Representation by sums of ascending even prime powers

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Abstract

In this work, we investigate the representation of numbers as sums of powers of prime numbers. It is shown that all sufficiently large integers are sums of some ascending even prime powers.

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1 Introduction

Roth [11] proved that all sufficiently large positive integers N can be expressed

$$\sum_{i=1}^{s} x_i^{i+1} = N, \tag{1.1}$$

with $s = 50$. Many predecessors, among them Thanigasalam [13, 14], Vaughan [15, 16], Brüdern [1, 2] and Ford [5, 6], struggled for half a century to improve the above result. The sharpest result was due to Liu and Zhao [10] ($s = 13$).

In [14], Thanigasalam also considered the equation (1.1) for prime powers and proved that

$$N_1 = \sum_{i=1}^{23} p_i^{i+1}, \quad N_2 = \sum_{i=1}^{24} p_i^{i+1},$$

for sufficiently large odd integers N_1, and even integers N_2.

In [3], Brüdern considered the equation (1.1) with even powers only. His result implies in particular that the set of integers representable as

$$N = x_1^2 + x_2^4 + x_3^6 + x_4^8, \quad x_1, x_2, x_3, x_4 \in \mathbb{N},$$

has positive density, but less than 1.

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In 2020, Kuan et al. [9] proved that

$$N = x_1^2 + x_2^4 + \cdots + x_{173}^{346}, \quad x_1, x_2, \cdots, x_{173} \in \mathbb{N},$$

for all sufficiently large natural number N.

Incorporating a powerful admissible exponents method developed by Davenport [4] and Thanigasalam [14], we establish the following theorem.

Theorem 1.1 *Let $\mathcal{N}_1 \equiv 1 \pmod 3$, $\mathcal{N}_2 \equiv 2 \pmod 3$, $\mathcal{N}_3 \equiv 0 \pmod 3$. All sufficiently large integers $\mathcal{N}_1$, $\mathcal{N}_2$, and $\mathcal{N}_3$ are representable in the forms*

$$\mathcal{N}_1 = \sum_{k=1}^{s} p_k^{2k}, \quad \mathcal{N}_2 = \sum_{k=1}^{s+1} p_k^{2k}, \quad \mathcal{N}_3 = \sum_{k=1}^{s+2} p_k^{2k}, \tag{1.2}$$

where s=2224 and the p's are primes.

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2 Preliminaries

Definition 1 *[14, Definition A] Given natural numbers $k_1, \cdots, k_s$ with $2 \le k_s \le \cdots \le k_2 \le k_1$ $(s \ge 2)$ and real numbers $\nu_1, \cdots, \nu_s$ with $0 < \nu_i \le 1$ $(i = 1, \cdots, s)$, the pairs (k_1, ν_1), (k_2, ν_2), $\cdots$, (k_s, ν_s) are said to form admissible exponents, if for every large positive Z and every $\epsilon > 0$, the number of solutions of the equation*

$$x_1^{k_1} + x_2^{k_2} + \cdots + x_s^{k_s} = y_1^{k_1} + y_2^{k_2} + \cdots + y_s^{k_s}$$

subject to

$$Z^{\nu_i/k_i} \le x_i \le 2Z^{\nu_i/k_i}, \quad Z^{\nu_i/k_i} \le y_i \le 2Z^{\nu_i/k_i}, \quad (i = 1, 2, \cdots, s),$$

is

$$\ll Z^{(\sum_{i=1}^{s} \nu_i/k_i)+\epsilon}.$$

Let

$$K_1 = \{12, 14, \cdots, 40\}, \quad K_2 = \{42, 44, \cdots, 4446\}. \tag{2.1}$$

By [14, Theorem 3], we get the following lemma.

Lemma 2.1 *There exist numbers* ν_k $(k \in K_1 \cup K_2)$ *satisfying*

$$\alpha_1 = \sum_{k \in K_1} \frac{\nu_k}{k} > 0.508926, \quad \alpha_2 = \sum_{k \in K_2} \frac{\nu_k}{k} > 0.905733, \tag{2.2}$$

such that

(1) $\{k, \nu_k\}$ *with* $k \in K_1$ *form pairs of admissible exponents;*
(2) $\{k, \nu_k\}$ *with* $k \in K_2$ *form pairs of admissible exponents.*

Let $K_3 = \{8, 10\} \cup K_2$. We have the following lemma.

