Low Regularity for Nonlinear Dirac Equation in the Half Line

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Abstract

We study the Initial-Boundary Value problem(IBVP) for a nonlinear Dirac equation with vector self-interaction (Thirring model), and obtain local existence, uniqueness, and continuous dependence on initial data in low-regularity spaces. Moreover, we get the global existence for initial data in $\mathbb{R} \times \mathbb{R}^+$.

Keywords: Initial-Boundary Value Problem, Dirac Equation, Contraction Map, Local Existence, Global Existence, Restricted Norm Method

1 Introduction

In this paper, we study the low-regularity for the initial-boundary value problem (IBVP) of the nonlinear Dirac equation(Thirring model)

$$\begin{cases} (-i\gamma^\mu \partial_\mu + m)\psi = \lambda \bar{\psi}\gamma^\mu \psi \gamma_\mu \psi, & (x,t) \in \mathbb{R}^+ \times (0,T), \\ \psi(0,x) = \psi_0(x), & x > 0, \\ \psi_2(t,0) = h(t), & t > 0. \end{cases} \tag{1.1}$$

The matrices α, β are given by

$$\alpha = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

and the γ^μ's are 2×2 Dirac matrices in the representation

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Take expansion for (1.1), we have

$$\begin{cases} -i\partial_t \psi_2 + i\partial_x \psi_2 + m\psi_1 = 2\lambda |\psi_1|^2 \psi_2, & (x,t) \in \mathbb{R}^+ \times (0,T), \\ -i\partial_t \psi_1 - i\partial_x \psi_1 + m\psi_2 = 2\lambda |\psi_2|^2 \psi_1, & (x,t) \in \mathbb{R}^+ \times (0,T), \\ \psi(0,x) = \psi_0(x), & x > 0, \\ \psi_2(t,0) = h(t), & t > 0, \end{cases} \tag{1.2}$$

with compatibility condition $h(0) = \psi_{02}(0)$, where

$$\psi_0(x) = \begin{pmatrix} \psi_{01}(x) \\ \psi_{02}(x) \end{pmatrix}.$$

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For more details about Dirac equation, we can refer to [7]-[13].

Notation:

$\langle\cdot\rangle = \sqrt{1+|\cdot|^2}$, $\tilde{u}(\tau,\xi) = \int_{\mathbb{R}^{1+1}} e^{-i(t\tau+x\xi)}u(t,x)\ dtdx$.

The characteristic function on $[0,\infty)$ is defined by χ. $A \sim B$ means that there exists constant C_1 and C_2 such that $C_1 B \le A \le C_2 B$.

Let $a, b \in \mathbb{R}$, define:

$$\|u\|_{X_\pm^{a,b}} = \|\langle\xi\rangle^a \langle\tau \pm \xi\rangle^b \tilde{u}(\tau,\xi)\|_{L^2_{\tau,\xi}},$$

$$\|u\|_{H^{a,b}} = \|\langle\xi\rangle^a \langle|\tau| - |\xi|\rangle^b \tilde{u}(\tau,\xi)\|_{L^2_{\tau,\xi}},$$

and

$$H^s(\mathbb{R}^+) = \left\{ g \in D(\mathbb{R}^+) : \text{there exists } \tilde{g} \in H^s(\mathbb{R}) \ \text{ with } \ \tilde{g}\chi = g \right\},$$

$$\|g\|_{H^s(\mathbb{R}^+)} = \inf\left\{ \|\tilde{g}\|_{H^s(\mathbb{R})} : \tilde{g}\chi = g \right\}.$$

Lemma 1.1. *(see [1]) Let $1/2 < b \le 1$, $a \in \mathbb{R}$, $0 < T \le 1$ and $0 \le \delta \le 1-b$. Then for all data $F \in X_\pm^{a,b-1+\delta}(S_T)$ and $f \in H^a$, the Cauchy problem*

$$-i(\partial_t \pm \partial_x)u = F(t,x) \quad in \quad (0,T)\times\mathbb{R}, u(0,x) = f(x)$$

has unique solution $u \in X_\pm^{a,b}(S_T)$. Moreover,

$$\|u\|_{X_\pm^{a,b}(S_T)} \le C(\|f\|_{H^a} + T^\delta \|F\|_{X_\pm^{a,b-1+\delta}(S_T)}) \tag{1.3}$$

where C depends only on b and $S_T = (0,\infty)\times(0,T)$.

To obtain the result, we need the following two lemmas, we can see the details in [1].

Lemma 1.2. *If $a_1, a_2, a_3 \in \mathbb{R}$ satisfying*

$$a_1 + a_2 + a_3 > \frac{1}{2}, a_1 + a_2 \ge 0, a_1 + a_3 \ge 0, a_2 + a_3 \ge 0, \tag{1.4}$$

then

$$\|fg\|_{H^{-a_3}} \lesssim \|f\|_{H^{a_1}} \|g\|_{H^{a_2}}. \tag{1.5}$$

Moreover, we can allow $a_1 + a_2 + a_3 = \frac{1}{2}$ if $a_j \ne \frac{1}{2}$ for $1 \le j \le 3$.

Lemma 1.3. *Suppose $a_1, a_2, a_3 \in \mathbb{R}$ satisfying (1.4), let $\alpha, \beta, \gamma \ge 0$ with $\alpha + \beta + \gamma > \frac{1}{2}$. Then*

$$\|uv\|_{H^{-a_3,-\gamma}} \lesssim \|u\|_{H^{a_1,\alpha}} \|v\|_{H^{a_2,\beta}}. \tag{1.6}$$

Moreover, we can allow $a_1 + a_2 + a_3 = \frac{1}{2}$ if $a_j \ne \frac{1}{2}$ for $1 \le j \le 3$.

Denote

$$\Gamma_\pm = \tau \pm \xi, \quad \Theta_\pm = \lambda \pm \eta, \quad \Sigma_\pm = \lambda - \tau \pm (\eta - \xi).$$

Then,

$$|\xi| \le \frac{3}{2}\max(|\Gamma_\pm|, |\Theta_\mp|, |\Sigma_\mp|).$$

Remark 1.4. *Naturally, there is a question that why we only give one boundary value problem? By I.P.Naumkin [6] we have that*

$$\mathcal{L}_t\psi_1(\xi,0) = \frac{\xi - i}{\langle\xi\rangle}(\mathcal{L}_t\psi_2)(\xi,0) + \frac{\xi - i}{\langle\xi\rangle}(\mathcal{L}_x\psi_{01})(\langle\xi\rangle) - (\mathcal{L}_x\psi_{01})(\langle\xi\rangle),$$

where $\mathcal{L}$ denotes the Laplace transform, that

$$(\mathcal{L}_x\phi)(p) = \int_0^\infty e^{-px}\phi(x)dx, \quad 1 \le p \le \infty,$$

$$(\mathcal{L}_t\phi)(q) = \int_0^\infty e^{-qt}\phi(t)dt, \quad 1 \le q \le \infty.$$

