Characterization of the continuity of Δ-space via the convergence

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Abstract

The concepts of Δ-convergence and Δ_L-convergence of a net are introduced in Δ-space defined by Zhang et.al. The characterization of the continuity of the Δ-space is obtained in terms of the Δ-convergence of the nets. The result that the continuity of the Δ-space implies the Δ_L-convergence being topological in Δ-space is given. An example is supplied to illustrate that the converse of the above result does not hold. Meantime, we prove that the Δ-space X is continuous if and only if the Δ_L-convergence is topological in X, X is meet-continuous and $\mathcal{O}(X) \bigvee \omega(X) = \tau_{\Delta_L}$. Moreover, we put forward the concept of weak continuity of the Δ-space and show that a sufficient and necessary condition for the Δ-space being weak continuous is that the Δ_L-convergence is topological.

2020 Mathematics Subject Classification: 06F30
Key words: Δ-convergence, Δ_L-convergencd, Δ-space, weak continuous

1 Introduction

The theory of T_0-spaces is a combination of order theory and general topology, each playing a crucial role, and each interacting with other in ways that both are enriched. In [1], the authors discussed the Scott topology and its connection with the convergence given in order theoretic terms by S-convergence and lower limits in directed complete posets(dcpos). They obtained the characterization of the continuity of the dcpos by the S-convergence structure, that is, a dcpo is continuous if and only if the S-convergence is topological. Afterwards, B. Zhao and D. Zhao gave a sufficient and necessary condition for the posets to be continuous utilizing the S-convergence structure in [2]. Moreover, there were many kinds of convergence class proposed to characterize various kinds of continuity of the posets or the dcpos in the papers [3, 4, 6–10]. As a generalization of the posets endowed with Scott topology, Z. Zhang et.al introduced the concept of

Z. Zheng and T. Wang (eds.), *Proceedings of the International Academic Summer Conference on Number Theory and Information Security (NTIS 2023)*, Advances in Physics Research 9,
https://doi.org/10.2991/978-94-6463-463-1_9

the Δ-space in [13]. They also defined the continuity of the Δ-space and presented topological characterizations of the continuity. The notions of a domain, a continuous poset, a quasi-continuous domain, an s_2-continuous poset given by Erne [6], an s_2-quasi-continuous poset introduced by Zhang and Xu [5], a strongly continuous poset proposed by Xu and Mao [12], and a θ-continuous poset defined by Zhang et.al [11] are special cases of the continuity of the Δ-space. It is natural to ask: Can we make use of the convergence of the nets to characterize the continuity of the Δ-space?

In this paper, we introduce the concepts of Δ-convergence and Δ_L-convergence of a net in Δ-space. The characterization of the continuity of the Δ-space is obtained in terms of the Δ-convergence of the nets. This result answers the above question. We obtain that the continuity of the Δ-space implies that the Δ_L-convergence is topological in Δ-space, but the converse does not hold. Meantime, we prove that the Δ-space X is continuous if and only if the Δ_L-convergence is topological in X, X is meet-continuous and $\mathcal{O}(X)\bigvee\omega(X)=\tau_{\Delta_L}$. At last, we put forward the concept of the weak continuity of the Δ-space and shows that a sufficient and necessary condition for the Δ-space being weak continuous is that the Δ_L-convergence is topological.

2 Preliminary

The following definitions can be seen in [1] and [14].

Let L be a poset, $A\subseteq L$. Let $A^u=\{b\in L:\ \forall a\in A,\ b\geqslant a\}$ be the set of upper bounds of A, $A^l=\{b\in L:\ \forall a\in A,\ b\leqslant a\}$ be the set of lower bounds of A, $\downarrow A=\{b\in L:\ \exists a\in A,\ b\leqslant a\}$ and $\uparrow A=\{b\in L:\ \exists a\in A,\ b\geqslant a\}$. A subset A is called a lower set(upper set) if $A=\downarrow A(A=\uparrow A)$. A subset D is called *directed*(*filtered*) if it is non-empty and for every non-empty and finite subset F of D, $F^u\cap D\neq\varnothing(F^l\cap D\neq\varnothing)$. L is called a *dcpo* if every directed subset has a sup. A subset A is called a *filter* if it is a filtered upper set. We call the topology generated by $\{L\backslash\uparrow x\mid x\in L\}\cup\{L\}$ the *lower topology*, and we denote it by $\omega(L)$. For a subset A of L, a net $(x_i)_{i\in I}\in A$ *usually* if for all $i\in I$, there exists a $i_0\in I$ with $i_0\geq i$ such that $x_{i_0}\in A$. And a net $(x_i)_{i\in I}\in A$ *eventually* if there exists a $i_0\in I$ such that $x_i\in A$ for all $i\geq i_0$. For any topological space $(X,\mathcal{O}(X))$, a net $(x_i)_{i\in I}$ in X *converges* to an element x in X if $(x_i)_{i\in I}\in U$ eventually for all U in $\mathcal{O}(X)$ with $x\in U$.

For a space $(X,\mathcal{O}(X))$, the specialization order $\leq$ on X is defined by

$$x\leq y \text{ if and only if } x\in cl(\{y\}).$$

In this paper, unless other stated otherwise, whenever an order-theoretical concept is mentioned in the context of a space X, it is to be interpreted with respect to the specialization order on X.

Lemma 2.1 *[15] Let $\mathcal{L}$ be a class of some pairs $((x_i)_{i\in I},x)$ of a net $(x_i)_{i\in I}$ and an element x in a poset L. Then the class $\mathcal{L}$ is topological, that is, there exists a topology τ on L such that $((x_i)_{i\in I},x)\in\mathcal{L}$ iff the net $(x_i)_{i\in I}$ converges to x with respect to the topology τ, if and only if it satisfies the following four conditions:*

(**Constants**) *If* $(x_i)_{i\in I}$ *is a constant net, that is, for all* $i \in I$, $x_i = x$, *then* $((x_i)_{i\in I}, x) \in \mathcal{L}$.
(**Subnets**) *If* $((x_i)_{i\in I}, x) \in \mathcal{L}$ *and* $(y_j)_{j\in J}$ *is a subnet of* $(x_i)_{i\in I}$, *then* $((y_j)_{j\in J}), x) \in \mathcal{L}$.
(**Divergences**) *If* $((x_i)_{i\in I}, x) \notin \mathcal{L}$, *then there exists a subnet* $(y_j)_{j\in J}$, *which has no subnet* $(z_k)_{k\in K}$ *so that* $((z_k)_{k\in K}), x) \in \mathcal{L}$.
(**Iterated limits**) *If* $((x_i)_{i\in I}, x) \in \mathcal{L}$, $((x_{i,j})_{j\in J(i)}, x_i) \in \mathcal{L}$ *for all* $i \in I$, *then* $((x_{i,f(i)})_{(i,f)\in I\times M}, x) \in \mathcal{L}$, *where* $M = \prod_{i\in I} J(i)$. *The order of* M *is defined as follows,* $f_1 \leq f_2$ *if and only if* $f_1(i) \leq f_2(i)$ *for all* $i \in I$, *the order of* $I \times M$ *is defined as follows,* $(a, f_1) \leq (b, f_2)$ *if and only if* $f_1 \leq f_2$ *and* $a \leq b$.