Lemma 2.2 *There exist numbers* $\nu_k^{'}$ $(k \in K_3)$ *satisfying*

$$0 < \nu_k^{'} \le 1, \quad \nu_8^{'} = 1, \quad 0.876157 < \nu_{10}^{'} < 0.876158, \quad \sum_{k \in K_3} \frac{\nu_k^{'}}{k} = \alpha_3 > \frac{89}{96} + \frac{1}{10^6} \tag{2.3}$$

such that $\{(k, \nu_k^{'})\}$ $(k \in K_3)$ *form pairs of admissible exponents.*

Proof *By [4, Theorem 2], we obtain this lemma.*

2.1 Notation

p denotes a prime number. Let θ be a small positive constant. α, β denote real numbers, and ϵ is a small positive number. N is a large positive number, and we write $L = \log N$. $C_0, C_1, \ldots,$ are positive constants. We use the abbreviations

$$e(\alpha) := e^{2\pi i \alpha}, \quad e_q(\alpha) := e(a/q).$$

ν_k $(k \in K_1 \cup K_2)$ be defined in Lemma 2.1, $\nu_k^{'}$ $(k \in K_3)$ be defined in Lemma 2.2 and

$$\nu_2 = \nu_4 = \nu_6 = 1.$$

We define ($a \le q$, and $(a, q) = 1$)

$$2P_k := N^{\frac{\nu_k}{k}}, \quad f_k := f_k(\alpha) = \sum_{P_k \le x \le 2P_k} e(\alpha x^k),$$

$$J_k := J_k(\beta) = \sum_{(P_k)^k \le y \le (2P_k)^k} \frac{1}{k} y^{\frac{1}{k}-1} e(\beta y), \quad S_k := S_k(a, q) = \sum_{x=1}^{q} e_q(a x^k), \tag{2.4}$$

$$h_k := h_k(\alpha, a, q) = q^{-1} S_k(a, q) J_k\left(\alpha - \frac{a}{q}\right).$$

Define (for $k \in K_3$)

$$u_k := u_k(\alpha) = \sum_{P_k^{'} \le x \le 2P_k^{'}} e(\alpha x^k), \quad 2P_k^{'} = N^{\nu_k^{'}/k},$$

$$G_k := G_k(\beta) = \sum_{(P_k^{'})^k \le y \le (2P_k^{'})^k} \frac{1}{k} y^{\frac{1}{k}-1} e(\beta y), \tag{2.5}$$

$$t_k := t_k(\alpha, a, q) = q^{-1} S_k(a, q) G_k\Big(\alpha - \frac{a}{q}\Big).$$

We denote by

$$F(\alpha) := F_1(\alpha) F_2(\alpha) \tag{2.6}$$

with

$$F_1(\alpha) := f_2\Big(\prod_{k \in K_1} f_k\Big), \quad F_2(\alpha) := f_4 f_6 F_3(\alpha), \quad F_3(\alpha) := f_8 u_{10}\Big(\prod_{k \in K_2} u_k\Big). \tag{2.7}$$

Let

$$Q := N^{7/8+\theta}, \ \tau := 1/8 - 2\theta, \tag{2.8}$$

and subdivide the interval

$$Q^{-1} \le \alpha \le 1 + Q^{-1} \tag{2.9}$$

as follows: For $q \le N^\tau$, let $\mathfrak{M}_{a,q}$ denote the interval $\alpha = \frac{a}{q} + \beta$, $|\beta| \le (qQ)^{-1}$, and denote the aggregate of all $\mathfrak{M}_{a,q}$'s by $\mathfrak{M}$. It can be proved in the standard way that any two $\mathfrak{M}_{a,q}$'s are disjoint. Let $\mathfrak{m}$ denote the complement of $\mathfrak{M}$ in (2.9). Also denote the complement of $\mathfrak{M}_{a,q}$ in (2.9) by $\mathfrak{m}_{a,q}(q \le N^\tau)$.

