Tensor C*-Algebra on Two Qubit Spin-1/2 System

Rivani Adistia Dewi[1], Imam Nugraha Albania[1,*], Rizky Rosjanuardi[1] and Sumanang Muhtar Gozali[1]

[1] Mathematics Study Program, Faculty of Mathematics and Natural Science Education, Universitas Pendidikan Indonesia, Bandung, Indonesia
albania@upi.edu

Abstract. Spin-1/2 system is one of the most important phenomenon in quantum mechanics. Such a system is known to be expressed as Pauli matrices, that is $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. By inserting the identity matrix I, we can construct a C*-algebra $C^*(\{I, \sigma_1, \sigma_2, \sigma_3\})$ with $|\{I, \sigma_1, \sigma_2, \sigma_3\} \otimes \{I, \sigma_1, \sigma_2, \sigma_3\}| = 16 = \left|\{E_{ij} : i, j = 1,2,3,4\}\right|$, where E_{ij} is the canonical basis of C*-algebra $M_4(\mathbb{C})$. In this study, by using the linear independence concept, we got $C^*(\{I, \sigma_1, \sigma_2, \sigma_3\}) \otimes C^*(\{I, \sigma_1, \sigma_2, \sigma_3\}) \cong M_4(\mathbb{C})$. This means, that for any observable in the composite of two spin-1/2 systems, by self-adjointness of Pauli matrices, it can be expressed by the field reduction from $\mathbb{C}$ to $\mathbb{R}$ in $span\{I, \sigma_1, \sigma_2, \sigma_3\} \otimes \{I, \sigma_1, \sigma_2, \sigma_3\}$ where the structure itself is non-associative algebra. Furthermore, given a Hamiltonian $H(t) = (\sigma_1 \otimes \sigma_1) + \mathcal{J}\, sin(\omega t)\, (\sigma_3 \otimes I) + \mathcal{J}\, sin\left(\omega t + \frac{\pi}{2}\right) (I \otimes \sigma_3)$ and the initial state which is a linear combination of Bell basis, by using the integration factor method and the commutation fact of the exponential operator, we got the following time-dependent state $|\psi(t)\rangle = \frac{1}{\sqrt{2}} \left(e^{\left(\frac{it}{\hbar}\right)\left(\frac{i\mathcal{J}}{\hbar\omega} cos(\omega t)\right)\left(\frac{i\mathcal{J}}{\hbar\omega} sin(\omega t)\right) \nu_{i(5-i)}} \right)_{i=1}^{4}$ with $\nu_{ij} = \delta_{(i+j)5}(-1)^{maks\{i,j\}}$, where δ_{ij} is Kronecker delta.

Keywords: Commutation Relation, Pauli Matrices, Tensor Product.

1 Introduction

Spin-1/2 is the "spin" of particles in a physical system where it takes a "rotation" of 720° for an object to return to its "initial position" [6]. In [8], the state of a particle in a spin-1/2 system corresponds to spin operators which are described in 2×2 matrices called the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1)$$

In determining the state of a particle in a spin-1/2 system, Pauli matrices are used to determine the state of one particle, so to determine the state of two particles, it is necessary to design the tensor product of Pauli matrices. Pauli matrices has become an

F. Khoerunnisa et al. (eds.), *Proceedings of the 9th Mathematics, Science, and Computer Science Education International Seminar (MSCEIS 2023)*, Advances in Social Science, Education and Humanities Research 860,
https://doi.org/10.2991/978-2-38476-283-5_31

active study in the past 5 years back as seen in [1]-[5], [7] and [9]-[12] since its significance to many applications. The scope of this article is the composite of two spin-1/2 systems, that is C*-algebra $C^*(\mathcal{A})\otimes C^*(\mathcal{A})$ with $\mathcal{A} = \{I, \sigma_1, \sigma_2, \sigma_3\}$, where I is 2×2 identity matrix, then define the time-dependent state $|\psi(0,t)\rangle; t > 0$ for the time-dependent Hamiltonian

$$H(t) = (\sigma_1\otimes\sigma_1) + \mathcal{J}\sin(\omega t)(\sigma_3\otimes I) + \mathcal{J}\sin\left(\omega t + \frac{\pi}{2}\right)(I\otimes\sigma_3), \quad (2)$$

with the initial state $|\xi\rangle = \frac{1}{\sqrt{2}}(|\phi^+\rangle + |\psi^+\rangle)$ which is a superposition (linear combination) of the Bell bases.