3 Δ-space

Recall that a topological space $(X, \mathcal{O}(X))$ is called a *weak monotone convergence space* if and only if $\mathcal{O}(X) \subseteq \sigma(X)$. And a space X is called a *monotone determined space* if a subset U of X is open if and only if for any directed subset D of X, $cl(D) \cap U \neq \emptyset$ implies $D \cap U \neq \emptyset$.

Definition 1 *[13] A space* X *is called a* Δ*-space if it is both a weak monotone convergence space and a monotone determined space.*

Example 1 *Examples of* Δ*-spaces:*

(1) Any poset with Scott topology(Posets endowed with the Scott topology);

(2) Sober C-spaces;

(3) Sober locally finitary compact space.

Definition 2 *[13] Let* X *be a* Δ*-space, and* $x, y \in X$. *We say that* x *approximates* y, *in symbols* $x \prec y$, *if for any directed* $D \subseteq X$, $y \in cl(D)$ *implies* $x \in {\downarrow}D$. *We write* ${\Downarrow}x = \{a \in X : a \prec x\}$, ${\Uparrow}x = \{a \in X : x \prec a\}$.

Proposition 1 *[13] Let* X *be a* Δ*-space. For any* $a, b, c, d \in X$, *the following statements hold:*

(1) $a \prec b$ *implies* $a \leq b$.

(2) $a \leq b \prec c \leq d$ *implies* $a \prec d$.

Definition 3 *[13] A* Δ*-space* X *is said to be continuous if* ${\Downarrow}x$ *is directed and* $x = \bigvee {\Downarrow}x$ *for all* $x \in X$.

Proposition 2 *A* Δ*-space* X *is continuous if and only if there exists a directed subset* D *of* ${\Downarrow}x$ *such that* $\vee D = x$ *for all* $x \in X$.

Proof *The necessity is obvious. Conversely, let $x \in X$ and suppose that there exists a directed subset D of $\Downarrow x$ such that $\bigvee D = x$ for all $x \in X$. Suppose $x_1, x_2 \in \Downarrow x$. Since $x = \vee D \in cl(D)$, we have that $x_1, x_2 \in \downarrow D$, i.e., there exist $d_1, d_2 \in D$ such that $x_1 \leq d_1$ and $x_2 \leq d_2$. Thus, $x_1 \leq d_1 \leq d$ and $x_2 \leq d_2 \leq d$ for some $d \in D$ by the directness of D. Hence, the set $\Downarrow x$ is directed. Besides, $\bigvee \Downarrow x = x$ by the assumption $\bigvee D = x$ and $D \subseteq \Downarrow x \subseteq \downarrow x$. Therefore, X is continuous.*

Proposition 3 *[13] If Δ-space X is continuous, then $\Uparrow x$ is open and $U = \bigcup\{\Uparrow u : u \in U\}$ for all $x \in X$ and $U \in \mathcal{O}(X)$.*

Definition 4 *[13] A Δ-space X is $meet-continuous$ if for any $x \in X$ and directed set $D \subseteq X$, $x \in cl(D)$ implies $x \in cl(\downarrow x \cap \downarrow D)$.*

Definition 5 *[13] A Δ-space X is called quasicontinuous if for all $x \in X$ and $U \in \mathcal{O}(X)$, $x \in U$ implies that there is a finite subset $F \subseteq X$ such that $x \in int(\uparrow F) \subseteq \uparrow F \subseteq U$.*

Theorem 3.1 *[13] A Δ-space X is continuous if and only if X is quasicontinuous and meet-continuous.*

Next we will characterize the continuity of Δ-space by Galois connections.

Definition 6 *Let X be a Δ-space. A filter $\mathcal{F}$ in X is said to converge to $x \in X$ denoted by $\mathcal{F} \longrightarrow x$ if there exists a non-empty directed subset D of X such that*

(1) $x \in cl(D)$;

(2) for each $d \in D$, $\uparrow d \in \mathcal{F}$.

Let $\theta(X)$ denote the family of all lower set of X. Then $\theta(X)$ is a completely distributive lattice under set inclusion. Define $\alpha : \theta(X) \longrightarrow \theta(X)$ by $\alpha(A) = \{y \in X : \exists\ a\ proper\ filter\ \mathcal{F} \longrightarrow y\ and\ A \in \mathcal{F}\}$ for each $A \in \theta(X)$, and $\beta : \theta(X) \longrightarrow \theta(X)$ by $\beta(B) = \Downarrow B = \bigcup\{\Downarrow b : b \in B\}$ for each $B \in \theta(X)$. Then both α and β are order-preserving.

Lemma 3.1 *Let X be a continuous Δ-space and $A \in \theta(X)$.*
Then $cl(A) = \{y \in X : \exists\ a\ proper\ filter\ \mathcal{F} \longrightarrow y\ and\ A \in \mathcal{F}\}$

Proof *Obviously.*

Theorem 3.2 *Let X be a Δ-space, the following statements are equivalent:*

(1) X is continuous.

(2) α and β form an adjunction,i.e, $\beta(\alpha(A)) \subseteq A \subseteq \alpha(\beta(A))$ for all $A \subseteq X$.

Proof *(1) ⇒ (2): Let* $x \in \beta(\alpha(A))$*, then there exists* $y \in \alpha(A)$ *such that* $x \prec y$*. Thus* $y \in cl(A)$ *and* $y \in \Uparrow x$*. Obviously,* $\Uparrow x$ *is open in* X *by the continuity of* X*. It follows that* $\Uparrow x \bigcap A \neq \emptyset$*. Since* A *is a lower set, we have that* $x \in A$*.*

Let $x \in A$*. We need to show that* $x \in \alpha(\beta(A)) = cl(\Downarrow A)$*. In fact, since* X *is continuous, we have that* $x = \bigvee \Downarrow x$*. Hence,* $x \in cl(\Downarrow x) \subseteq \bigcup_{a \in A} cl(\Downarrow a) = cl(\Downarrow A)$*.*

(2) ⇒ (1): First, we claim that $\Downarrow x$ *is directed for all* $x \in X$*. In fact, let* F *be a finite subset of* $\Downarrow x$*, we want to show that* $F^u \bigcap \Downarrow x \neq \emptyset$*. By (2),* $x \in \downarrow x \subseteq \alpha(\beta(\downarrow x)) = \alpha(\Downarrow x)$*, which implies that there exists a proper filter* $\mathcal{F}$ *such that* $\mathcal{F} \longrightarrow x$ *and* $\Downarrow x \in \mathcal{F}$*. Hence, there exists a directed subset* D *of* X *such that* $x \in cl(D)$ *and* $\uparrow d \in \mathcal{F}$ *for all* $d \in D$*. We conclude that* $F \subseteq \downarrow D$*,i.e,* $\uparrow a \in \mathcal{F}$ *for all* $a \in F$*. And there exists* $F_a \in \mathcal{F}$ *such that* $a \in F_a^l$*, which implies* $F \subseteq (\cap F_a)^l$*. Let* $E = \cap F_a$*, it is obvious that* $E \in \mathcal{F}$ *and* $E \subseteq F^u$*. Thus* $F^u \in \mathcal{F}$ *and* $F^u \bigcap \Downarrow x \in \mathcal{F}$*. Since* $\mathcal{F}$ *is proper,* $F^u \bigcap \Downarrow x \neq \emptyset$*. Second, it is obvious that* $\downarrow x \subseteq \alpha(\Downarrow x) \subseteq (\Downarrow x)^{ul} \subseteq (\downarrow x)^{ul} = \downarrow x$*, thus* $x = \bigvee \Downarrow x$*. Therefore,* X *is continuous.*

4 Δ-convergence in Δ-spaces

In this section, we introduce and study the Δ-convergence in Δ-spaces. It is proved that a Δ-space is continuous if and only if the Δ-convergence is topological.