We define

$$\begin{aligned}
&B_1(q) = \sum\nolimits^* q^{-5-3/4-1/5}|S_6|^2|S_8|^2; \quad B_2(q) = \sum\nolimits^* q^{-5-5/12-1/5}|S_8|^2;\\
&B_3(q) = \sum\nolimits^* q^{-5-7/12-1/5}|S_6||S_8|^2; \quad B_4(q) = \sum\nolimits^* q^{-4+2\theta-1/5}|S_6|^2;\\
&B_5(q) = \sum\nolimits^* q^{-3-2/3+2\theta-1/5}; \quad B_6(q) = \sum\nolimits^* q^{-4+1/6-1/5+2\theta}|S_6|;\\
&B_7(q) = \sum\nolimits^* q^{-5-1/5}|S_4|^2|S_8|^2; \quad B_8(q) = \sum\nolimits^* q^{-3-1/4-1/5}|S_4|^2;\\
&B_9(q) = \sum\nolimits^* q^{-5-3/4-1/5+\theta}|S_4|^2|S_6|^2; \quad B_{10}(q) = \sum\nolimits^* q^{-6-1/5}|S_4|^2|S_6|^2|S_8|^2,
\end{aligned} \tag{2.10}$$

where the notation $\sum^*$ means $\sum_{\substack{1 \le a \le q\\(a,q)=1}}$.

Lemma 2.3 *[15, Lemma 4.1] Suppose that $p \nmid a$ and $k \ge 3$. Then*

$$|S_k(a, p)| \le k\sqrt{p}.$$

Lemma 2.4 *[15, Lemma 4.4] Suppose that $k \ge 3$ and $(a, q) = 1$. Then $S_k(a, q) \ll q^{1-1/k}$.*

Lemma 2.5 *[15, Lemma 4.5] Suppose that $(a_1, q_1) = (a_2, q_2) = (q_1, q_2) = 1$. Then*

$$S_k(a_1q_2 + a_2q_1, q_1q_2) = S_k(a_1, q_1)S_k(a_2, q_2).$$

Lemma 2.6 *[15, Lemma 10.1] Suppose that $f(q)$ is a non-negative multiplicative function of q and $\sum_{h=0}^\infty f(p^h)$ exists for every prime p. Then, when $X \ge 1$,*

$$\sum_{l \le X} f(l) \le \prod_{p \le X} \sum_{h=0}^\infty f(p^h).$$

Lemma 2.7 *$B_1(q)$, $B_2(q)$, $\cdots$, $B_{10}(q)$ are multiplicative.*

Proof *If follows easily from* (2.10) *and Lemma 2.5.*

Lemma 2.8 *For $k = 4$, 6 and 8, $|\beta| \le 1/2$,*

$$J_k(\beta) \ll \min(N^{1/k}, N^{1/k-1}|\beta|^{-1}), \quad h_k(\alpha, a, q) \ll q^{-1/k} N^{1/k} \min(1, N^{-1}|\beta|^{-1}),$$

$$f_k(\alpha) - h_k(\alpha, a, q) \ll q^{1-1/k+\theta} \quad (if\; q \le N^{1/k-\theta},\ |\beta| \le q^{-1} N^{1/k-1-\theta}),$$

and

$$t_{10}(\alpha, a, q) \ll q^{-1/10} N^{\nu_{10}'/10} \min(1, N^{-\nu_{10}'}|\beta|^{-1}).$$

Proof *See Lemma 5 in [14].*

Lemma 2.9

$$\sum_{1\le x\le P} e_q(ax^k) - \frac{P}{q} S_k(a, q) \ll q^{1/2+\epsilon}.$$

Proof *See the main theorem in [7].*

Lemma 2.10 *For $k = 4$, 6 and 8,*

$$f_k(\alpha) - h_k(\alpha, a, q) \ll q^{1/2+\epsilon}\{\max(1, N|\beta|)\},$$

and

$$u_{10}(\alpha) - t_{10}(\alpha, a, q) \ll q^{1/2+\epsilon} \max(1, N^{\nu_{10}'}|\beta|).$$

