrate just a handful of them along with the current status of this subject.

The goal of the present paper is to provide a more in-depth overview of the overall kinematics of the 3-DOF manipulator, a process that would allow for a better theoretical and practical understanding of the device's motion. A broad analysis of both the forward and inverse kinematics will be conducted in the following research, with the study utilizing previous discoveries and discoveries in the targeted area to cover the most recent findings on the topic. The present section will cover the materials used to conduct the study, with a focus on average base points and minimal mathematical models capable of describing the treatment in real-world settings. This section will also comment on the results obtained as the object analysis was completed, and examples of selected points and solutions should be regarded to illustrate the kinematic solutions' appropriateness in practice. And finally, the work will look to make it support the overall field of robotics and the development of more relative and highly advanced robotic systems in the future.

2 Forward Kinematic and Inverse Kinematic Analysis

2.1 Parameters

The 3-degree-of-freedom (3-DOF) manipulator is a robotics system that consists of two links and three rotational joints. There is a rotational base which can rotate on the y-axis, it is connected to the second rotational joint which is connected between the first link and the base. The third rotational joint is connected to both the end of link 1 and the head end of link 2. There is an end effector connected to the link 2. Fig. 1 depicts a basic sketch of the three-degree of freedom (DOF) manipulator. Table 1 shows the parameters of the manipulator.

Fig. 1. A 3-Dimension Model of the 3-DOF Manipulator (Photo credited: Original)

Table 1. The Parameter of Manipulator

Physical Dimensions	The Length of Link 1	L1 = 0.5 m
	The Length of Link 2	L2 = 0.5 m
	The Height of Base	H1 = 0.1 m
Coordinate Systems	The Coordinate System of Base	$[x_0, y_0, z_0]$
	The Coordinate System of Joint 1	$[x_1, y_1, z_1]$
	The Coordinate System of Joint 2	$[x_2, y_2, z_2]$
Variables	The Coordinate System of End Effector	$[x_3, y_3, z_3]$
	The Angle of The Base	θ_1
	The Angle of Joint 1	θ_2
	The angle of Joint 2	θ_3

2.2 Forward Kinematic Analysis

Determine the position and the orientation of the manipulator, the rotation matrices and transformation matrices are used and shown in equations (1), (2), (3), (4), (5), (6).

$$\mathrm{Rotx}, \alpha = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C_\alpha & -S_\alpha & 0 \\ 0 & S_\alpha & C_\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}$$

$$\mathrm{Roty}, \beta = \begin{bmatrix} C_\beta & 0 & S_\beta & 0 \\ 0 & 1 & 0 & 0 \\ -S_\beta & 0 & C_\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

$$\mathrm{Rotz}, \gamma = \begin{bmatrix} C_\gamma & -S_\gamma & 0 & 0 \\ S_\gamma & C_\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$

$$\mathrm{Transx, a} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

$$\mathrm{Transy, b} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

$$\mathrm{Transz, c} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$

A commonly used method to convert a selecting frame in robotics is called the Denavit-Hartenberg convention [4]. In this convention, Ai is shown as the result of four fundamental developments. (equations (1), (2), (3), (4), (5), (6)) as homogeneous transformations in equation (7).

$$Ai = Rotz, \theta iTransz, diTransx, aiRotx, \alpha i \tag{7}$$

Forward kinematics aims to use the manipulator's joint variable values to determine the end effector's location and orientation. To input joint variables, a set of DH parameters ai, di, αi, θi is given in Table 2. Fig. 2 shows the coordinate systems of each joint and the end effector.

ai = distance from the point where the xi and zi-1 axes intersect to oi along xi.

di = distance from oi-1 to the point where the xi and zi-1 axes connect along zi-1. di is variable if joint i is prismatic.

αi = the angle measured about xi between zi-1 and zi.

θi = the angle measured about zi-1 between xi-1 and xi. θi is variable if joint i is a revolute joint [5].

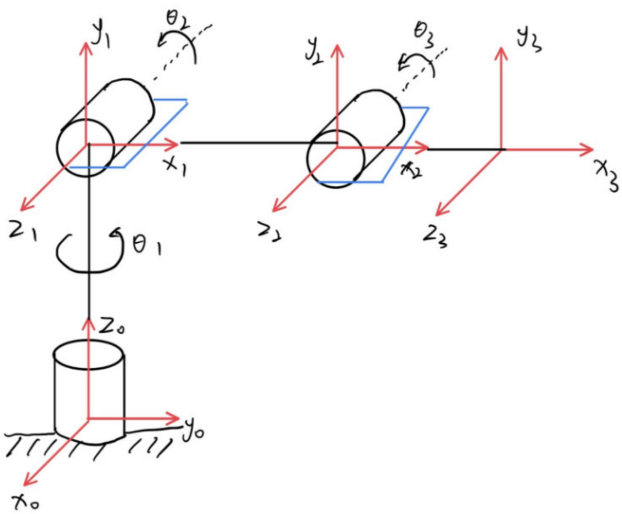

Fig. 2. The Coordinate Systems of Manipulator (Photo credited: Original)