Definition 7 *Let* X *be a* Δ*-space. A net* $(x_i)_{i \in I}$ *in* X *is said to* Δ*-converge to* $x \in X$ *if there exists a non-empty directed subset* D *of* X *such that*

(1) $x \in cl(D)$*;*

(2) for each $d \in D$*,* $x_i \in \uparrow d$ *eventually.*

In this case, we write $(x_i)_{i \in I} \xrightarrow{\Delta} x$.

Remark 4.1 *Let* X *be a* Δ*-space. Then*

(1) The constant net $(x_i)_{i \in I}$ *in* X *with value* x Δ*-converges to* x*.*

(2) For any net $(x_i)_{i \in I}$ *in* X*. If* $(x_i)_{i \in I} \xrightarrow{\Delta} x$*, then* $(x_i)_{i \in I} \xrightarrow{\Delta} y$ *for every* $y \leq x$*.*

(3) For any $U \in \mathcal{O}(X)$*,* $U = \uparrow U$*.*

Definition 8 *Let* X *be a* Δ*-space. We consider the following family of the subset of* X*.*

$\tau_\Delta = \{U \subseteq X :$ *whenever* $(x_i)_{i \in I} \xrightarrow{\Delta} x$ *and* $x \in U$*, then* $x_i \in U$ *eventually*$\}$*. It is easy to prove that* τ_Δ *is a topology. It is called the* Δ*-topology on* X*. Each* $U \in \tau_\Delta$ *is called* Δ*-open set. Complements of* Δ*-open sets are called* Δ*-closed sets.*

Proposition 4 *Let* X *be a* Δ*-space and* $A \subseteq X$*. Then* A *is a* Δ*-closed set if and only if for any net* $(x_i)_{i \in I}$ *in* A*, if* $(x_i)_{i \in I} \xrightarrow{\Delta} x$*, then* $x \in A$*.*

Proof (⇒): *Assume* $(x_i)_{i\in I} \subseteq A$ *and* $(x_i)_{i\in I} \xrightarrow{\Delta} x$. *Suppose that* $x \in X \setminus A$. *Since* A *is* Δ*-closed, we have that* $X\setminus A$ *is* Δ*-open. Thus* $(x_i)_{i\in I} \in X\setminus A$ *eventually, a contradiction.*

(⇐):*Suppose not,* A *is not* Δ*-closed. Then* $X \setminus A$ *is not* Δ*-open. Thus there exists* $x \in X \setminus A$ *and a net* $(x_i)_{i\in I}$ *such that* $(x_i)_{i\in I} \xrightarrow{\Delta} x$, *but* $(x_i)_{i\in I}$ *is not eventually in* $X \setminus A$. *It follows that* $(x_i)_{i\in I} \in A$ *usually.*

Proposition 5 *Let* X *be a* Δ*-space and* $U \subseteq X$. *Then* $U \in \tau_\Delta$ *if and only if for any directed subset* D *of* X, $cl(D) \cap U \neq \emptyset$ *implies* $D \cap U \neq \emptyset$.

Proof (⇒): *Assume that* $U \in \tau_\Delta$. *Let* D *be a directed subset of* X *and* $cl(D) \cap U \neq \emptyset$. *Then there exists* $x \in U$ *such that* $x \in cl(D)$. *Obviously,* $(d)_{d\in D} \xrightarrow{\Delta} x$. *Hence,* $(d)_{d\in D} \in U$ *eventually. It follows that* $D \cap U \neq \emptyset$.

(⇐): *Assume that for any directed subset* D *of* X, $cl(D)\cap U \neq \emptyset$ *implies* $D\cap U \neq \emptyset$. *Let* $(x_i)_{i\in I} \xrightarrow{\Delta} x$ *and* $x \in U$. *By the definition of* Δ*-convergence, there exists a directed subset* D_0 *of* X *such that* $x \in cl(D_0)$ *and for each* $d \in D_0$, $d \leq x_i$ *holds eventually. From the assumption, we can see easily that* U *is an upper set. And* $x \in cl(D_0)\cap U \neq \emptyset$. *Hence, there exists* $d_0 \in D_0$ *such that* $\uparrow d_0 \subseteq U$. *It is tantamount to* $x_i \in \uparrow d_0 \subseteq U$ *eventually. Thus* $U \in \tau_\Delta$.

Remark 4.2 *Let* X *be a* Δ*-space. Then* $\mathcal{O}(X) \subseteq \tau_\Delta$.

Proposition 6 *Let* X *be a* Δ*-space. Then for* $x, y \in X$, $x \prec y$ *if and only if for every net* $(x_i)_{i\in I}$ *in* X, $(x_i)_{i\in I} \xrightarrow{\Delta} y$ *implies* $x_i \in \uparrow x$ *eventually.*

Proof (⇒): *Assume that* $x \prec y$. *Let the net* $(x_i)_{i\in I} \xrightarrow{\Delta} y$. *By the definition of* Δ*-convergence, there exists a directed subset* D *of* X *such that* $y \in cl(D)$ *and for all* $d \in D$, $x_i \in \uparrow d$ *eventually. Since* $x \prec y$, *we have that* $x \leq d_0$ *for some* $d_0 \in D$. *Moreover,* $x_i \in \uparrow d_0 \subseteq \uparrow x$ *eventually.*

(⇐): *Let* D_0 *be a directed subset of* X *with* $y \in cl(D_0)$. *Obviously,* $(d)_{d\in D_0} \xrightarrow{\Delta} y$. *By assumption,* $(d)_{d\in D_0} \in \uparrow x$ *eventually. It follows that* $x \in \downarrow D$. *Hence,* $x \prec y$.

Lemma 4.1 *If the* Δ*-space* X *is continuous. Then* $(x_i)_{i\in I} \xrightarrow{\Delta} x$ *if and only if* $(x_i)_{i\in I} \xrightarrow{\mathcal{O}(X)} x$.

Proof (⇒): *Suppose that* $(x_i)_{i\in I} \xrightarrow{\Delta} x$ *and* $x \in U \in \mathcal{O}(X)$. *By the definition of* Δ*-convergence, there exists a directed set* D *of* X *such that* $x \in cl(D)$ *and for all* $d \in D$, $x_i \in \uparrow d$ *eventually. Since* $x \in cl(D) \cap U \neq \emptyset$, *we have that* $D \cap U \neq \emptyset$, *i.e, there exists* $d_0 \in D \cap U$ *such that* $x_i \in \uparrow d_0 \subseteq U$ *eventually. Thus* $(x_i)_{i\in I} \xrightarrow{\mathcal{O}(X)} x$.