Proof *The lemma can be proved by a partial summation with Lemma 2.9.*

Lemma 2.11 *On $\mathfrak{m}$,*

$$f_4(\alpha) \ll N^{7/32+\theta/4}, \quad f_6(\alpha) \ll N^{31/192+\theta/6}.$$

Proof *It follows a similar argument from [15, Lemma 8.2].*

Lemma 2.12 *[14, Lemma 21] Let $K_1 = \{12, 14, \cdots, 40\}$ and $t = \sum_{k\in K_1}(x_k^k - y_k^k)$ with*

$$P_k \le x_k \le 2P_k, \quad P_k \le y_k \le 2P_k \quad for \quad k \in K_1.$$

Then

$$\sum_{t\ne 0} d(|S|) \ll N^{\alpha_1}(\log N)^{C_1},$$

where $d(n)$ denotes the divisor function and α_1 is defined in (2.2).

Lemma 2.13 *The number of solutions of*

$$x_2^2 + \left(\sum_{k \in K_1} x_k^k \right) = y_2^2 + \left(\sum_{k \in K_1} y_k^k \right) \tag{2.11}$$

with the x_k's, y_k's subject to

$$P_k \le x_k \le 2P_k, \quad P_k \le y_k \le 2P_k$$

for $k \in K_1$ and $k = 2$ is

$$\ll N^{\alpha_1 + 1/2 + \epsilon} (\log N)^{C_1}.$$

Proof *Using Lemma 2.12 and [14, Lemma 22], we have this lemma.*

Proposition 1

$$\int_0^1 |F_1(\alpha)|^2 \, d\alpha \ll N^{-1} (\log N)^{C_1} \{F_1(0)\}^2,$$

where C_1 is a positive contant.

Proof *By $N^{2(\alpha_1 + 1/2)} \ll \{F_1(0)\}^2$, this proposition follows from Lemma 2.13.*

By (2.5) and Lemma 2.2, since $N^{\beta_2} \ll F_3(0)$,

$$\int_0^1 |F_3(\alpha)|^2 \, d\alpha \ll N^{\beta_2 + \epsilon} \ll N^{-\beta_2 + \epsilon} \{F_3(0)\}^2. \tag{2.12}$$

By (2.5), (2.12) and Lemma 2.11, since $N^{1/4} \ll f_4(0)$, $N^{1/6} \ll f_6(0)$,

$$\begin{aligned} \int_{\mathfrak{m}} |F_2(\alpha)|^2 \, d\alpha &\ll N^{2\times(7/32+\theta/4)} N^{2\times(31/192+\theta/6)} \int_0^1 |F_3(\alpha)|^2 \, d\alpha \\ &\ll N^{-1+2/4+2/6} \{F_3(0)\}^2 \ll N^{-1} \{F_2(0)\}^2. \end{aligned} \tag{2.13}$$

Lemma 2.14 *On $\mathfrak{M}_{a,q}$,*

$$u_{10} \ll q^{-1/10} N^{\nu_{10}'/10}.$$

Proof *The lemma now follows from (2.3) and Lemma 2.8.*

Lemma 2.15 *On $\mathfrak{M}_{a,q}$,*

$$f_k - h_k \ll q^{-1/2} N^{1/8}; \quad h_k \ll N^{1/k} q^{-1} |S_k| \min(1, |N|^{-1} |\beta|^{-1}); \quad (k = 4, 6, 8)$$

$$f_4 + h_4 \ll q^{-1/4} N^{1/4}; \quad f_6 + h_6 \ll q^{-1/6} N^{1/6}; \quad f_8 + h_8 \ll q^{-1/8} N^{1/8};$$

$$f_6^2 \ll |h_6|^2 + q^{-1} N^{1/4} + |h_6| q^{-1/2} N^{1/8}; \quad f_8^2 \ll |h_8|^2 + q^{7/4+2\theta}.$$