Table 2. DH Parameters of Manipulator

Link	**a_i**	**α_i**	**d_i**	**θ_i**
1	H_1	90	0	θ_1
2	L_1	0	0	θ_2
3	L_2	0	d3	θ_3

Each transformation matrices (A1, A2, A3) are given in equations (8), (9) and (10), and the forward kinematic result T30 is given in equation (11). The equations of forward kinematic of the 3-DOF manipulator are shown in Table 3.

$$\mathbf{A1} = \begin{bmatrix} C_1 & 0 & S_1 & 0 \\ S_1 & 0 & -C_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{8}$$

$$\mathbf{A2} = \begin{bmatrix} C_2 & -S_2 & 0 & 0 \\ S_2 & C_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{9}$$

$$\mathbf{A3} = \begin{bmatrix} C_3 & -S_3 & 0 & 0 \\ S_3 & C_3 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{10}$$

$$\boldsymbol{T_3^0} = \begin{bmatrix} C_1S_2S_3 + C_1C_2C_3 & -C_1C_2C_3 - C_1S_2C_3 & S_1 & 0 \\ -S_1S_2S_3 + S_1C_2C_3 & -S_1C_2S_3 - S_1S_2C_3 & -C_1 & 0 \\ C_2S_3 + S_2C_3 & -S_2S_3 + C_2C_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{11}$$

Table 3. Forward Kinematic Equations

$r_{11} = \cos(\theta_1)\sin(\theta_2)\sin(\theta_3) + \cos(\theta_1)\cos(\theta_2)\cos(\theta_3)$

$r_{12} = -\cos(\theta_1)\cos(\theta_2)\cos(\theta_3) - \cos(\theta_1)\sin(\theta_2)\cos(\theta_3)$

$r_{13} = \sin(\theta_1)$

$r_{14} = 0$

$r_{21} = -\sin(\theta_1)\sin(\theta_2)\sin(\theta_3) + \sin(\theta_1)\cos(\theta_2)\cos(\theta_3)$

$r_{22} = -\sin(\theta_1)\cos(\theta_2)\sin(\theta_3) - \sin(\theta_1)\sin(\theta_2)\cos(\theta_3)$

$r_{23} = -\cos(\theta_1)$

$r_{24} = 0$

$r_{31} = \cos(\theta_2)\sin(\theta_3) + \sin(\theta_2)\cos(\theta_3)$

$r_{32} = -\sin(\theta_2)\sin(\theta_3) + \cos(\theta_2)\cos(\theta_3)$

$r_{33} = 0$

$r_{34} = 0$

$r_{41} = 0$

$$r_{42} = 0$$

$$r_{43} = 0$$

$$r_{44} = 1$$

2.3 Inverse Kinematic Analysis

A geometric approach is used to solve the inverse kinematic problem for a 3-DOF RRR manipulator. Since the manipulator has a frame rotating on the base, the problem is reduced from 3D to 2D and converted to a k-z coordinate system. There is substantial research that shows that geometric methods are more effective at reducing computational burden without losing the accuracy of the kinematic analysis of robotic systems. Craig discusses mechanics and control in robotic systems, which provides several avenues for the application of geometric methods [6]. And Siciliano et al. provide comprehensive work on modeling, planning, and controlling robotics [7]. Latombe (1991) explores robot motion planning extensively [8]. These advanced researches can be combined with inverse kinematics of geometric methods to realize efficient path planning algorithms, control algorithms of robot systems, and force analysis of robot joints. The system is displayed in Fig. 3. Geometric techniques are well-recognized for their efficiency and optimization of computational burden along with the higher degree of precision required for kinematic analysis of robots. Objective references include the study by Craig, which develops the mechanics and control in robotics, Siciliano et al., which provides details for modeling, planning, and control in robotics; and again Latombe, that studies motion planning in detail.

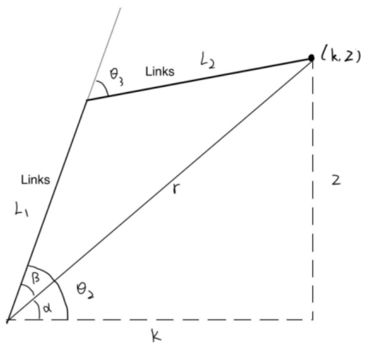

Fig. 3. 2D Representation of 3-DOF RRR Manipulator (Photo credited: Original)