To obtain (1.2), we begin by constructing the solution of the linear initial-boundary-value problem:

$$\begin{cases} -i\partial_t\psi_2 + i\partial_x\psi_2 = 2\lambda|\psi_1|^2\psi_2, & (x,t) \in \mathbb{R}^+ \times (0,T), \\ \psi(0,x) = \psi_0(x) = \phi(x), & x > 0, \\ \psi_2(t,0) = h(t), & t > 0. \end{cases} \tag{1.7}$$

For extension ϕ^R to the full line $\mathbb{R}$ of the function ϕ. $\rho \in C^\infty$ be a cut-off function such that

$$\rho = \begin{cases} 1, & [0,\infty), \\ 0, & (-\infty,0), \end{cases} \qquad \text{supp } \rho \subset [-1,\infty).$$

$\eta \in C^\infty$ be a bump function such that

$$\eta = \begin{cases} 1, & [-1,1], \\ \text{const}, & [-2,-1) \cup (1,2], \\ 0, & (-\infty,-2) \cup (2,+\infty). \end{cases}$$

D_0 represents evaluation at $x = 0$, i.e.,

$$D_0[u(x,t)] = u(0,t).$$

Denote the solution of (1.7) by $W_0^t(\phi,h)$,

$$W_0^t(\phi,h) = W_0^t(0,h-p_2) + W_t^R(\phi^R),$$

$p_2(t) = D_0[W_t^R(\phi_2^R)]$, W_t^R is the Fourier multiplier operator with multiplier Re $e^{it|\xi|}$, $W_t^{R_2}$ is the Fourier multiplier operator with multiplier Re $e^{-it\xi_2}$, $W_t^{R_1}$ is the Fourier multiplier operator with multiplier Re $e^{it\xi_1}$.

We decompose the solution operator as a sum of a modified boundary operator and the free propagator defined on the whole real line. Note that $W_0^t(0,h)$ is the solution of the following problem:

$$\begin{cases} -i\partial_t\psi_2 + i\partial_x\psi_2 = 2\lambda|\psi_1|^2\psi_2, & (x,t) \in \mathbb{R}^+ \times (0,T), \\ \psi(0,x) = \psi_0(x) = \phi(x), & x > 0, \\ \psi_2(t,0) = h(t), & t > 0. \end{cases} \tag{1.8}$$

$W_0^t(0,h_1)$ is the solution of the following problem:

$$\begin{cases} -i\partial_t\psi_1 - i\partial_x\psi_1 = 2\lambda|\psi_2|^2\psi_1, & (x,t) \in \mathbb{R}^+ \times (0,T), \\ \psi(0,x) = \psi_0(x) = \phi(x), & x > 0, \\ \psi_1(t,0) = h_1(t), & t > 0, \end{cases} \tag{1.9}$$

where $h_1(t)$ can be expressed by $h(t)$.

Lemma 1.5. *Suppose h is a Schwartz function, the solution to (1.8) on $\mathbb{R}^+ \times \mathbb{R}^+$ can be written in the form:*

$$\psi_2(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{h} e^{-iw_2 x}\rho(x)dw_2,$$

$$\psi_1(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{h_1} e^{iw_1 x}\rho(x)dw_1.$$

Proof. Taking the Laplace transform in time of (1.8) yields the equation

$$\begin{cases} -\tau_2\widetilde{\psi_2}(x,\tau) - \partial_x\widetilde{\psi_2}(x,\tau) = 0, \\ \widetilde{\psi_2}(\tau,0) = \tilde{h}(\tau), & \tau > 0. \end{cases}$$

The characteristic equation of this is $-\tau_2 - \xi_2 = 0$, which has root satisfying $\xi_2 = -\tau_2$.

So,

$$\widetilde{\psi_2}(x,\tau) = \tilde{h}e^{\xi_2 x}, \quad \xi_2 \in \mathbb{C}.$$

Because we are interested in solution which decay at infinity, so, Re $\tau_2 \geq 0$.

By Mellin inversion, we have for any $c \geq 0$ the equality,

$$\psi_2(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{h}e^{\xi_2 x}e^{-\tau_2 t}d\tau_2,$$

analogously,

$$\psi_1(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \widetilde{h_1}e^{-\xi_1 x}e^{-\tau_1 t}d\tau_1.$$

We can write this as an integral along the imaginary axis plus integrals along a key hole contour about the branch cut and integrals along $s \pm iR$ for $s \in [0,c]$ with $R \to \infty$.

By Jordan's lemma, we have

$$\psi_2(x,t) = 2\mathrm{Re}\frac{1}{2\pi i}\int_{0}^{i\infty} \tilde{h}e^{-\tau_2 x}e^{-\tau_2 t}d\tau_2.$$

Make the change of variable $\tau_2 = i\mu_2$, then $d\tau_2 = id\mu_2$. On the positive imaginary axis

$$\begin{aligned}\psi_2(x,t) &= \frac{1}{\pi}\mathrm{Re}\int_0^{\infty} \tilde{h}e^{-i\mu_2 x}e^{-i\mu_2 t}d\mu_2 \\ &= \frac{1}{2\pi}(\int_0^{\infty} \tilde{h}e^{-i\mu_2 x}d\mu_2 + \int_0^{\infty} \tilde{h}e^{i\mu_2 x}d\mu_2) \\ &= \frac{1}{2\pi}(\int_0^{\infty} \tilde{h}e^{-i\mu_2 x}d\mu_2 + \int_{-\infty}^{0} \tilde{h}e^{-i\mu_2 x}d\mu_2) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{h}e^{-iw_2 x}e^{-iw_2 t}\rho(x)dw_2,\end{aligned}$$

where $\mu_2 = \rho(x)w_2$.

□

The explicit form will be used to establish bound on $W_0^t(0,h)$ in the subsequent sections. It is clear that the solution to (1.8) satisfies $\Phi_2(\psi_2) = \psi_2$, where the operator Φ_2 is given by

$$\begin{aligned}\Phi_2(\psi_2(x,t)) = \eta(\frac{t}{T})W_R^t(\phi_2^R) + \eta(\frac{t}{T})\int_0^t W_R^{t-t'}G_2(\psi_2)dt' \\ + \eta(\frac{t}{T})W_0^t(0,h-p_2),\end{aligned} \tag{1.10}$$

$$G_2(\psi_2) = -im\psi_1 + 2i\lambda|\psi_1|^2\psi_2. \tag{1.11}$$

Analogously, we have the equation Φ_1,

$$\begin{aligned}\Phi_1(\psi_1(x,t)) = \eta(\frac{t}{T})W_R^t(\phi_1^R) + \eta(\frac{t}{T})\int_0^t W_R^{t-t'}G_1(\psi_1)dt' \\ + \eta(\frac{t}{T})W_0^t(0, h_1 - p_1),\end{aligned} \tag{1.12}$$

where

$$p_1(t) = \eta(\frac{t}{T})D_0[W_R^t(\phi_1^R)], \quad p_2(t) = \eta(\frac{t}{T})D_0[W_R^t(\phi_2^R)], \tag{1.13}$$

$$G_1(\psi_1) = -im\psi_2 + 2i\lambda|\psi_2|^2\psi_1. \tag{1.14}$$

Next, we will use the fixed point argument to obtain the unique solution to $\Phi_1(\psi_1) = \psi_1$ and $\Phi_2(\psi_2) = \psi_2$ separately in a suitable function space on $\mathbb{R}\times\mathbb{R}$ for sufficiently small T. The restriction of ψ_1, ψ_2 to $\mathbb{R}^+\times\mathbb{R}$ is a distributional solution of (1.9) and (1.8) separately.