Proof *The lemma is deduced from Lemmas 2.14 and 2.8.*

Lemma 2.16 *On* $\mathfrak{M}_{a,q}$,

$$(f_4^2 f_6^2 f_8^2 - h_4^2 h_6^2 h_8^2) u_{10}^2 \ll q^{-1/5} N^{2\nu_{10}'/10} (\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3),$$

where

$$\begin{aligned}\mathcal{L}_1 =& (f_4 - h_4)(f_4 + h_4) f_6^2 f_8^2 \\ \ll& N^{2\varrho} \big\{ q^{-5-3/4} |S_6|^2 |S_8|^2 + q^{-5-5/12} |S_8|^2 + q^{-5-7/12} |S_6||S_8|^2 \\ &+ q^{-4+2\theta} |S_6|^2 \big\} \min(1, |N|^{-1}|\beta|^{-1})^2 + N^{2/4+2/6+1/8} \big\{ q^{-8/3+2\theta} + q^{-2-5/6+2\theta} |S_6| \big\}, \\ \mathcal{L}_2 =& (f_6 - h_6)(f_6 + h_6) h_4^2 f_8^2 \ll N^{2\varrho} \big\{ q^{-5} |S_4|^2 |S_8|^2 + q^{-3-1/4} |S_4|^2 \big\} \min(1, |N|^{-1}|\beta|^{-1})^2, \\ \mathcal{L}_3 =& (f_8 - h_8)(f_8 + h_8) h_4^2 h_6^2 \ll q^{-5-3/4+\theta} N^{2\varrho} |S_4|^2 |S_6|^2 \min(1, |N|^{-1}|\beta|^{-1})^2,\end{aligned}$$

and $\varrho := 1/4 + 1/6 + 1/8$.

Proof *This is deduced in a standard way from the above two lemmas.*

Lemma 2.17 *For* $i = 1, 2, \cdots, 10$,

$$\sum_{q \le N^\tau} B_i(q) \ll 1. \tag{2.14}$$

Proof *By Lemma 2.4 and* (2.10),

$$0 \le B_1(q) \ll q^{-5-3/4-1/5+7/4+5/3+1} \ll q^{-23/15}. \tag{2.15}$$

In fact, we can get the desired bound from this upper bound since the power is less than -1. *But it is not the case for some of others,* B_8 *for example. Thus, we write here a general argument suitable for all.*

It follows that $\sum_{h=0}^\infty B_1(p^h)$ *exists for every prime* p *and*

$$\sum_{h=2}^\infty B_1(p^h) \ll p^{-46/15}. \tag{2.16}$$

By Lemma 2.3 and (2.10),

$$0 \le B_1(p) \ll p^{-5-3/4-1/5} (\sqrt{p})^2 (\sqrt{p})^2 p \ll p^{-59/20}. \tag{2.17}$$

Clearly $B_1(1) = 1$. *Hence, by* (2.16) *and* (2.17),

$$\sum_{h=0}^\infty B_1(p^h) \ll 1 + C_2 p^{-59/20}. \tag{2.18}$$

By Lemmas 2.6, 2.7 and (2.18), *we have*

$$\sum_{q \le N^\tau} B_1(q) \le \prod_{p \le N^\tau} \sum_{h=0}^\infty B_1(p^h) \le \prod_{p \le N^\tau} (1 + C_2 p^{-59/20}) \ll 1.$$

The other terms in (2.14) *can be treated in the same way.*

Lemma 2.18

$$\sum_{q\le N^\tau}\sum_{\substack{1\le a\le q\\(a,q)=1}}\int_{\mathfrak{M}_{a,q}}|(f_4^2f_6^2f_8^2-h_4^2h_6^2h_8^2)u_{10}^2|\,d\alpha\ll N^{2\eta-1}, \tag{2.19}$$

and

$$\sum_{q\le N^\tau}\sum_{\substack{1\le a\le q\\(a,q)=1}}\int_{\mathfrak{M}_{a,q}}|h_4^2h_6^2h_8^2u_{10}^2|\,d\alpha\ll N^{2\eta-1}. \tag{2.20}$$