(⇐): *Suppose that* $(x_i)_{i\in I} \xrightarrow{\mathcal{O}(X)} x$. *Since* X *is continuous,* $\Downarrow x$ *is directed and* $x = \bigvee \Downarrow x$. *It follows that* $x \in cl(\Downarrow x)$. *Let* $a \in \Downarrow x$. *By Proposition 3.7,* $x \in \Uparrow a \in \mathcal{O}(X)$. *Hence, we have that* $x_i \in \Uparrow a \subseteq \uparrow a$ *eventually and* $(x_i)_{i\in I} \xrightarrow{\Delta} x$.

Lemma 4.2 *If the Δ-space X is continuous. Then $(x_i)_{i\in I} \xrightarrow{\Delta} x$ if and only if $(x_i)_{i\in I} \xrightarrow{\tau_\Delta} x$.*

Proof *From the proof of Lemma 4.8 and Remark 4.6, it is easy to show that.*

Corollary 4.3 *If the Δ-space X is continuous. Then the Δ-convergence is topological in X. In particular, $\mathcal{O}(X) = \tau_\Delta$.*

Lemma 4.3 *Let X be a Δ-space. If the Δ-convergence is topological in X. Then X is continuous.*

Proof *Since the Δ-convergence is topological. We have that there exists a topology τ such that $(x_i)_{i\in I} \xrightarrow{\Delta} x \Leftrightarrow (x_i)_{i\in I} \xrightarrow{\tau} x$. Let $x \in X$. Set $I = \{(U,a) \in \mathcal{N}(x) \times X : a \in U\}$, where $\mathcal{N}(x) = \{U \in \tau : x \in U\}$. Define an order on I as follows:*

$$\forall (U_1,a_1),(U_2,a_2) \in I,\ (U_1,a_1) \le (U_2,a_2) \text{ if and only if } U_1 \supseteq U_2.$$

Then $(I,\le)$ is a preordered set. Obviously, I is directed. Let $x_i = a$ for any $i = (U,a) \in I$. Then it is easy to see that $(x_{(U,a)})_{(U,a)\in I} \xrightarrow{\tau} x$. Thus $(x_{(U,a)})_{(U,a)\in I} \xrightarrow{\Delta} x$. By the definition of Δ-convergence, we conclude that there exists a directed subset D of X such that $x \in cl(D)$ and for any $d \in D$, $x_{(U,a)} \in \uparrow d$ eventually. In particular, for any $d \in D$, there exists $W_d \in \tau$ such that $x \in W_d \subseteq \uparrow d$.

We claim that $D \subseteq \Downarrow x$.

Assume that $a \in D$. We need to prove $a \prec x$. In fact, for any net $(x_i)_{i\in I}$ with $(x_i)_{i\in I} \xrightarrow{\Delta} x$, we know that there exists $W_a \in \tau$ such that $x \in W_a \subseteq \uparrow a$. Thus $x_i \in W_a \subseteq \uparrow a$ eventually. By Proposition 4.7, $a \prec x$. Moreover, $\bigvee D = x$.

By the Proposition 3.6, X is continuous.

Theorem 4.4 *Let X be a Δ-space. Then X is continuous if and only if the Δ-convergence is topological in X.*

Proof *It follows from Corollary 4.10 and Lemma 4.11.*

5 Δ_L-convergence in Δ-spaces

In this section, the concept of Δ_L-convergence in Δ-spaces is introduced. It is proved that a Δ-space X is continuous if and only if the Δ_L-convergence is topological, $\mathcal{O}(X) \bigvee \omega(X) = \tau_{\Delta_L}$, and X is meet-continuous. Moreover, we give a characterization for the Δ_L-convergence being topological.

Definition 9 *Let X be a Δ-space. A net $(x_i)_{i\in I}$ in X is said to Δ_L-converge to $x \in X$ if there exists a non-empty directed subset D of X such that*

(1) $\vee D$ exists and $x = \vee D$;

(2) for each $d \in D$, $x_i \in \uparrow d$ eventually;

(3) for each $a \in X$, if $x_i \in \uparrow a$ usually, then $x \in \uparrow a$.

In this case, we write $(x_i)_{i\in I} \xrightarrow{\Delta_L} x$.

Definition 10 *Let X be a Δ-space. We consider the family of subsets of X below.*

$\tau_{\Delta_L} = \{U \subseteq X :$ *whenever* $(x_i)_{i\in I} \xrightarrow{\Delta_L} x$ *and* $x \in U$, *then* $x_i \in U$ *eventually*$\}$. *Obviously, it is a topology. It is called the Δ_L-topology on X. Each $U \in \tau_{\Delta_L}$ is called Δ_L-open set. Complements of Δ_L-open sets are called Δ_L-closed sets.*

Proposition 7 *Let X be a Δ-space. Then $\mathcal{O}(X) \subseteq \tau_{\Delta_L}$ and $\omega(X) \subseteq \tau_{\Delta_L}$.*

Proof *First, let $U \in \mathcal{O}(X)$ and $(x_i)_{i\in I} \xrightarrow{\Delta_L} x \in U$. By the definition of the Δ_L-convergence, there exists a directed subset D of X such that*

(1) $\vee D$ exists and $x = \vee D$;

(2) for each $d \in D$, $x_i \in \uparrow d$ eventually;

(3) for each $a \in X$, if $x_i \in \uparrow a$ usually, then $x \in \uparrow a$.

Since $x \in cl(D) \cap U \neq \emptyset$, we have that $D \cap U \neq \emptyset$, i.e., there exists $a \in D \cap U$ such that $x_i \in \uparrow a \subseteq U$ eventually. Thus $\mathcal{O}(X) \subseteq \tau_{\Delta_L}$.

Second, let $x \in X$. Suppose that $(x_i)_{i\in I}$ is a net and it Δ_L-converges to an element $y \in X \setminus \uparrow x$. Then $(x_i)_{i\in I}$ is not usually in $\uparrow x$; otherwise, $y \in \uparrow x$. So we conclude that the net $(x_i)_{i\in I} \in X \setminus \uparrow x$ eventually. Therefore, $X \setminus \uparrow x \in \tau_{\Delta_L}$ and $\omega(X) \subseteq \tau_{\Delta_L}$.