We will argue the contraction in Bourgain spaces $X^{s,b}$. To obtain the solution to the linear Dirac on $\mathbb{R}$ and Duhamel term, we will use the following estimates:

$$\|\eta(t)W_R^t(\phi)\|_{X^{s,b}} = \|\eta(t)e^{it|\xi|}\phi\|_{X^{s,b}} \le \|\phi\|_{H^s} \tag{1.15}$$

for any s and b.

Next, we will give the Duhamel estimate:

Lemma 1.6. *For any $-\frac{1}{2} < b' \le 0 \le b \le b'+1$, and $0 < T \le 1$, we have*

$$\|\eta(\frac{t}{T})\int_0^t W_{R_2}^{t-t'}(\psi_2)dt'\|_{H^{s,b}} \lesssim T^{1-(b-b')}\|M(G_2(\psi_2))\|_{H^{s,b'}},$$

where $\big(\hat{M}(f)\big)(\xi_2) = |\xi_2|\hat{f}$.

Proof. Since

$$\begin{aligned}&\mathcal{F}_x\big[\eta(t/T)\int_0^t W_{R_2}^{t-t'}G_2(\psi_2)dt'\big] \\ &= e^{-it\xi_2}[\eta(t/T)\int_0^t e^{it'\xi_2}\mathcal{F}_x(-im\psi_1 + 2i\lambda|\psi_1|^2\psi_2)dt'] = e^{-it\xi_2}\mathcal{F}_x w_2(t,\xi_2),\end{aligned}$$

therefore,

$$\mathcal{F}_{x,t}[\eta(t/T)\int_0^t W_{R_2}^{t-t'}G_2(\psi_2)dt'](\tau_2,\xi_2) = \widetilde{w_2}(\tau_2+\xi_2,\xi_2).$$

Analogously,

$$\mathcal{F}_{x,t}(\eta(t/T)\int_0^t W_{R_1}^{t-t'}G_1(\psi_1)dt')(\tau_1,\xi_1) = \widetilde{w_1}(\tau_1-\xi_1,\xi_1).$$

Now using the definition of $X^{s,b}$ we have

$$\begin{aligned}&\|\eta(t/T)\int_0^t W_{R_2}^{t-t'}G_2(\psi_2)dt'\|^2_{X^{s,b}} \\ &\lesssim \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle\xi_2\rangle^{2s}\langle\tau_2-\xi_2\rangle^{2b}|\widetilde{w_2}(\xi_2,\tau_2)|^2 d\xi_2 d\tau_2 \\ &\lesssim \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle\xi_2\rangle^{2s}\langle\tau_2\rangle^{2b}\langle\xi_2\rangle^{2b}|\widetilde{w_2}(\xi_2,\tau_2)|^2 d\xi_2 d\tau_2 \\ &\lesssim \int_{-\infty}^{\infty}\langle\xi_2\rangle^{2s+2b}\|\mathcal{F}_x w_2\|^2_{H_t^b}d\xi_2 \\ &\lesssim T^{1-(b-b')}\iint_{\mathbb{R}^2}\langle\tau_2+\xi_2\rangle^{2b'}\langle\xi_2\rangle^{2b+2s}\big|\mathcal{F}_{x,t}(-im\psi_1 + 2i\lambda|\psi_1|^2\psi_2)\big|d\xi_2 d\tau_2,\end{aligned}$$

hence we have

$$\|\eta(t/T)\int_0^t W_{R_2}^{t-t'}G_2(\psi_2)dt'\|_{X^{s,b}} \lesssim T^{1-(b-b')}\|\mathcal{N}_2(G_2(\psi_2))\|_{X^{s+b,b'}}, \tag{1.16}$$

where

$$\mathcal{F}_x\big(\mathcal{N}_2(f)\big)(\xi_2) = -\xi_2\hat{f}(t,\xi_2) = M(f),$$

similarly, we have

$$\|\eta(t/T)\int_0^t W_{R_1}^{t-t'}G_1(\psi_1)dt'\|_{X^{s,b}} \lesssim T^{1-(b-b')}\|\mathcal{N}_1(G_1(\psi_1))\|_{X^{s+b,b'}},$$

$$\|\eta(t/T)\int_0^t W_{R_1}^{t-t'}G_1(\psi_1)dt'\|_{X^{s,b}} \lesssim T^{1-(b-b')}\|\mathcal{M}(G_1(\psi_1))\|_{H^{s,b'}},$$

where

$$\hat{\mathcal{M}}(f)(\xi_1) = |\xi_1|\hat{f}(\xi,t).$$

□

From [4], we have the following two lemmas.

Lemma 1.7.

$$\|\eta(\frac{t}{T})F\|_{H^{s_1,b_1}} \lesssim T^{b_2-b_1}\|F\|_{H^{s_2,b_2}}$$

for any $-\frac{1}{2} \le b_1 < b_2 < \frac{1}{2}$.

Lemma 1.8. *Assume* $h \in H^s(\mathbb{R}^+)$,

(i) If $-\frac{1}{2} < s < \frac{1}{2}$, *then* $\|\chi h\|_{H^s(\mathbb{R})} \lesssim \|h\|_{H^s(\mathbb{R}^+)}$;

(ii) If $\frac{1}{2} < s < \frac{3}{2}$, $h(0) = 0$, *then* $\|\chi h\|_{H^s(\mathbb{R})} \lesssim \|h\|_{H^s(\mathbb{R}^+)}$.

2 The a-prior estimates

The estimates include two parts: linear estimates and nonlinear estimates. First, we will give the linear estimates.

2.1 Linear estimates

Lemma 2.1. *(Kato Smoothing) For any* $s \in \mathbb{R}$,

$$\|\eta(t)W_R^t(\phi)\|_{L_x^\infty H_t^{\frac{2s-1}{2}}} \lesssim \|\phi\|_{H_x^s}.$$

Proof. In the following we only consider the evaluation at $x = 0$, since Sobolev norm is invariant under translations,

$$\begin{aligned}
\mathcal{F}_t(\eta W_{R_2}^t(\phi))(0,\tau) &= \mathcal{F}_t(\eta(t)e^{-it\xi_2}\hat{\phi}(\xi_2))(0,\tau)\\
&= \int_{\mathbb{R}} e^{-it\tau}\eta(t)e^{-it\xi_2}\hat{\phi}(\xi_2)d\xi_2\\
&= \int_{\mathbb{R}} \hat{\eta}(\tau+\xi_2)\hat{\phi}(\xi_2)d\xi_2\\
&= \int_{|\xi_2|\le 1} \hat{\eta}(\tau+\xi_2)\hat{\phi}(\xi_2)d\xi_2 + \int_{|\xi_2|>1} \hat{\eta}(\tau+\xi_2)\hat{\phi}(\xi_2)d\xi_2.
\end{aligned}$$

On the first region, the term can easily be bounded in $H_t^{\frac{2s+1}{4}}$, since η is a Schwartz function. When $|\xi_2| > 1$, it is obvious to obtain the result.