where $\eta := 1/4+1/6+1/8+\nu_{10}'/10.$

Proof (2.19) *follows in a similar manner from Lemmas 2.16 and 2.17.* (2.20) *is again deduced in a standard way from Lemmas 2.14, 2.15, and 2.17 using the estimate*

$$h_4^2h_6^2h_8^2u_{10}^2\ll N^{2\eta}q^{-6-1/5}|S_4|^2|S_6|^2|S_8|^2\min(1,N^{-1}|\beta|^{-1})^2.$$

Thus we have the result.

It now follows from Lemma 2.18 that,

$$\sum_{q\le N^\tau}\sum_{\substack{1\le a\le q\\(a,q)=1}}\int_{\mathfrak{M}_{a,q}}|f_4^2f_6^2f_8^2u_{10}^2|\,d\alpha\ll N^{-1}\left\{f_4(0)f_6(0)f_8(0)u_{10}(0)\right\}^2 \tag{2.21}$$

by $N^\eta\ll f_4(0)f_6(0)f_8(0)u_{10}(0)$.

Now, using the trivial estimate

$$\prod_{k\in K_2}u_k(\alpha)\ll\prod_{k\in K_2}u_k(0),$$

we have from (2.21),

$$\int_{\mathfrak{M}}|F_2(\alpha)|^2\,d\alpha\ll N^{-1}\{F_2(0)\}^2. \tag{2.22}$$

Proposition 2

$$\int_0^1|F_2(\alpha)|^2\,d\alpha\ll N^{-1}\{F_2(0)\}^2.$$

Proof *With* Q *defined by* (2.8), *we make the same subdivision of the interval* $Q^{-1}\le\alpha\le 1+Q^{-1}$ *(into* $\mathfrak{M}$ *and* $\mathfrak{m}$ *).*
Write

$$\int_{Q^{-1}}^{1+Q^{-1}}|F_2(\alpha)|^2\,d\alpha=\int_{\mathfrak{M}}|F_2(\alpha)|^2\,d\alpha+\int_{\mathfrak{m}}|F_2(\alpha)|^2\,d\alpha.$$

The required result follows from (2.13) *and* (2.22).

3 Proof of Theorem 1.1

Let

$$2P_{4448} = N^{1/4448}, \quad U_k = \begin{cases} P_k, & \text{if } k \in \{2,4,6,4448\} \cup K_1; \\ P_k', & \text{if } k \in \{8,10\} \cup K_2, \end{cases} \tag{3.1}$$

where K_1, K_2, P_k and P_k' are defined by (2.1), (2.4) and (2.5).

Write

$$\hat{f}_k = \hat{f}_k(\alpha) = \sum_{U_k \le p \le 2U_k} e(\alpha p^k) \quad \text{for} \quad k = 2i, \quad 1 \le i \le 2224, \tag{3.2}$$

and

$$C_2 = C_1/2, \quad C_3 = 2^{6\times 4448}(C_2 + 2226), \quad L = \log N, \quad \hat{F}(\alpha) = \prod_{i=1}^{2224} \hat{f}_{2i}(\alpha). \tag{3.3}$$

Subdivide the interval

$$N^{-1}L^{C_3} \le \alpha \le 1 + N^{-1}L^{C_3} \tag{3.4}$$

into basic intervals $\mathfrak{M}^{\dagger}_{a,q}$ for $q \le L^{C_3}$ with $\alpha = a/q + \beta$, $|\beta| \le q^{-1}N^{-1}L^{C_3}$ and denote the union of $\mathfrak{M}^{\dagger}_{a,q}$'s (these being disjoint) by $\mathfrak{M}^{\dagger}$; the supplementary intervals $\mathfrak{m}^{\dagger}$ denotes the complement of $\mathfrak{M}^{\dagger}$ in (3.4).