Theorem 5.1 *Let X be a Δ-space. If X is continuous, then $(x_i)_{i\in I} \xrightarrow{\Delta_L} x$ if and only if $(x_i)_{i\in I} \xrightarrow{\tau_{\Delta_L}} x$.*

Proof *The necessity is obvious. Conversely, let $x \in X$ and suppose that the net $(x_i)_{i\in I} \xrightarrow{\tau_{\Delta_L}} x$. Since X is continuous, we have that $\Downarrow x$ is directed and $x = \bigvee \Downarrow x$. Let $a \in \Downarrow x$. It follows that $x \in \Uparrow a \in \mathcal{O}(X) \subseteq \tau_{\Delta_L}$ by Proposition 5.3. Thus $x_i \in \Uparrow a \subseteq \uparrow a$ eventually. Let $b \in X$, $x_i \in \uparrow b$ usually. Suppose that $x \notin \uparrow b$ i.e., $x \in X \setminus \uparrow b \in \omega(X)$. We conclude that $x_i \in X \setminus \uparrow b$ eventually by Proposition 5.3, which is a contradiction. Hence, $x \in \uparrow b$. In a word, $(x_i)_{i\in I} \xrightarrow{\Delta_L} x$.*

Theorem 5.2 *Let X be a Δ-space. If X is continuous, then $(x_i)_{i\in I} \xrightarrow{\Delta_L} x$ if and only if $(x_i)_{i\in I} \xrightarrow{\mathcal{O}(X)\bigvee\omega(X))} x$.*

Proof *It can be proved by Theorem 5.4 and Proposition 5.3.*

Immediately, we obtain the following conclusion.

Corollary 5.3 *If the Δ-space X is continuous. Then the Δ_L-convergence is topological in X. In particular, $\mathcal{O}(X)\bigvee\omega(X)=\tau_{\Delta_L}$.*

From the above corollary, we know that if the Δ-space X is continuous, then the Δ_L-convergence is topological in X and $\mathcal{O}(X)\bigvee\omega(X)=\tau_{\Delta_L}$. The following example shows that the converse does not hold.

Example 2 *Let $L=\mathbb{N}\cup\{a,\omega_1,\omega_2\}$ and $X=(L,\sigma_2(L))$. (See Figure 1). The partial order $\leqslant$ on L is defined as follows:*

- $n\leqslant\omega_1$ *and* $n\leqslant\omega_2$ *for all* $n\in\mathbb{N}$
- $0\leqslant 1\leqslant 2...\leqslant n\leqslant ...$
- $a\leqslant\omega_1$, $a\leqslant\omega_2$
- $x\leqslant x$ *for all* $x\in L$

(1) X is a Δ-space;

(2) X is not continuous.

Claim 1: $cl(\mathbb{N})=\mathbb{N}\cup\{a\}$. In fact, $\omega_1,\omega_2\notin cl(\mathbb{N})$, because $\{\omega_1,\omega_2\}\in\sigma_2(L)$ and $\{\omega_1,\omega_2\}\bigcap\mathbb{N}=\emptyset$. Hence, $cl(\mathbb{N})\subseteq\mathbb{N}\cup\{a\}$. Conversely, we only to prove that $a\in cl(\mathbb{N})$. Let $U\in\sigma_2(L)$ with $a\in U$. Since $\mathbb{N}^{ul}=\mathbb{N}\cup\{a\}$, we have that $\mathbb{N}\bigcap U\neq\emptyset$. Thus $a\in cl(\mathbb{N})$.

Claim 2: $a\not\ll a$. Indeed, $a\in cl(\mathbb{N})=\mathbb{N}\cup\{a\}$, but $a\notin\downarrow\mathbb{N}$.

So $\Downarrow a=\emptyset$ and $a\neq\bigvee\Downarrow a$. Thus X is not continuous.

(3) The Δ_L-convergence is topological in X.

Claim 1: $(x_i)_{i\in I}\xrightarrow{\Delta_L}x$ if and only if $x_i\in\{x\}$ eventually. In fact, suppose that $x_i\in\{x\}$ eventually. Let $D=\{x\}$. We have that $x=\vee D$, $x_i\in\uparrow d$ eventually for all $d\in D$ and for each $y\in X, x_i\in\uparrow y$ usually implies $x\in\uparrow y$. By the definition of Δ_L-convergence, $(x_i)_{i\in I}\xrightarrow{\Delta_L}x$. Conversely, assume that $(x_i)_{i\in I}\xrightarrow{\Delta_L}x$. Then there exists a directed subset $D\subseteq X$ such that $x=\vee D$, $x_i\in\uparrow d$ eventually for all $d\in D$ and for each $y\in X, x_i\in\uparrow y$ usually, so $x\in\uparrow y$. Next we discuss the following situations. Suppose $x\in\mathbb{N}$. Since $x+1\not\leq x$, we have that x_i is not usually in $\uparrow\{x+1\}$ and thus $x_i\in X\setminus\uparrow\{x+1\}$ eventually. From $x=\vee D$, $max(D)$ exists, we have that $x\in D$. Thus $x_i\in\uparrow x$ eventually. Hence we can conclude that $x_i\in\uparrow x\bigcap X\setminus\uparrow\{x+1\}=\{x\}$ eventually. If $x=a$, then $x_i\in\uparrow a$ eventually. Since $\omega_1\not\leq a$ and $\omega_2\not\leq a$, we have that $x_i\in X\setminus\uparrow\omega_1\bigcap X\setminus\uparrow\omega_2$ eventually. Thus $x_i\in\uparrow a\bigcap X\setminus\uparrow\omega_1\bigcap X\setminus\uparrow\omega_2=\{x\}$ eventually. If $x\in\{\omega_1,\omega_2\}$, without loss of generality, suppose $x=\omega_1$, then $x_i\in\uparrow\omega_1$ eventually, that is $x_i\in\{x\}$ eventually.

It is easy to prove the following results. $\tau_{\Delta_L}=\mathbb{P}(X)$ and $(x_i)_{i\in I}\xrightarrow{\tau_{\Delta_L}}x$ if and only if $x_i\in\{x\}$ eventually.

From the claim 1 and the results above, we have that $(x_i)_{i\in I}\xrightarrow{\tau_{\Delta_L}}x$ if and only if $(x_i)_{i\in I}\xrightarrow{\Delta_L}x$, and thus the Δ_L-convergence is topological in X.

(4) $\mathbb{P}(X)=\sigma_2(X)\bigvee\omega(X)$. Let $x\in X$. If $x\in\{\omega_1,\omega_2\}$, then $\{x\}\in\sigma_2(X)$. Suppose $x\in\mathbb{N}$. Since $\uparrow n\in\sigma_2(X)$ and $X\setminus\uparrow\{n+1\}\in\omega(X)$, we have that $\{n\}=\uparrow n\setminus\uparrow\{n+1\}=\uparrow n\bigcap X\setminus\uparrow\{n+1\}\in\sigma_2(X)\bigvee\omega(X)$. If $x=a$, then $\{a\}=X\setminus\uparrow 0\in\omega(X)$. Hence, $\mathbb{P}(X)\subseteq\sigma_2(X)\bigvee\omega(X)$. So by the Proposition 5.3, $\mathbb{P}(X)=\sigma_2(X)\bigvee\omega(X)$.