□

Lemma 2.2. *For any compactly supported smooth function η and any $s \geq -\frac{1}{2}$ with $b < \frac{1}{2}$,*

$$\|\eta(t)W_0^t(0,h)\|_{H^{s,b}} \lesssim \|\chi h\|_{H_t^s(\mathbb{R})}.$$

Proof. By Lemma 1.1, we obtain that

$$\begin{aligned} 2\pi\psi_2 &= \int_{-\infty}^{\infty} \tilde{h}(iw_2)e^{-iw_2x}e^{-iw_2t}\rho(x)dw_2 \\ &= \mathcal{L}_t\int_{-\infty}^{\infty} \tilde{h}(w_2)e^{-iw_2x}\rho(x)dw_2 = \mathcal{L}_t\phi_A, \end{aligned}$$

where $\mathcal{L}_t$ is the laplace transform in time, $\widehat{\phi_A} = \hat{h}(w_2)$.

Because of (1.15),

$$\|\eta(t)\mathcal{L}_tA\|_{H^{s,b}} \lesssim \|A\|_{H_x^s}.$$

Now,

$$\|A\|_{H_x^s} = \int_{-\infty}^{\infty} \langle w\rangle^{2s}|\hat{h}(w)|^2dw \lesssim \|\chi h(t)\|_{H_t^s(\mathbb{R})}.$$

□

Lemma 2.3. *For any $s \geq 0$ and initial data (h, h_1) such that $(\chi h, \chi h_1) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$, we have*

$$\begin{aligned} W_0^t(0,h) \in C_t^0H_x^s, \quad W_0^t(0,h_1) \in C_t^0H_x^s, \\ \eta W_0^t(0,h) \in C_x^0H_t^s, \quad \eta W_0^t(0,h_1) \in C_x^0H_t^s. \end{aligned}$$

Proof. Note that

$$2\pi W_0^t(0,h) = A = \mathcal{L}_t\int_{-\infty}^{\infty} \tilde{h}(w_2)e^{-iw_2x}\rho(x)dw_2 = \mathcal{L}_t\phi_A(w),$$

and

$$\|\phi_A\|_{H_x^s}^2 = \int_{-\infty}^{\infty} \langle w\rangle^{2s}|\hat{h}(w)|^2dw \lesssim \|\chi h\|_{H_x^s}^2,$$

$\mathcal{L}_t$ is the Fourier multiplier operator with multiplier e^{-iwt}. Thus, using time continuity of the linear operator $\mathcal{L}_t$, it suffices to show that the map

$$g \mapsto T(g) = \int_{-\infty}^{\infty} \hat{g}(w)f(wx)dw$$

is bounded from H^s to H^s, where $f(x) = e^{-x}$. Consider first $s = 0$. Rewrite $Tg(x)$ as follows $(z = iwx) \Rightarrow w = \frac{z}{ix}$,

$$Tg(x) = \int_{-\infty}^{\infty} f(iwx)\hat{g}(w)dw = \int_{-\infty}^{\infty} f(z)\hat{g}(\frac{z}{ix})\frac{1}{ix}dz.$$

Then,

$$\|Tg\|_{L_x^2} \lesssim \int_{-\infty}^{\infty} |f(x)|\|\chi_{[0,iz]}\hat{g}(\frac{z}{ix})\frac{1}{ix}\|_{L_x^2}dz,$$

and expanding the L_x^2 norm, combining $\frac{z}{ix} = y$, we obtain

$$\int_0^{iz} |\hat{g}(\frac{z}{ix})|^2\frac{1}{x^2}dx \lesssim \int_0^{\infty} |\hat{g}(y)|^2dy,$$

$$\|\int_{-\infty}^{\infty} f(wx)\hat{g}(w)dw\|_{L_x^2}^2 \lesssim \|g\|_{L^2}^2 \|\chi_{[0,1]}(w)f(wx)\|_{L_{x,w}^2}^2$$
$$= \|g\|_{L^2}^2 \int_0^1 \int_0^\infty f^2(y)\frac{1}{w}dydw \lesssim \|g\|_{L^2}^2.$$

Thus completes the proof that

$$2\pi W_0^t(0,h) \in C_t^0 H_x^s \ \text{ for } \ s = 0.$$

For any $s \in \mathbb{N}$ and $s > 0$, we have

$$\partial_x^s Tg(x) = \int_0^\infty w^s f^{(s)}(wx)\hat{g}(w)dw.$$

This and interpolation imply the desired bounds for positive s.

It remains to prove that $\eta W_0^t(0,h) \in C_x^0 H_t^s$. Applying Lemma 1.1 we obtain the result. □

2.2 Nonlinear Estimates

Lemma 2.4. *(bilinear $H^{s,b}$ estimate) Let M be the Fourier multiplier operator with multiplier ξ. For $\frac{1}{2} - b > 0$ sufficiently small, we have*

$$\big\|M|u|^2 v\big\|_{X^{s,-b}} \lesssim \big\|u\|_{X^{s,b}}^2 \big\| v\|_{X^{s,b}}.$$

Proof. By duality, it suffices to show that

$$\Big|\iint_{\mathbb{R}^2} M(|u|^2 v)\bar{\phi}dxdt\Big| \lesssim \|u\|_{X^{s,b}}^2 \|v\|_{X^{s,b}} \|\phi\|_{X^{-(s+a),b}} \tag{2.1}$$

for any $\phi \in X^{-(s+a),b}$.

The left hand side of (2.1) is equal to

$$\begin{aligned}
&|\iint_{\mathbb{R}^2} -\xi\widehat{(|u|^2 v)}(\xi,\tau)\overline{\hat{\phi}(\xi,\tau)}d\xi d\tau| \\
&= |\iint\iint_{\mathbb{R}^4} -\xi|\hat{u}|^2(\xi-\xi_1,\tau-\tau_1)v(\xi_1,\tau_1)(\xi,\tau)\overline{\hat{\phi}(\xi,\tau)}d\xi_1 d\tau_1 d\xi d\tau| \\
&= |\iint\iint\iint_{\mathbb{R}^6} -\xi\hat{u}(\xi-\xi_2,\tau-\tau_2)\hat{u}(\xi_2-\xi_1,\tau_2-\tau_1)v(\xi_1,\tau_1)(\xi,\tau)\overline{\hat{\phi}(\xi,\tau)}d\xi_2 d\tau_2 d\xi_1 d\tau_1 d\xi d\tau\big|.
\end{aligned}$$

Now we define

$$f(\xi,\tau) = \langle\xi\rangle^s \langle\tau-\xi\rangle^b \hat{u}(\xi,\tau), \quad g(\xi,\tau) = \langle\xi\rangle^s \langle\tau-\xi\rangle^b \overline{\hat{u}(\xi,\tau)},$$
$$h(\xi,\tau) = \langle\xi\rangle^s \langle\tau-\xi\rangle^b \hat{v}(\xi,\tau), \quad r(\xi,\tau) = \langle\xi\rangle^{-(s+a)} \langle\tau-\xi\rangle^b \overline{\hat{\phi}(\xi,\tau)}.$$