This article explains that $\mathcal{A}\otimes\mathcal{A}$ is linearly independent and the number of matrices from the set $\mathcal{A}\otimes\mathcal{A}$ is 16 which is same as the canonical bases of $M_4(\mathbb{C})$, so that $C^*(\mathcal{A}\otimes\mathcal{A})$ (since $C^*(\mathcal{A})$ is finite dimensional and so $C^*(\mathcal{A}\otimes\mathcal{A})$ is isomorphic to $C^*(\mathcal{A})\otimes C^*(\mathcal{A})$) and $M_4(\mathbb{C})$ are interchangeable. Furthermore, it is found that the Heisenberg uncertainty is smaller in the composite of two spin-1/2 system than in the single qubit system. This article also provides the state $|\psi(t)\rangle$ for the time-dependent Hamiltonian.

$$H(t) = (\sigma_1\otimes\sigma_1) + \mathcal{J}\sin(\omega t)(\sigma_3\otimes I) + \mathcal{J}\sin\left(\omega t + \frac{\pi}{2}\right)(I\otimes\sigma_3), \quad (3)$$

and the initial state $|\xi\rangle = \frac{1}{\sqrt{2}}(|\phi^+\rangle + |\psi^+\rangle)$.

2 Method

To find out the interchangeability between $C^*(\mathcal{A})\otimes C^*(\mathcal{A})$ and $M_4(\mathbb{C})$, we will prove that $C^*(\mathcal{A})\otimes C^*(\mathcal{A}) \cong M_4(\mathbb{C})$. Consider that the number of elements of $\mathcal{A}\otimes\mathcal{A}$ is 16 which is same as the dimension of $M_4(\mathbb{C})$, so it is sufficient to prove the linear independence of $\mathcal{A}\otimes\mathcal{A}$. The method for solving this problem is classify forms of "similar" $\mathcal{A}\otimes\mathcal{A}$ elements into subsets of $\mathcal{A}\otimes\mathcal{A}$, then analyze whether each element of each subset is a linear combination of other elements in that subset or not.

To find out $|\psi(t)\rangle$ we need a time evolution operator $U(t)$ with initial state $|\xi\rangle$. So, it is necessary to find the solution of the Schrödinger equation.

$$i\hbar\frac{d}{dt}U(t) = \left[\sigma_1\otimes\sigma_1 + \mathcal{J}\sin(\omega t)(\sigma_3\otimes I) + \mathcal{J}\sin\left(\omega t + \frac{\pi}{2}\right)(I\otimes\sigma_3)\right]U(t) \quad (4)$$

to get $U(t)$, then multiply it by $|\xi\rangle$. The method for finding the solutions of the Schrödinger equation is by the integration factor of homogeneous linear differential equations, the fact that $e^{A+B} = e^A e^B$ if and only if $AB - BA = 0$ for compatible matrices A and B, and the fact that $\sum_{n=0}^{\infty}\frac{1}{n!}A^n$ is convergent for every square matrix A.

3 Result and Discussion

3.1 The Structure of span $\mathcal{A}\otimes\mathcal{A}$

For spin-1/2 system, the tensor product $P\otimes Q$ where P and Q are 2×2 matrices is defined as follows:

$$P\otimes Q = \begin{pmatrix} p_{11}Q & p_{12}Q \\ p_{21}Q & p_{22}Q \end{pmatrix} \tag{5}$$

which is a 4×4 matrix. Consider that

$$\mathcal{A} = \{I, \sigma_1, \sigma_2, \sigma_3\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}. \tag{6}$$

For technical reasons and ease of notation, define $I = \sigma_0$ and $\sigma_i\otimes\sigma_j = \sigma_{ij}$ for each $i, j \in \{0,1,2,3\}$.

Lemma 1. The eigenvalues of σ_{ij} are 1 and -1, for each $i, j \in \{1,2,3\}$ with multiplicity 2.

Proof. It is easy to see that the eigenvalues of σ_i and σ_j are 1 and -1, for each $i \in \{1,2,3\}$. Let λ_1 and λ_2 are the eigenvalues of σ_i and σ_j respectively with the corresponding eigenvectors are v_1 and v_2 respectively. We have $(\sigma_i\otimes\sigma_j)(v_1\otimes v_2) = (\sigma_i v_1)\otimes(\sigma_j v_2) = (\lambda_1 v_1)\otimes(\lambda_2 v_2) = \lambda_1\lambda_2(v_1\otimes v_2)$, so that the eigenvalues of σ_{ij} are $\{-1, -1, 1, 1\}$.

Proposition 2. *For each* $i, j, k \in \{1,2,3\}$, 1) σ_{ij} *are self-adjoint.*; 2) $\operatorname{tr}\sigma_{ij} = \operatorname{tr}\sigma_k$.; and 3) $\det\sigma_{ij} = -\det\sigma_k$.