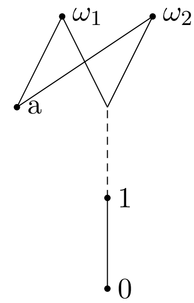

Figure 1: a Δ-space in which Δ_L-convergence is topological but not continuous

Definition 11 *Let X be a Δ-space. For $x, y \in X$, define $x \prec_{\Delta_L} y$ if for every net $(x_i)_{i\in I}$ in X which Δ_L-converges to y, $x_i \in \uparrow x$ eventually. We write $\Downarrow_{\Delta_L} x = \{a \in X : a \prec_{\Delta_L} x\}$, $\Uparrow_{\Delta_L} x = \{a \in X : x \prec_{\Delta_L} a\}$.*

Proposition 8 *Let X be a Δ-space. For all $x, y \in X$, $x \prec_{\Delta_L} y$ if and only if for any directed subset D of X, $cl(\{y\}) = cl(D)$ implies $x \in \downarrow D$.*

Proof *Suppose that $x \prec_{\Delta_L} y$. Let D be a directed subset of X and $cl(\{y\}) = cl(D)$. Clearly, $(d)_{d\in D} \xrightarrow{\Delta_L} y$. Thus $x \in \downarrow D$. Conversely, suppose that for any directed subset D of X, $cl(\{y\}) = cl(D)$ implies $x \in \downarrow D$. Let the net $(x_i)_{i\in I} \xrightarrow{\Delta_L} y$. Then there exists a directed subset D of X such that*

(1) $\vee D$ exists and $y = \vee D$;

(2) for each $d \in D$, $x_i \in \uparrow d$ eventually;

(3) for each $a \in X$, if $x_i \in \uparrow a$ usually, then $y \in \uparrow a$.

Since $y = \vee D$, we have that $cl(\{y\}) = cl(D)$. By assumption, $x \in \downarrow D$, which implies $x_i \in \uparrow d_0 \subseteq \uparrow x$ eventually for some $d_0 \in D$. Thus $x \prec_{\Delta_L} y$.

Definition 12 *A Δ-space X is called Δ_L-continuous if for any $x \in X$, there exists a directed subset D_x of $\Downarrow_{\Delta_L} x$ and $\vee D_x = x$.*

Theorem 5.4 *If the Δ_L-convergence is topological in a Δ-space X. Then X is Δ_L-continuous.*

Proof *Since the Δ_L-convergence is topological, there exists a topology τ such that $(x_i)_{i\in I} \xrightarrow{\Delta_L} x \Leftrightarrow (x_i)_{i\in I} \xrightarrow{\tau} x$. Let $x \in X$. Set $I = \{(U, a) \in \mathcal{N}(x) \times X : a \in U\}$, where $\mathcal{N}(x) = \{U \in \tau : x \in U\}$. Define an order on I as follows:*

$$\forall (U_1, a_1), (U_2, a_2) \in I,\ (U_1, a_1) \leq (U_2, a_2) \text{ if and only if } U_1 \supseteq U_2.$$

Then $(I, \leq)$ is a preordered set. Obviously, I is directed. Let $x_i = a$ for any $i = (U, a) \in I$. Then it is easy to see that $(x_{(U,a)})_{(U,a)\in I} \xrightarrow{\tau} x$. Thus $(x_{(U,a)})_{(U,a)\in I} \xrightarrow{\Delta_L} x$. By the definition of Δ_L-convergence, we can conclude that there exists a directed subset D of X such that

(1) $\vee D$ exists and $x = \vee D$;

(2) for each $d \in D$, $x_{(U,a)} \in \uparrow d$ eventually;

(3) for each $t \in X$, if $x_{(U,a)} \in \uparrow t$ usually, then $x \in \uparrow t$.

In particular, for any $d \in D$, there exists $W_d \in \tau$ such that $x \in W_d \subseteq \uparrow d$.

We claim that $D \subseteq \Downarrow x$. Assume $d_0 \in D$. Then we need to prove $d_0 \prec x$. In fact, for any net $(x_i)_{i \in I}$ with $(x_i)_{i \in I} \xrightarrow{\Delta_L} x$, there exists $W_{d_0} \in \tau$ such that $x \in W_{d_0} \subseteq \uparrow d_0$. Thus $x_i \in W_{d_0} \subseteq \uparrow d_0$ eventually. So $d_0 \prec x$. Moreover, $\bigvee D = x$. Therefore, X is Δ_L-continuous.

From the above theorem, we know that if the Δ_L-convergence is topological in a Δ-space X, then X is Δ_L-continuous. However, the example below reveals that the converse does not hold.

Example 3 *Let $L = (\mathbb{N} \times (\mathbb{N} \cup \{w\})) \cup \{a, \omega_1, \omega_2\} \cup \mathbb{N}$ and $X = (L, \sigma(L))$ (See Figure 2). The order on L is defined as follows:*

- $(n_1, m_1) \leqslant (n_2, m_2)$ *if* $n_1 = n_2$ *and* $m_1 \leqslant m_2$ *for all* $n_1, m_1, n_2, m_2 \in \mathbb{N}$
- $(n_1, m_1) \leqslant (n_2, w)$ *if* $n_1 \geq n_2$ *for all* $n_1, n_2 \in \mathbb{N}$ *and* $m_1 \in \mathbb{N}$
- $a \leqslant (n, \omega)$ *for all* $n \in \mathbb{N}$
- $(n_1, \omega) \leqslant (n_2, \omega)$ *if* $n_1 \geq n_2$ *for all* $n_1, n_2 \in \mathbb{N}$
- $n \leqslant \omega_1$ *and* $n \leqslant \omega_2$ *for all* $n \in \mathbb{N}$
- $0 \leqslant 1 \leqslant 2 ... \leqslant n \leqslant ...$
- $(n, \omega) \leqslant \omega_1$, $(n, \omega) \leqslant \omega_2$ *for all* $n \in \mathbb{N}$
- $x \leqslant x$ *for all* $x \in L$

(1) X is a Δ_L-continuous Δ-space. Indeed, for all $n \in \mathbb{N}$, $\Downarrow_{\Delta_L}(n, w) = \{(n, m) : m \in \mathbb{N}\}$, $\Downarrow_{\Delta_L} a = \{a\}$, and for all $n, m \in \mathbb{N}$, $\Downarrow_{\Delta_L}(n, m) = \downarrow(n, m)$, $\Downarrow_{\Delta_L} n = \downarrow n$, $\Downarrow_{\Delta_L} \omega_1 = \downarrow \omega_1$, $\Downarrow_{\Delta_L} \omega_2 = \downarrow \omega_2$. Obviously, X is Δ_L-continuous.

(2) The Δ_L-convergence is not topological in X. Let $x_n = (n, w)$ for all $n \in \mathbb{N}$. It is easy to show that $(x_n)_{n \in \mathbb{N}} \xrightarrow{\Delta_L} a$ and $\Uparrow_{\Delta_L} a = \{a, \omega_1, \omega_2\} \notin \tau_{\Delta_L}$. By Theorem 5.20 and Corollary 5.21, the Δ_L-convergence is not topological in X.