Then the inequality (2.1) is equivalent to

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^6} M(\xi,\xi_1,\xi_2,\tau,\tau_1,\tau_2)f(\xi_1,\tau_1)g(\xi_2-\xi_1,\tau_2-\tau_1)h(\xi-\xi_2,\tau-\tau_2) \\
&r(\xi,\tau)d\xi_1 d\xi_2 d\xi d\tau_1 d\tau_2 d\tau\Big| \lesssim \|f\|_{L_{\xi,\tau}^2}\|g\|_{L_{\xi,\tau}^2}\|h\|_{L_{\xi,\tau}^2}\|r\|_{L_{\xi,\tau}^2},
\end{aligned} \tag{2.2}$$

where

$$M = \frac{|\xi|\langle\xi\rangle^{s+a}\langle\xi_1\rangle^{-s}\langle\xi_2-\xi_1\rangle^{-s}\langle\xi-\xi_2\rangle^{-s}}{\langle\tau-\xi\rangle^b\langle(\tau_2-\tau_1)-(\xi_2-\xi_1)\rangle^b\langle(\tau-\tau_2)-(\xi-\xi_2)\rangle^b}.$$

By Cauchy Schwartz and Young's inequalities, we have

$$\text{LHS of (2.2)} \lesssim \sup_{\rho,2} \|M\|_{L^2_{\xi_1,\tau_1}} \|f\|_{L^2_{\xi,\tau}} \|g\|_{L^2_{\xi,\tau}} \|h\|_{L^2_{\xi,\tau}}$$

then it suffices to show that

$$\sup_{\xi,\tau} \iint\iint_{\mathbb{R}^4} \frac{\langle\xi\rangle^{2(s+a)+2}\langle\xi_1\rangle^{-2s}\langle\xi_2-\xi_1\rangle^{-2s}\langle\xi-\xi_2\rangle^{-2s}d\xi_1 d\xi_2 d\tau_1 d\tau_2}{\langle\tau-\xi\rangle^{2b}\langle(\tau_2-\tau_1)-(\xi_2-\xi_1)\rangle^{2b}\langle(\tau-\tau_2)-(\xi-\xi_2)\rangle^{2b}\langle\tau_1-\xi_1\rangle^{2b}} \lesssim 1. \tag{2.3}$$

Using the triangular inequality $\langle a\rangle\langle b\rangle \geqslant \langle a+b\rangle$ we have

$$\text{LHS of (2.3)} \lesssim \sup_{\xi} \int\iint_{\mathbb{R}^3} \frac{\langle\xi\rangle^{2(s+a)+2}\langle\xi_1\rangle^{-2s}\langle\xi_2-\xi_1\rangle^{-2s}\langle\xi-\xi_2\rangle^{-2s}d\xi_1 d\xi_2 d\tau_2}{\langle\tau_2-\xi_2\rangle^{2b}\langle\tau_2-(\xi-\xi_2)-\xi\rangle^{2b}}. \tag{2.4}$$

Applying lemma A.l of Erdogan[5] in τ_2 integral, we are reduced to prove

$$\begin{aligned} &\sup_{\xi} \iint_{\mathbb{R}^2} \frac{\langle\xi\rangle^{2(s+a)+2}\langle\xi_1\rangle^{-2s}\langle\xi_2-\xi_1\rangle^{-2s}\langle\xi-\xi_2\rangle^{-2s}d\xi_1 d\xi_2}{\langle 2(-\xi+\xi_2)\rangle^{1-}} \\ &\lesssim \sup_{\xi} \iint_{\mathbb{R}^2} \langle\xi\rangle^{2(s+a)+2}\langle\xi_1\rangle^{-2s}\langle\xi_2-\xi_1\rangle^{-2s}\langle\xi-\xi_2\rangle^{-2s-1}d\xi_1 d\xi_2 \lesssim 1. \end{aligned} \tag{2.5}$$

Due to the symmetry of $\xi_1, \xi_2-\xi_1, \xi-\xi_2$, we may assume that $|\xi_1| \lesssim |\xi_2-\xi_1| \lesssim |\xi-\xi_2|$. We will discuss (2.5) in the following cases,

Case I: $|\xi_1| > 1, |\xi-\xi_2| \sim |\xi_1-\xi_2| \sim |\xi| \sim |\xi_1|$. Combining triangular inequality,

$$\text{LHS of (2.5)} \lesssim \sup_{\xi}\langle\xi\rangle^2 \cdot \int_{\mathbb{R}} \langle\xi_1\rangle^{-1-4s+2a} d\xi_1 \lesssim 1$$

provided that $s > \frac{1}{2}$.

Case II: $|\xi_2-\xi_1| > 1, \quad |\xi_1| \ll |\xi_2-\xi_1| \sim |\xi-\xi_2| \sim |\xi| \sim |\xi_2|$.

Case II-a: $|\xi_1| \leq 1$. Then,

$$\text{LHS of(2.5)} \lesssim \sup_{\xi}\langle\xi\rangle^{-2s+1} \int_{\mathbb{R}} \langle\xi_1\rangle^{-2s} d\xi_1 \lesssim \sup_{\xi}\langle\xi\rangle^{-2s+1} \lesssim 1$$

provided $s > \frac{1}{2}$.

Case II-b: $|\xi_1| > 1$.

$$\text{LHS of (2.5)} \lesssim \sup_{\xi}\langle\xi\rangle^{2(a-s)+1} \int_{\mathbb{R}} \langle\xi_1\rangle^{-2s} d\xi_1.$$

- If $s < 0$, then$\langle\xi_1\rangle^{-2s} \lesssim \langle\xi\rangle^{-2s}$.

$$\text{LHS of (2.5)} \lesssim \sup_{\xi}\langle\xi\rangle^{2(a-2s)+1} \lesssim 1.$$

- If $s > 0$,

$$\text{LHS of (2.5)} \lesssim \int_{\mathbb{R}} \langle\xi_1\rangle^{-4s+1} d\xi_1 \lesssim 1$$

provided $s > \frac{1}{2}$.

Case III: $|\xi_1| > 1, |\xi| \ll |\xi_1| \sim |\xi-\xi_2|$.

Case III-a: $|\xi| > 1$. Then,

$$\text{LHS of (2.5)} \lesssim \sup_{\xi}\langle\xi\rangle^{2(s+1)} \int_{\mathbb{R}} \langle\xi_1\rangle^{-6s-1} d\xi_1 \lesssim \sup_{\xi}\langle\xi\rangle^{-4s+2} \lesssim 1$$

provided $s > \frac{1}{2}$.

Case III-b: $|\xi| \leq 1$.

$$\text{LHS of (2.5)} \lesssim \int_{\mathbb{R}} \langle \xi_1 \rangle^{-6s-1} d\xi_1 \lesssim 1$$

provided $s > 0$. So, from above estimates we have $s > \frac{1}{2}$.