Proof. 1) Clear.; 2) Since the trace of a matrix is equal to the sum of all its eigenvalues, then by Lemma 1, we have $\operatorname{tr}\sigma_{ij} = 0 = \operatorname{tr}\sigma_k$ for each $i, j, k \in \{1,2,3\}$.; 3) Since the determinant of a matrix is equal to the product of all its eigenvalues, then by Lemma 1, we have $\det\sigma_{ij} = 1 = -(-1) = -\det\sigma_k$ for each $i, j, k \in \{1,2,3\}$.

Theorem 3. $C^*(\mathcal{A})\otimes C^*(\mathcal{A}) \cong M_4(\mathbb{C})$ with norm operator (matrix) and transpose conjugate involution.

Proof. We classify $\mathcal{A}\otimes\mathcal{A}$ into subsets $(\mathcal{A}\otimes\mathcal{A})_1$, $(\mathcal{A}\otimes\mathcal{A})_2$, $(\mathcal{A}\otimes\mathcal{A})_3$, and $(\mathcal{A}\otimes\mathcal{A})_4$ as follows:

$$(\mathcal{A}\otimes\mathcal{A})_1 = \{\sigma_{00}, \sigma_{03}, \sigma_{30}, \sigma_{33}\}; (\mathcal{A}\otimes\mathcal{A})_2 = \{\sigma_{01}, \sigma_{02}, \sigma_{31}, \sigma_{32}\}; \tag{6}$$

$$(\mathcal{A}\otimes\mathcal{A})_3 = \{\sigma_{10}, \sigma_{13}, \sigma_{20}, \sigma_{23}\}; (\mathcal{A}\otimes\mathcal{A})_4 = \{\sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22}\}. \tag{7}$$

It can be showed that $(\mathcal{A}\otimes\mathcal{A})_i$ is linearly independent for every $i \in \{1, 2,3,4\}$. Consider that for any $\sigma_{mn} \in (\mathcal{A}\otimes\mathcal{A})_i$, we have $\sigma_{mn} \notin \operatorname{span}(\mathcal{A}\otimes\mathcal{A})_i\backslash\{\sigma_{mn}\}$ for each i, and based on the classification of the subset $(\mathcal{A}\otimes\mathcal{A})_i$ of $\mathcal{A}\otimes\mathcal{A}$, we obtain that $\sigma_{mn} \notin \operatorname{span}(\mathcal{A}\otimes\mathcal{A})_j\,; j \neq i$. So, $\mathcal{A}\otimes\mathcal{A}$ is linearly independent, and because of the number of elements of $\mathcal{A}\otimes\mathcal{A}$ is 16 which is same as the dimension of $M_4(\mathbb{C})$, then $\operatorname{span}\mathcal{A}\otimes\mathcal{A} = M_4(\mathbb{C})$. This means $C^*(\mathcal{A}\otimes\mathcal{A}) \cong M_4(\mathbb{C})$. Since $C^*(\mathcal{A}) \cong M_2(\mathbb{C})$, then $C^*(\mathcal{A}\otimes\mathcal{A}) \cong M_4(\mathbb{C}) = M_2(\mathbb{C})\otimes M_2(\mathbb{C}) \cong C^*(\mathcal{A})\otimes C^*(\mathcal{A})$.

3.2 State $|\psi(t)\rangle$

To obtain the time dependent state $|\psi(t)\rangle$, we need time evolution operator that presented in the following lemma.

Lemma 4. *If* $H(t) = \sigma_{11} + \mathcal{J}\sin(\omega t)\,(\sigma_{30}) + \mathcal{J}\sin\left(\omega t + \frac{\pi}{2}\right)(\sigma_{03})$, *then* $U(t) = e^{\frac{i}{\hbar}(-\sigma_{11})t + \frac{i\mathcal{J}}{\hbar\omega}(\cos(\omega t)(\sigma_{30}) - \sin(\omega t)(\sigma_{03}))}$.

Proof. By solving the Schrödinger equation

$$\frac{d}{dt}U(t) + \frac{i\mathcal{J}}{\hbar}\left[\frac{\sigma_{11}}{\mathcal{J}} + \sin(\omega t)\,(\sigma_{30}) + \cos(\omega t)\,(\sigma_{03})\right]U(t) = 0 \tag{8}$$

using the integration factor $e^{\frac{i}{\hbar}\sigma_{11}t + \frac{i\mathcal{J}}{\hbar\omega}(\sin(\omega t)(\sigma_{03}) - \cos(\omega t)(\sigma_{30}))}$, then we obtain

$$\frac{d}{dt}\left(e^{\frac{i}{\hbar}\sigma_{11}t + \frac{i\mathcal{J}}{\hbar\omega}(\sin(\omega t)(\sigma_{03}) - \cos(\omega t)(\sigma_{30}))}U(t)\right) = 0.$$

$$\Rightarrow e^{\frac{i}{\hbar}\sigma_{11}t + \frac{i\mathcal{J}}{\hbar\omega}(\sin(\omega t)(\sigma_{03}) - \cos(\omega t)(\sigma_{30}))}U(t) = 1$$

$$\Rightarrow U(t) = e^{\frac{i}{\hbar}(-\sigma_{11})t + \frac{i\mathcal{J}}{\hbar\omega}(\cos(\omega t)(\sigma_{30}) - \sin(\omega t)(\sigma_{03}))}. \tag{9}$$