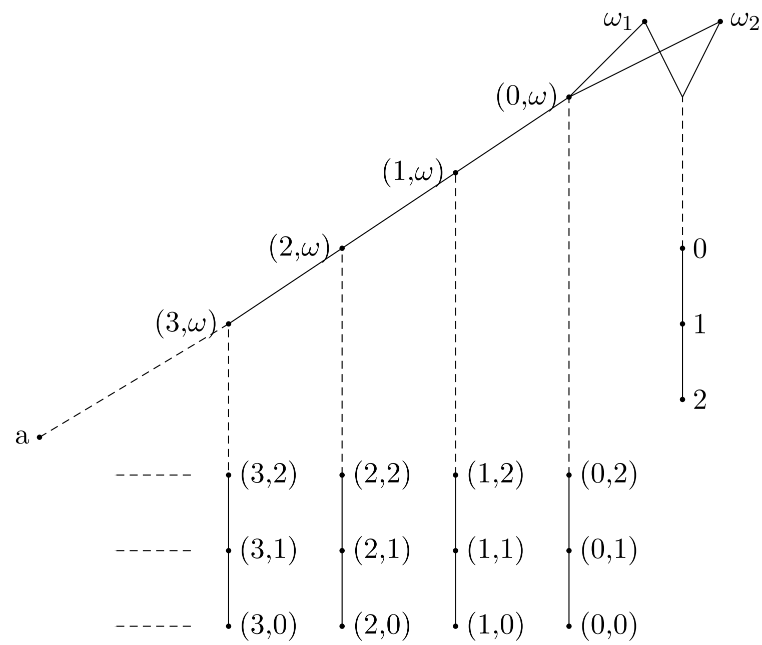

Figure 2: A Δ-space is Δ_L-continuous but the Δ_L-convergence is not topological in it

Theorem 5.5 *Let X be a Δ-space, the following statements are equivalent:*

(1) The Δ_L-convergence is topological, $\mathcal{O}(X)\bigvee\omega(X)=\tau_{\Delta_L}$, and X is meet-continuous.

(2) X is continuous.

Proof *(1) $\Rightarrow$ (2): It suffices to prove that $\Downarrow_{\Delta_L}x=\Downarrow x$ for all $x\in X$. Let $y\in X$, suppose that $y\in\Downarrow_{\Delta_L}x$. Let D be a directed subset of X and $x\in cl(D)$. Since X is meet-continuous, we have that $x\in cl(\downarrow D\cap\downarrow x)$. Define*

$$I=\{(U,a)\in\mathcal{N}(x)\times X: a\in U\cap\downarrow D\cap\downarrow x\},\ \textit{where}\ \mathcal{N}(x)=\{U\in\mathcal{O}(X): x\in U\},$$

and a preorder $\leq$ on I as follows, $\forall(U_1,a_1),(U_2,a_2)\in I$, $(U_1,a_1)\leq(U_2,a_2)$ if and only if $U_1\supseteq U_2$. Let $x_i=a$ for any $i=(U,a)\in I$. Then it is self-evident that $(x_{(U,a)})_{(U,a)\in I}\xrightarrow{\mathcal{O}(X)}x$ and $(x_{(U,a)})_{(U,a)\in I}\subseteq\downarrow x$. Thus $(x_{(U,a)})_{(U,a)\in I}\subseteq X\setminus\uparrow m$ for all $x\in X\setminus\uparrow m$. This implies $(x_{(U,a)})_{(U,a)\in I}\xrightarrow{\omega(X)}x$. So $(x_{(U,a)})_{(U,a)\in I}\xrightarrow{\mathcal{O}(X)\bigvee\omega(X)}x$. Since the Δ_L-convergence is topological and $\mathcal{O}(X)\bigvee\omega(X)=\tau_{\Delta_L}$, we have that $(x_{(U,a)})_{(U,a)\in I}\xrightarrow{\Delta_L}x$. We conclude that $x(U,a)\in\uparrow y$ eventually by $y\prec_{\Delta_L}x$, i.e., there exists $(U_0,a_0)\in I$ such that $x_{(U,a)}\in\uparrow y$ for all $(U,a)\geq(U_0,a_0)$. In particular, we have $(U_0,a)\geq(U_0,a_0)$ for all $a\in U_0\cap\downarrow D\cap\downarrow x$ and then $U_0\cap\downarrow D\cap\downarrow x\subseteq\uparrow y$. Hence, $y\in\downarrow D$. It follows that $y\in\Downarrow x$. Conversely, suppose that $y\in\Downarrow x$. Let the net $(x_i)_{i\in I}$ in X which Δ_L-converges to x. Then there exists a directed subset D of X such that

(1) $\vee D$ exists and $x=\vee D$;

(2) for each $d \in D$, $x_i \in \uparrow d$ eventually;

(3) for each $a \in X$, if $x_i \in \uparrow a$ usually, then $x \in \uparrow a$.

Since $x = \vee D \in cl(D)$ and $y \in \Downarrow x$, we have that $y \in \downarrow D$, which implies that there exists $d_0 \in D$ such that $x_i \in \uparrow d_0 \subseteq \uparrow y$ eventually. Thus $y \in \Downarrow_{\Delta_L} x$. X is Δ_L-continuous by Theorem 5.11, we conclude that X is continuous.

(2) $\Rightarrow$(1): It can be proved by Theorem 3.10 and Corollary 5.6.

Definition 13 *Let X be a Δ-space. For $x, y \in X$, define $x \prec_\Delta y$ if for every net $(x_i)_{i\in I}$ in X which Δ_L-converges to y, $x_i \in \Uparrow_{\Delta_L} x$ eventually. We write $\Downarrow_\Delta x = \{a \in X : a \prec_\Delta x\}$, $\Uparrow_\Delta x = \{a \in X : x \prec_\Delta a\}$.*

The following example illustrates that $\prec_\Delta$ and $\prec$ are different:

Example 4 *Let $L = (\mathbb{N} \cup \{\omega\} \times \mathbb{N} \cup \{\omega\})$ and $X = (L, \sigma(L))$. (See Figure 3). The order on L is defined by the following rules:*

- *$(n_1, m_1) \leqslant (n_2, m_2)$ if $n_1 = n_2$ and $m_1 \leqslant m_2$ or $m_2 = \omega$ for all $n_1, n_2 \in \mathbb{N}, m_1, m_2 \in \mathbb{N} \cup \{\omega\}$*
- *$(\omega, n_1) \leqslant (\omega, n_2)$ if $n_1 \leqslant n_2$ or $n_2 = \omega$ for all $n_1, n_2 \in \mathbb{N} \cup \{\omega\}$*
- *$(\omega, m) \leqslant (n, \omega)$ if $m \leq n$ for all $m, n \in \mathbb{N}$*

(1) We claim that $\prec \not\subseteq \prec_\Delta$. It is easy to see that $(\omega, 0) \prec (\omega, \omega)$, but $(\omega, 0) \not\prec_\Delta (\omega, \omega)$. In fact, let $x_n = (n, \omega)$ for all $n \in \mathbb{N}$, then $(x_n)_{n\in\mathbb{N}} \xrightarrow{\Delta_L} (\omega, \omega)$ and $\Uparrow_{\Delta_L}(\omega, 0) = \{(\omega, n) : n \in \mathbb{N} \cup \{\omega\}\}$. However, $(x_n)_{n\in\mathbb{N}}$ is not in $\Uparrow_{\Delta_L}(\omega, 0)$ eventually. Thus we have that $(\omega, 0) \not\prec_\Delta (\omega, \omega)$.