□

Lemma 2.5. *For $\frac{1}{2} - b > 0$ sufficiently small, we have*

$$\|\eta(t) \int_0^t W_{R,2}^{t-t'} G dt'\|_{L_x^\infty H_t^{s-\frac{1}{2}}} \lesssim \|M(G)\|_{X^{s,-b}}, \qquad \frac{1}{2} < s < 1.$$

Proof. It suffices to consider evaluation at $x = 0$. We have

$$\begin{aligned}\int_0^t W_{R,2}^{t-t'} G dt' &= \int_{\mathbb{R}} \int_0^t \frac{e^{-i(t-t')\xi} + e^{i(t-t')\xi}}{2} \hat{G}(\xi, t') dt' d\xi \\ &= \iint_{\mathbb{R}^2} \int_0^t \frac{e^{-i(t-t')\xi} + e^{i(t-t')\xi}}{2} e^{i\tau t'} \hat{G}(\xi, \tau) dt' d\tau d\xi\end{aligned}$$

and

$$\int_0^t e^{it'(\tau \pm \xi)} dt' = \frac{e^{it'(\tau \pm \xi)}}{i(\tau \pm \xi)} \Big|_0^t = \frac{e^{it(\tau \pm \xi)} - 1}{i(\tau \pm \xi)}.$$

Thus, we will bound

$$\iint_{\mathbb{R}^2} \frac{e^{it(\tau \pm \xi)}}{i(\tau \pm \xi)} \hat{G}(\xi, \tau) d\tau d\xi.$$

Define ψ being a smooth cut-off function such that

$$\psi = \begin{cases} 1, & [-1, 1], \\ 0, & (-\infty, -2) \cup (2, +\infty). \end{cases}$$

Then,

$$\begin{aligned}\eta(t) \int_0^t W_{R,2}^{t-t'} G dt' = \eta(t) &\iint_{\mathbb{R}^2} \frac{e^{it(\tau \pm \xi)} \psi(\tau \pm \xi)}{i(\tau \pm \xi)} d\xi d\tau \\ &+ \eta(t) \iint_{\mathbb{R}^2} \frac{e^{\mp it\xi} \psi^C(\tau \pm \xi)}{i(\tau \pm \xi)} d\xi d\tau \\ &+ \eta(t) \iint_{\mathbb{R}^2} \frac{e^{it\tau} \psi^C(\tau \pm \xi)}{i(\tau \pm \xi)} d\xi d\tau \\ &:= \text{I} + \text{II} + \text{III}.\end{aligned}$$

By Taylor Expanding, we have

$$\frac{e^{it\tau} - e^{\mp it\xi}}{i(\tau \pm \xi)} = -e^{it\tau} \sum_{k=1}^{\infty} \frac{(-it)^k}{k!} (\tau \pm \xi)^{k-1}.$$

Therefore, $\|\text{I}\|_{H_t^s}$ is bounded by

$$\begin{aligned}&\sum_{k=1}^{\infty} \frac{\|\eta(t) t^k\|_{H^1}}{k!} \| \iint_{\mathbb{R}^2} e^{it\tau} (\tau \pm \xi)^{k-1} \psi(\tau \pm \xi) \hat{G}(\xi, \tau) d\xi d\tau \|_{H_t^{s-\frac{1}{2}}} \\ &\lesssim \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \|\langle \tau \rangle^{s-\frac{1}{2}} \int_{\mathbb{R}} (\tau \pm \xi)^{k-1} \psi(\tau \pm \xi) \hat{G}(\xi, \tau) d\xi\|_{L_\tau^2} \\ &\lesssim \|\langle \tau \rangle^s \int_{\mathbb{R}} (\tau \pm \xi) \hat{G}(\xi, \tau) d\xi\|_{L_\tau^2}.\end{aligned}$$

Using the Cauchy-Schwartz inequality in τ, this can be banded by

$$\Big(\int_{\mathbb{R}}\langle\tau\rangle^{2s-1}(\int_{|\tau\pm\xi|<1}\langle\xi\rangle^{-2s}d\xi)(\int_{|\tau\pm\xi|<1}\langle\xi\rangle^{2s}|\hat{G}(\xi,\tau)|^2d\xi)d\tau\Big)^{\frac{1}{2}}$$
$$\lesssim \sup_{\tau}\Big(\langle\tau\rangle^{2s-1}\int_{|\tau\pm\xi|<1}\langle\xi\rangle^{-2s}d\xi\Big)^{\frac{1}{2}}\|M(G)\|_{X^{s,-b}}$$
$$\lesssim \|M(G)\|_{X^{s,-b}},$$

where we have used that

$$1\approx\frac{1}{\langle\tau-\xi\rangle^{2b}}.$$

The supreme bound holds since

$$\langle\tau\rangle^{2s-1}\int_{|\tau\pm\xi|<1}\langle\xi\rangle^{-2s}d\xi\lesssim\begin{cases}1, & |\tau|\lesssim 1,\\ \langle\tau\rangle^{2s-1}\int_{|\tau\pm\xi|<1}\langle\tau\rangle^{-2s}d\xi, & |\tau|>1,\end{cases}$$

the latter bound comes from $|\tau|\to|\xi|$.

Next, consider II. When $|\xi|\le 1$, since $b<\frac{1}{2}$, we have

$$\Big\|\eta(t)\iint_{\mathbb{R}^2}\frac{e^{\mp it\xi}\psi^C(\tau\pm\xi)}{i(\tau\pm\xi)}d\xi d\tau\Big\|_{H_t^{s-\frac{1}{2}}}$$
$$\lesssim\iint_{|\xi|\le 1}\frac{\|\eta(t)e^{\mp it\xi}\|_{H_t^{s-\frac{1}{2}}}}{|\tau\pm\xi|}\psi^C(\tau\pm\xi)|\widehat{M(G)}(\xi,\tau)|d\xi d\tau$$
$$\lesssim\iint_{\mathbb{R}^2}\frac{\chi_{[-1,1]}(\xi)}{\langle\tau\pm\xi\rangle}|\widehat{M(G)}(\xi,\tau)|d\xi d\tau$$
$$\lesssim\|M(G)\|_{X^{s,-b}}\Big\|\frac{\chi_{[-1,1]}(\xi)}{\langle\tau\pm\xi\rangle}\Big\|^2_{L_{\tau,\xi}}\lesssim\|M(G)\|_{X^{s,-b}}.$$

To control the part of II where $|\xi|\ge 1$,

$$\Big\|\eta(t)\iint_{\mathbb{R}^2}\frac{e^{\mp it\xi}\psi^C(\tau\pm\xi)}{i(\tau\pm\xi)}d\xi d\tau\Big\|_{H_t^{s-\frac{1}{2}}}\lesssim\|\langle\xi\rangle^{s-\frac{1}{2}}\int_{\mathbb{R}}\frac{|\hat{G}(\xi,\tau)|}{|\tau\pm\xi|}d\tau\|_{L^2_{|\xi|\ge 1}}.$$

By Cauchy-Schwartz inequality in the τ integral, using the fact that $b<\frac{1}{2}$, this is bounded by

$$\|\langle\xi\rangle^{s-\frac{1}{2}}\frac{|\hat{G}(\xi(z),\tau)|}{\langle\tau-z\rangle^{b}}\|_{L^2_{|z|\ge 1}L^2_\tau},$$

the above is bounded by $\|M(G)\|_{X^{s,-b}}$.