From the converted coordinate system, the variable ‘r’ and θ_1 are represented in equations (12) and (13) below,

$$r = \sqrt{k^2 + z^2} = \sqrt{x^2 + y^2 + z^2} \tag{12}$$

$$\theta_1 = tan^{-1}\left(\frac{y}{x}\right) \tag{13}$$

also, according to Cosine rule, equation (14) is derived below,

$$r^2 = L_1^2 + L_2^2 - 2L_1 \cdot L_2 \cdot \cos(180 - \theta_3) \tag{14}$$

then the equation of 3 is shown in equation (15) below,

$$\theta_3 = -cos^{-1}\left(\frac{x^2+y^2+z^2-L_1^2 \cdot L_2^2}{2 \cdot L_1 \cdot L_2}\right) \tag{15}$$

from Fig. 3. and from trigonometry principles, angle and angle can be determined as in equations (16) and (17) shown below,

$$\alpha = \sin^{-1}\left(\frac{z}{r}\right) \tag{16}$$

$$\beta = \tan^{-1}\left(\frac{L_2 \cdot \sin(\theta_3)}{L_1 + L_2 \cdot \cos(\theta_3)}\right) \tag{17}$$

according to equation (16) and (17), 2 is derived and given in equation (18).

$$\theta_2 = \sin^{-1}\left(\frac{z}{r}\right) + \tan^{-1}\left(\frac{L_2 \cdot \sin(\theta_3)}{L_1 + L_2 \cdot \cos(\theta_3)}\right) \tag{18}$$

Therefore, the inverse kinematic results and equations for a 3-DOF RRR manipulator are given in Table 4.

Table 4. Inverse Kinematic Results and Equations

$$\theta 1 = tan^{-1}\left(\frac{y}{x}\right)$$

$$\theta_2 = \sin^{-1}\left(\frac{z}{r}\right) + \tan^{-1}\left(\frac{L_2 \cdot \sin(\theta_3)}{L_1 + L_2 \cdot \cos(\theta_3)}\right)$$

$$\theta_3 =- cos^{-1}\left(\frac{x^2+y^2+z^2-L_1^2\cdot L_2^2}{2\cdot L_1\cdot L_2}\right)$$

$$\text{r} = \sqrt{k^2+z^2} = \sqrt{x^2+y^2+z^2}$$

3 Discussion

In this paper, the forward kinematics and inverse kinematics of the 3-DOF manipulator are studied extensively. An accurate mathematical model and an efficient algorithm are given. The mathematical method mentioned therein requires calculating the exact position and orientation of the end effector based on the joint parameters provided (forward kinematics) and using a transformation matrix and computational algorithm to calculate the joint parameters required to implement the end effector (inverse kinematics). These methods help improve our understanding of robot motion control and provide assistance for precise operation in practical applications. The advantage of forward kinematics analysis is that it is simple to apply, and the position of the end-effector in three-dimensional space is determined from a known joint Angle. This method is the basis of predictive modeling of robot path planning and is the most effective solution for joint parameter control. However, it does not take into account external forces or changes in the real-world environment, which may affect accuracy in real-world scenarios.

Inverse kinematics solutions, on the other hand, are made more complex by their nonlinear nature and the possibility of multiple solutions or no solutions at all. In this paper, the geometric method and trigonometric function are used to simplify the problem, and the specific configuration of the 3-DOF manipulator is solved effectively. This method can solve the angle and position of each joint of the robot arm using the geometric method and the position of the end effector and convert it into a useful formula for the robot's program.

However, the geometric approach to inverse kinematics, while simplifying the problem, also limits the flexibility of dealing with more complex configurations or higher degrees of freedom. The geometric method is more suitable for some robot structures with simple structures and low precision requirements. This method usually also needs to consider some structural errors, such as the effects of joint elasticity or mechanical slip. As suggested by Johnson and Murphy (2019) [9], future research could improve on this by integrating machine learning techniques to predict potential motion errors and dynamically optimize control parameters.

With the continuous development of robot technology, the integration of adaptive control systems and real-time feedback mechanisms explored by Franklin et al. (2018) is crucial to enhance the adaptability of kinematic models [10]. Going forward, extending these kinematic analyses to more complex robotic systems, combined with advanced computational methods, will open up new avenues for research and application in the field of robotics, providing deeper insights and more efficient solutions.

4 Conclusion

In this paper, the forward kinematics and inverse kinematics of the three-degree-of-freedom manipulator are studied and calculated deeply, which provides an accurate mathematical model and powerful algorithm for solving the kinematics problems. Through the establishment of a detailed mathematical model, the forward kinematics solution process is deeply studied, and the position and direction of the end actuator of the manipulator are precisely deduced by using the transformation matrix and the rotation matrix. This method ensures that the position of the end effector can be accurately known when the Angle of the relevant joint is known and provides a reliable method for the path planning and accurate movement of the robot. In addition, this paper explores a relatively simple inverse kinematic solution that uses geometric and trigonometric principles to calculate the necessary joint angles to achieve the desired end-effector position. This approach is crucial because it addresses the nonlinear nature of the robotic arm's motion and provides a formula for obtaining joint angles, improving the robot's operational flexibility and accuracy when performing specific tasks. The analysis and calculation of forward kinematics and inverse kinematics enhance the understanding of robot motion control and provide a solid theoretical foundation for the precise operation of robots in various applications. These include industrial automation, medical robotics, bionic robotics, and other fields that require high precision and adaptability. Future research could further refine these kinematic models by considering more dynamic environmental interactions and advanced control strategies, such as machine learning algorithms, to adapt to real-time changes and further improve the efficiency and reliability of robotic systems.

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