Let

$$\begin{aligned} R^{\dagger}(N) &:= \int_{N^{-1}L^{C_3}}^{1+N^{-1}L^{C_3}} \hat{F}(\alpha)e(-N\alpha)\,d\alpha \\ &= \int_{\mathfrak{M}^{\dagger}} \hat{F}(\alpha)e(-N\alpha)\,d\alpha + \int_{\mathfrak{m}^{\dagger}} \hat{F}(\alpha)e(-N\alpha)\,d\alpha. \end{aligned} \tag{3.5}$$

As in Lemma 7 and its corollary in [12] (with slight modifications), we have on $\mathfrak{m}^{\dagger}$,

$$\hat{f}_{4448}(\alpha) \ll N^{1/4448}L^{-C_2-2225}. \tag{3.6}$$

Replacing (33) and (34) in [12] by Propositions 1 and 2, and arguing as in section 8 of [12], we have from (3.3) and (3.6)

$$\int_{\mathfrak{m}^{\dagger}} |\hat{F}(\alpha)|\,d\alpha \ll N^{-1+1/4448}\{F_1(0)F_2(0)\}L^{-2225}. \tag{3.7}$$

Also, by (2.4), (2.5) and (3.1),

$$L^{C_3} \ll (\log U_k)^{C_3} \ll L^{C_3} \quad \text{and} \quad N^{-1}L^{C_3} \ll U_k^{-k}(\log U_k)^{C_3},$$

for $1 \le i \le 2224$, $k = 2i$.

Let $\varphi(q)$ be Euler's totient function, we have by Lemma 8 of [12], on $\mathfrak{M}^{\dagger}$,

$$\hat{f}_k(\alpha) - \hat{h}_k(\alpha) \ll Ne^{-C_4\sqrt{L}} \quad \text{for} \quad 1 \le i \le 2224, \quad k = 2i \quad (C_5 > 0), \tag{3.8}$$

where $\hat{h}_k(\alpha)$ is the approximating function corresponding to $\hat{f}_k$ given by

$$\hat{h}_k(\alpha, a, q) = \{\varphi(q)\}^{-1}\Big\{\sum_{\substack{x=1\\(x,q)=1}}^{q} e_q(ax^k)\Big\}\Big\{\sum_{U_k^k\le y\le(2U_k^k)^k} y^{1/k-1}(\log y)^{-1}e(\beta y)\Big\}.$$

Let $M_s(p^\gamma, N) = M_s(p^\gamma, N; k_1, k_2, \cdots, k_s)$ denote the number of solutions of the congruence

$$x_1^{k_1} + x_2^{k_2} + x_3^{k_3} + \cdots + x_s^{k_s} \equiv N \pmod{p^\gamma}, \tag{3.9}$$

where $0 < x_i < p^\gamma$ and $p \nmid x_1x_2\cdots x_s$.

Let $p^{\lambda_i}||k_i$, and write

$$\gamma_i = \begin{cases} \lambda_i + 2, & \text{if } p = 2 \text{ and } 2|k_i;\\ \lambda_i + 1, & \text{otherwise,}\end{cases} \quad \text{and} \quad \gamma = \min(\gamma_1, \gamma_2, \cdots, \gamma_s). \tag{3.10}$$

Then $\gamma = 3$ for $p = 2$ and $k_i \in \{2, 4, \cdots, 4448\}$. Also, $\gamma = 1$ for $p \ge 3$ and $k_i \in \{2, 4, \cdots, 4448\}$. Hence if $M_{2224}(p^\gamma, N) > 0$ for each prime p (note that the premises of Lemmas 16, 19, 20 in [12] are satisfied), it would follow as in [12] that

$$\Re\left(\int_{\mathfrak{M}^\dagger} \hat{F}(\alpha)e(-N\alpha)\,d\alpha\right) \gg N^{-1+1/4448}\{F_1(0)F_2(0)\}L^{-2224}. \tag{3.11}$$

Then, from (3.5), (3.7) and (3.11) we get $R^\dagger(N) > 0$ for large N provided $M_{2224}(p^\gamma, N) > 0$ for each prime p.