It remains to bound III.

$$\|\mathrm{III}\|_{H_t^{s-\frac{1}{2}}}\lesssim\|\langle\tau\rangle^{s-\frac{1}{2}}\int_{\mathbb{R}}\frac{|\hat{G}(\xi,\tau)|}{\langle\tau\pm\xi\rangle}d\xi\|_{L^2_\tau}$$
$$\lesssim\|\int_{\mathbb{R}}\chi_R\langle|\tau|-|\xi|\rangle^{s-\frac{1}{2}}\langle\xi\rangle^{s-\frac{1}{2}}\frac{|\hat{G}(\xi,\tau)|}{\langle\tau\pm\xi\rangle}d\xi\|_{L^2_\tau}$$
$$+\|\int_{\mathbb{R}}\chi_{R^C}\langle|\tau|-|\xi|\rangle^{s-\frac{1}{2}}\langle\xi\rangle^{s-\frac{1}{2}}\frac{|\hat{G}(\xi,\tau)|}{\langle\tau\pm\xi\rangle}d\xi\|_{L^2_\tau}$$
$$=A+B.$$

For the first term of the right hand side of the above inequality,

$$\begin{aligned}
A &\lesssim \| \int_{\mathbb{R}} \chi_R \langle\tau\rangle^{s-\frac{1}{2}} \langle\xi\rangle^{s-\frac{1}{2}} \frac{|\hat{G}(\xi,\tau)|}{\langle\tau\pm\xi\rangle} d\xi \|_{L^2_\tau} \\
&\lesssim \| \int_{\mathbb{R}} \langle\tau\rangle^{s-\frac{3}{2}} \langle\xi\rangle^{s-\frac{1}{2}} |\hat{G}(\xi,\tau)| d\xi \|_{L^2_\tau} \\
&\lesssim \| \langle\tau\rangle^{s-\frac{3}{2}+b} \big(\int_{\mathbb{R}} \langle\xi\rangle^{-\frac{3}{2}} \langle\tau-\xi\rangle^{-b} \langle\xi\rangle^{s+1} |\hat{G}(\xi,\tau)|^2 d\xi \big)^{\frac{1}{2}} \|_{L^2_\tau} \\
&\lesssim \sup_\tau \langle\tau\rangle^{s-\frac{3}{2}+b} \|M(G)\|_{X^{s,-b}} \lesssim \|M(G)\|_{X^{s,-b}},
\end{aligned}$$

this is finite for $\frac{1}{2} < s \leq 1$.

The second term can be bounded by

$$\begin{aligned}
&\| \int_{\mathbb{R}} \langle\xi\rangle^{2s-1} \langle\tau-\xi\rangle^{-b-1} \langle\tau-\xi\rangle^{b} |\hat{G}(\xi,\tau)| d\xi \|_{L^2_\tau} \\
&\lesssim \| \big(\int_{\mathbb{R}} \langle\xi\rangle^{2s-3} \langle\tau-\xi\rangle^{2b-2} d\xi \big)^{\frac{1}{2}} \big(\int_{\mathbb{R}} \langle\xi\rangle^{2s+1} \langle\tau-\xi\rangle^{-2b} |\hat{G}(\xi,\tau)|^2 d\xi \big)^{\frac{1}{2}} \|_{L^2_\tau} \\
&\lesssim \|M(G)\|_{X^{s,-b}},
\end{aligned}$$

provided that $\sup_\tau \int_{\mathbb{R}} \langle\xi\rangle^{2s-3} \langle\tau-\xi\rangle^{2b-2} d\xi < \infty$, since $b < \frac{1}{2}$, $|\tau| < |\xi|$, then $\langle\tau-\xi\rangle^{2b-2} \lesssim \langle\tau\rangle^{2b-2} \lesssim 1$, then the above is bounded by $\sup_\tau \int_{\mathbb{R}} \langle\xi\rangle^{2s-3} d\xi \lesssim 1$ provided $\frac{1}{2} < s \leq 1$.

□

Lemma 2.6. *(see [4]) Assume $h \in H^s(\mathbb{R}^+)$,*
(i) If $-\frac{1}{2} < s < \frac{1}{2}$, then $\|\chi h\|_{H^s(\mathbb{R})} \lesssim \|h\|_{H^s(\mathbb{R})}$;
(ii) If $\frac{1}{2} < s < \frac{3}{2}$, $h(0) = 0$, then $\|\chi h\|_{H^s(\mathbb{R})} \lesssim \|h\|_{H^s(\mathbb{R})}$.

3 Proof of theorem

We will first show that the map Φ_2 defined in (1.10) has a unique fixed point in $X^{s,b}$. Let $\phi^R \in H^s(\mathbb{R})$ be the extension of ϕ such that $\|\phi^R\|_{H^s(\mathbb{R})} \lesssim \|\phi\|_{H^s(\mathbb{R})}$. Recall that

$$\begin{aligned}
\Phi_2(\psi_2(x,t)) &= \eta\left(\frac{t}{T}\right) W^t_{R,2}\left(\phi^R_2\right) + \eta\left(\frac{t}{T}\right) \int_0^t W^{t-t'}_{R,2} G_2(\psi_2)\, dt' \\
&\quad + \eta\left(\frac{t}{T}\right) W^t_0(0, h-p_2),
\end{aligned}$$

where $G_2(\psi_2)$ and p_2 are defined in (1.11) and (1.13). To bound the first term in Φ_2, apply (1.15) to obtain

$$\left\| \eta\left(\frac{t}{T}\right) W^t_{R,2}\left(\phi^R_2\right) \right\|_{X^{s,b}} \lesssim \|\phi^R\|_{H^s(\mathbb{R})} \lesssim \|\phi\|_{H^s(\mathbb{R}^+)}.$$

For the Duhamel term, we apply (1.16) and lemma 2.4 to obtain

$$\begin{aligned}
\left\| \eta(t/T) \int_0^t W^{t-t'}_{R,2} G_2(\psi_2)\, dt' \right\|_{X^{s,b}} &\lesssim T^{1-2b} \left\| M(|\psi_1|^2 \psi_2) \right\|_{X^{s,-b}} \\
&\lesssim T^{1-2b} \|\psi_1\|^2_{X^{s,b}} \|\psi_2\|_{X^{s,b}}.
\end{aligned}$$

Finally, for the W^t_0 term, we apply lemma 2.2 and lemma 2.6 to obtain

$$\begin{aligned}
\|\eta(t/T) W^t_0(0, h-p_2)\|_{X^{s,b}} &\lesssim \|\chi(h-p_2)\|_{H_t^{\frac{2s-1}{2}}(\mathbb{R})} \\
&\lesssim \|h-p_2\|_{H_t^{s-\frac{1}{2}}(\mathbb{R}^+)}.
\end{aligned}$$

By Kato smoothing, lemma 2.1, we have

$$\|p_2\|_{H_t^{\frac{2s-1}{2}}(\mathbb{R})} \lesssim \left\|\phi^R\right\|_{H^s(\mathbb{R})} \lesssim \|\phi\|_{H^s(\mathbb{R}^+)}.$$