Theorem 5. *If* $H(t) = \sigma_{11} + \mathcal{J}\sin(\omega t)\,(\sigma_{30}) + \mathcal{J}\cos(\omega t)\,(\sigma_{03})$ *with initial state* $|\xi\rangle = \frac{1}{\sqrt{2}}(|\phi^+\rangle + |\psi^+\rangle)$, *then*

$$\sqrt{2}|\psi(t)\rangle = \left(e^{\left(\frac{it}{\hbar}\right)\left(\frac{i\mathcal{J}}{\hbar\omega}cos(\omega t)\right)\left(\frac{i\mathcal{J}}{\hbar\omega}sin(\omega t)\right)v_{i(5-i)}}\right)_{i=1}^{4}, \tag{10}$$

with $v_{ij} = \delta_{(i+j)5}(-1)^{maks\{i,j\}}$, where δ_{ij} is Kronecker delta.

4 Conclusion

Our result $C^*(\mathcal{A})\otimes C^*(\mathcal{A}) \cong M_4(\mathbb{C})$ says that the observables in the composite of two spin-1/2 system includes all the self-adjoint matrices in $M_4(\mathbb{C})$. The canonical basis E_{ij} of $M_4(\mathbb{C})$ is not self-adjoint for each $i \neq j$, while the observables are represented by self-adjoint operators/matrices. Because $\mathcal{A}\otimes\mathcal{A}$ is a self-adjoint set, then restrict the span $\mathcal{A}\otimes\mathcal{A}$ to its self-adjoint elements is sufficient to restrict the field $\mathbb{C}$ to $\mathbb{R}$.

Acknowledgments. This research is supported by PDUPT 2023 no: 1199/UN40.LP/PT.01.03/2023.

References

1. Andoni, S.: Spin 1/2 One- and Two-particle System in Physical Space without Eigen-algebra or Tensor Product. Mathematical Methods in the Applied Sciences, 1-14 (2022).

2. Duarte, F. J., Taylor, T. S., Slaten, J. C.: On the Probability Amplitude of Quantum Entanglement and the Pauli Matrices. Optical and Quantum Electronics **52**, 106 (2020).
3. Jin, L., Song, Z.: Bulk-Boundary Correspondence in a Non-Hermitian System in One Dimension with Chiral Inversion Symmetry. Physical Review B **99**, (2019).
4. Lin, S., Jin, L., Song, Z.: Symmetry Protected Topological Phases Characterized by Isolated Exceptional Points. Physical Review B **99**, 16 (2019).
5. Miller, M., Miller, D.: GraphStateVis: Interactive Visual Analysis of Qubit Graph State and Their Stabilizer Groups. IEEE International Conference on Quantum Computing and Engineering (QCE), pp. 378-384. (2021).
6. Nielsen, M. A., Chuang, I. L.: Quantum Computation and Quantum Information. Cambridge University Press, (2010).
7. Nishimura, J., Tsuchiya, A.: Complex Langevin Analysis of the Space-Time Structure in the Lorentzian Type IIB Matrix Model. Journal of High Energy Physics **2019**(6), 1-15 (2019).
8. Sakurai, J. J.: Modern Quantum Mechanics (Revised Edition). Addison-Wesley Publishing Company, (1994).
9. Schlatter, A.: On the Vacuum Energy in Bohmian Mechanics. Journal of Physics Communications **2**, (2018).
10. Surawy-Stepney, T., Kahn, J., Kueng, R., Guta, M.: Projected least-squares quantum process tomography. Quantum **6**, 844 (2022).
11. Tempel, D. G., Aspuru-Guzik, A.: Quantum Computing Without Wavefunctions: Time-Dependent Density Functional Theory for Universal Quantum Computation. Scientific Reports **2**, 391 (2012).
12. Willette, T. Z., Wilkowski, D., Lefevre, R., Taichenachev, A. V., Yudin, V. I.: SU(2) Hyperclocks: Quantum Engineering of Spinor Interferences for Times and Frequency Metrology. Physical Review Research **4**, 2 (2022).

.