The argument is completed as follows: Let $\mathcal{N}_1 \equiv 1 \pmod 3$, and $p \ge 5$. $M_{2224}(p, N)$ denotes the number of solutions of the congruence

$$\Big(\sum_{k=1}^{2224} x_k^{2k}\Big) \equiv N \pmod p, \quad 0 < x_k < p \quad \text{for} \quad 1 \le k \le 2224. \tag{3.12}$$

By Lemma 8.4 of [8], for each prime p, x_k^{2k} $(0 < x_k < p)$ has $\frac{p-1}{(2k,p-1)}$ distinct residue classes mod p $(1 \le k \le 2224)$. Hence, applying Lemma 8.4 of [8] repeatedly (2223 times), we see that the number $J(p)$ of distinct residue classes mod p of $(\sum_{k=1}^{2224} x_k^{2k})$ $(0 < x_k < p)$ satisfies

$$J(p) \ge \min\left(\sum_{k=1}^{2224}\left\{\frac{p-1}{(2k,p-1)}\right\} - 2223, p\right). \tag{3.13}$$

Now

$$\frac{p-1}{(2k,p-1)} \ge \frac{p-1}{2k} \quad \text{and} \quad \sum_{k=1}^{2224}\frac{1}{2k} > 4.$$

Hence, from (3.13),

$$J(p) \ge \min\left(4(p-1) - 2223, p\right). \tag{3.14}$$

Also $4(p-1)-2223>p$ if $p \geq 743$; so that by (3.14),

$$M_{2224}(p,N)>0 \quad \text{for} \quad p \geq 743. \tag{3.15}$$

For primes $5 \leq p < 743$, it is an easy verification (by use of Lemmas 8.4 and 8.7 of [8]) that

$$M_{2224}(p,N)>0. \tag{3.16}$$

For $p=3$, by use of Lemma 8.4 in [8], when x_k runs through $1,2 \pmod 3$, x_k^{2k} runs through $(3-1)/(2k,3-1)$ mutually incongruent numbers mod 3 (for $1 \leq k \leq 2224$). We see that

$$x_k^{2k} \equiv 1 \pmod 3 \quad \text{for} \quad 1 \leq k \leq 2224,\ x_k \equiv 1 \,\text{or}\, 2 \pmod 3.$$

Hence

$$x_1^2+x_2^4+x_3^6+\cdots+x_{2224}^{4448} \equiv \mathcal{N}_1 \pmod 3, \tag{3.17}$$

where $\mathcal{N}_1 \equiv 1 \pmod 3$.

For $p=2$, we note that $1^2 \equiv 1 \pmod 8$, $2^2 \equiv 4 \pmod 8$, $3^2 \equiv 1 \pmod 8$, $4^2 \equiv 0 \pmod 8$, $5^2 \equiv 1 \pmod 8$, $6^2 \equiv 4 \pmod 8$, $7^2 \equiv 1 \pmod 8$. The congruence

$$x_1^2+x_2^4+x_3^6+\cdots+x_{2224}^{4448} \equiv \mathcal{N}_1 \pmod{2^3} \tag{3.18}$$

has solutions for all sufficiently large integers $\mathcal{N}_1$.

Combining (3.15), (3.16), (3.17) and (3.18), we can get $M_{2224}(p^{\gamma},\mathcal{N}_1)>0$ for all sufficiently large integers $\mathcal{N}_1$.

Similar argument holds for the cases $\mathcal{N}_2 \equiv 2 \pmod 3$ and $\mathcal{N}_3 \equiv 0 \pmod 3$.

Thus, all sufficiently large integers $\mathcal{N}_1 \equiv 1 \pmod 3$, $\mathcal{N}_2 \equiv 2 \pmod 3$, $\mathcal{N}_3 \equiv 0 \pmod 3$ are representable in the forms

$$\mathcal{N}_1=\sum_{k=1}^{s} p_k^{2k}, \quad \mathcal{N}_2=\sum_{k=1}^{s+1} p_k^{2k}, \quad \mathcal{N}_3=\sum_{k=1}^{s+2} p_k^{2k},$$

where $s=2224$ and the p's are primes.

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