Combining these estimates. we find that

$$\|\Phi_2(\psi_2)\|_{X^{s,b}} \lesssim \|\phi_2\|_{H^s(\mathbb{R}^+)} + \|h\|_{H_t^{\frac{2s-1}{2}}(\mathbb{R}^+)} + T^{\frac{1}{2}-b-} \|\psi_1\|^2_{X^{s,b}} \|\psi_2\|_{X^{s,b}} .$$

Analogously, we have

$$\|\Phi_1(\psi_1)\|_{X^{s,b}} \lesssim \|\phi_1\|_{H^s(\mathbb{R}^+)} + \|h_1\|_{H_t^{\frac{2s-1}{2}}(\mathbb{R}^+)} + T^{\frac{1}{2}-b-} \|\psi_2\|^2_{X^{s,b}} \|\psi_1\|_{X^{s,b}} .$$

This, together with similar estimates for the difference $\Phi_2(\psi_2) - \Phi_2\left(\tilde{\psi}_2\right)$, yields the existence of a fixed point of Φ_2 of T_2 sufficiently small:

$$T_2 = T_2\left(\|\phi\|_{H^s(\mathbb{R}^+)}, \|\vec{h}\|_{H_t^{\frac{2s-1}{2}}(\mathbb{R}^+)}\right).$$

To obtain the uniqueness, we should show that,
1. $\psi_2 \leftrightarrow \Phi_2(\psi_2)$ is onto $X^{s,b}$,
2. the map $\psi_2 \leftrightarrow \Phi_2(\psi_2)$ is a contraction in $X^{s,b}$.

From the above estimates, we obtain the uniqueness easily.

Proof of Global well-posedness:

It suffices to show that if $0 < T < \infty$ and

$$(\psi_1, \psi_2) \in C([0,T); H^s) \times C([0,T), H^s)$$

solves (1.2) on S_T, then

$$\|\psi_1\|_{L^\infty(S_T)} + \|\psi_2\|_{L^\infty(S_T)} < \infty. \tag{3.1}$$

Indead, if (3.1) is satisfied then global existence can be shown as follows: Denote

$$A(t) = \|\psi_1(t)\|_{H^s} + \|\psi_2(t)\|_{H^s},$$

then we have

$$A(t) \le A(0) + C\int_0^t \left(A(t') + \left\||\psi_2|^2\psi_1(t')\right\|_{H^s} + \left\||\psi_1|^2\psi_2(t')\right\|_{H^s}\right) dt'.$$

Now, we use the inequality (Ponce, 1993, lemma 1)

$$\|fg\|_{H^s} \lesssim \|f\|_{H^s}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|g\|_{H^s} \quad \text{for} \quad s > 0,$$

to obtain

$$\begin{aligned}\left\||\psi_2|^2\psi_1(t')\right\|_{H^s} &\lesssim \left\||\psi_2|^2(t')\right\|_{H^s}\|\psi_1(t')\|_{L_x^\infty} + \||\psi_2|^2(t')\|_{L_x^\infty}\|\psi_1(t')\|_{H^s} \\ &\lesssim \|\psi_2(t')\|_{H^s}\|\psi_2(t')\|_{L_x^\infty}\|\psi_1(t')\|_{L_x^\infty} + \|\psi_2(t')\|^2_{L_x^\infty}\|\psi_1(t')\|_{H^s}.\end{aligned}$$

For the term $\||\psi_1|^2\psi_2(t')\|_{H^s}$, we have the similar result. So we have

$$\left\||\psi_1|^2\psi_2(t')\right\|_{H^s} + \left\||\psi_2|^2\psi_1(t')\right\|_{H^s} \lesssim \left(\|\psi_1(t')\|_{L_x^\infty} + \|\psi_2(t')\|_{L_x^\infty}\right)^2 A(t').$$

Therefore,

$$A(t) \leq A(0) + C\left(1 + \|\psi_1\|_{L^\infty(S_T)} + \|\psi_2\|_{L^\infty(S_T)}\right)^2 \int_0^t A\left(t'\right) dt'.$$

Gronwall's Lemma then implies

$$A(t) \leq A_0 e^{c(1+\|\psi_1\|_{L^\infty(S_T)}+\|\psi_2\|_{L^\infty(S_T)})^2 t}$$

for $0 \leq t < T$, hence $\sup_{0\leq t<T}(\|\psi_1(t)\|_{H^s} + \|\psi_2(t)\|_{H^s}) < \infty$ allowing us to extend the solution to $[0, T+\varepsilon] \times \mathbb{R}, \varepsilon > 0$, global existence then follows.

Finally, we will prove (3.1). By (1.2), we derive

$$(\partial_t + \partial_x)|\psi_1|^2 = -2m \operatorname{Im}\left(\psi_1 \overline{\psi_2}\right), \tag{3.2}$$

$$(\partial_t - \partial_x)|\psi_2|^2 = 2m \operatorname{Im}\left(\psi_1 \overline{\psi_2}\right). \tag{3.3}$$

By Duhamel formula,

$$|\psi_1(t,x)|^2 = |\psi_1(0, x-t)|^2 - 2m \operatorname{Im} \int_0^t \left(\psi_1 \bar{\psi}_2\right)\left(t', x - t + t'\right) dt',$$
$$|\psi_2(t,x)|^2 = |\psi_2(0, x+t)|^2 + 2m \operatorname{Im} \int_0^t \left(\psi_1 \bar{\psi}_2\right)\left(t', x + t - t'\right) dt'.$$

We then estimate

$$\begin{aligned}
\left\||\psi_1(t,x)|^2\right\|_{L_x^\infty} + \left\||\psi_2(t,x)|^2\right\|_{L_x^\infty} &\leq \left\||\psi_1(0,x)|^2\right\|_{L_x^\infty} + \left\||\psi_2|^2(0,x)\right\|_{L_x^\infty} \\
&\quad + 2m \int_0^t \left(\left\||\psi_1(t,x)|^2\right\|_{L_x^\infty} + \left\||\psi_2(t,x)|^2\right\|_{L_x^\infty}\right) dt' \\
&\lesssim \|\psi_1(0,x)\|_{H^s}^2 + \|\psi_2(0,x)\|_{H^s}^2 \\
&\quad + 2m \int_0^t \left(\left\||\psi_1(t,x)|^2\right\|_{L_x^\infty} + \left\||\psi_2(t,x)|^2\right\|_{L_x^\infty}\right) dt',
\end{aligned}$$

Gronwall's lemma implies

$$\left\||\psi_1(t,x)|^2\right\|_{L_x^\infty} + \left\||\psi_2(t,x)|^2\right\|_{L_x^\infty} \lesssim e^{2mt}\left(\|\psi_1(0,x)\|_{H^s}^2 + \|\psi_2(0,x)\|_{H^s}^2 \right),$$

hence

$$\left\||\psi_1|^2\right\|_{L^\infty(S_T)} + \left\||\psi_2(t,x)|^2\right\|_{L_x^\infty} \lesssim \left(\|\psi_1(0,x)\|_{H^s}^2 + \|\psi_2(0,x)\|_{H^s}^2\right) e^{2mt},$$

therefore $\|\psi_1\|_{L^\infty(S_T)} + \|\psi_2\|_{L^\infty(S_T)} < \infty$ for $0 < T < \infty$. Here, the proof is completed.

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