(2) We need to prove that $\prec_\Delta \not\subseteq \prec$. First, we claim that $(\omega, 0) \prec_\Delta (\omega, 1)$. Indeed, for any net $(x_i)_{i\in I}$ with $(x_i)_{i\in I} \xrightarrow{\Delta_L} (\omega, 1)$, since $\mathcal{O}(X) \bigvee \omega(X) \subseteq \tau_{\Delta_L}$, we have that $X \setminus \uparrow (\omega, 2) \in \tau_{\Delta_L}$ and $(\omega, 1) \in X \setminus \uparrow (\omega, 2)$. Thus $x_i \in X \setminus \uparrow (\omega, 2)$ eventually. Since $(\omega, 1) \prec_{\Delta_L} (\omega, 1)$, we have that $x_i \in \uparrow (\omega, 1)$ eventually. By the fact that $(1, \omega) \nleq (\omega, 1)$, we can conclude that $x_i \in X \setminus \uparrow (1, \omega)$ eventually. Hence, $x_i \in (X \setminus \uparrow (\omega, 2)) \bigcap (X \setminus \uparrow (1, \omega)) \bigcap \uparrow (\omega, 1) = \{(\omega, 1)\} \subseteq \Uparrow_{\Delta_L}(\omega, 0)$ eventually. Let $D = \{(1, n) : n \in \mathbb{N}\}$. It is easy to see that $(\omega, 1) \in cl(D)$, but $(\omega, 0) \notin \downarrow D$. Thus $(\omega, 0) \not\prec (\omega, 1)$.

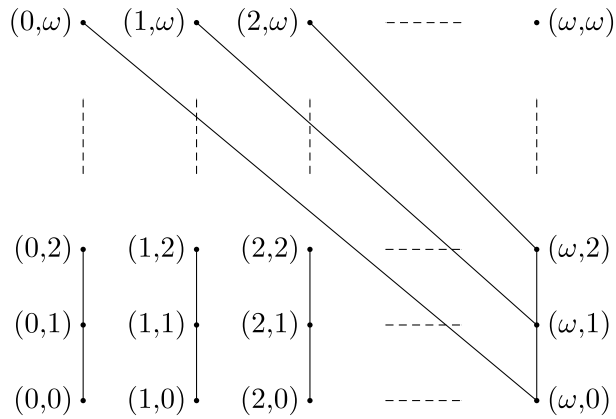

Figure 3

Proposition 9 *Let X be a Δ-space. Then the following statements hold for all $x, y, z \in X$.*

(1) $x \prec_\Delta y$ implies $x \prec_{\Delta_L} y$;

(2) $z \leq x \prec_\Delta y$ implies $z \prec_\Delta y$.

Proof *It is obvious.*

Definition 14 *We call a Δ-space X weak continuous if $\Downarrow_\Delta x$ is directed and $\bigvee \Downarrow_\Delta x = x$ for all $x \in X$.*

Lemma 5.1 *If the Δ-space X is weak continuous. Then X is Δ_L-continuous.*

Proof *Since $\Downarrow_\Delta x \subseteq \Downarrow_{\Delta_L} x$ and $\Downarrow_\Delta x$ is directed, we have that X is Δ_L-continuous.*

Proposition 10 *A Δ-space X is weak continuous if and only if there exists a directed subset D_x of $\Downarrow_\Delta x$ such that $\bigvee D_x = x$ for all $x \in X$.*

Proof *The necessity is easy to be proved. Conversely, let $x \in X$ and suppose that there exists a directed subset D_x of $\Downarrow_\Delta x$ such that $\bigvee D_x = x$ for all $x \in X$. Let $x_1, x_2 \in \Downarrow_\Delta x$. It is obvious that $(d)_{d \in D} \xrightarrow{\Delta_L} x$, and hence, $(d)_{d \in D} \in \Downarrow_{\Delta_L} x_1 \cap \Downarrow_{\Delta_L} x_2$. It follows that $x_1 \leq d$ and $x_2 \leq d$. Hence, the set $\Downarrow_\Delta x$ is directed. Meanwhile, $\bigvee \Downarrow_\Delta x = x$ since $\bigvee D_x = x$ and $D_x \subseteq \Downarrow_\Delta x \subseteq \downarrow x$. Therefore, X is weak continuous.*

Theorem 5.6 *Let X be a Δ-space. Then X is weak continuous if and only if the Δ_L-convergence is topological in X.*

Proof *Suppose that X is weak continuous.*

Step 1: We claim that $\Uparrow_{\Delta_L} x \in \tau_{\Delta_L}$ for all $x \in X$.

Let $y \in \Uparrow_{\Delta_L} x$. For any net $(x_i)_{i\in I}$ in X with $(x_i)_{i\in I} \xrightarrow{\Delta_L} y$, we need to prove that $x_i \in \Uparrow_{\Delta_L} x$ eventually. Since X is weak continuous, we have that $\Downarrow_{\Delta} y$ is directed and $\bigvee \Downarrow_{\Delta} y = y$. Thus there exists $d \in \Downarrow_{\Delta} y$ such that $x \leq d$ by Proposition 5.9. Hence, $x \leq d \prec_{\Delta} y$. It follows that $\Uparrow_{\Delta_L} d \subseteq \Uparrow_{\Delta_L} x$. So $x_i \in \Uparrow_{\Delta_L} d \subseteq \Uparrow_{\Delta_L} x$ eventually.

Step 2: The Δ_L-convergence is topological in X.

Let $(x_i)_{i\in I}$ be a net in X and $x \in X$. It suffices to prove that $(x_i)_{i\in I} \xrightarrow{\Delta_L} x$ if and only if $(x_i)_{i\in I} \xrightarrow{\tau_{\Delta_L}} x$. Suppose that $(x_i)_{i\in I} \xrightarrow{\tau_{\Delta_L}} x$. Since X is weak continuous, we have that X is Δ_L-continuous by Lemma 5.18. Thus, $y = \bigvee \Downarrow_{\Delta_L} y$. Let $a \in \Downarrow_{\Delta_L} y$, i.e., $y \in \Uparrow_{\Delta_L} a$. From $(x_i)_{i\in I} \xrightarrow{\tau_{\Delta_L}} x$, we conclude that $x_i \in \Uparrow_{\Delta_L} a \subseteq \uparrow a$ eventually. For any $b \in X$, if the net $(x_i)_{i\in I} \in \uparrow b$ usually, then $x \in \uparrow b$ because $X \setminus \uparrow b \in \tau_{\Delta_L}$. Therefore, $(x_i)_{i\in I} \xrightarrow{\Delta_L} x$. Conversely, suppose that $(x_i)_{i\in I} \xrightarrow{\Delta_L} x$, it is obvious that $(x_i)_{i\in I} \xrightarrow{\tau_{\Delta_L}} x$. So the Δ_L-convergence is topological in X.

Conversely, suppose that the Δ_L-convergence is topological in X. It follows that X is Δ_L-continuous from Theorem 5.11. It is enough to prove that $\Downarrow_{\Delta_L} x \subseteq \Downarrow_{\Delta} x$ for any $x \in X$. In fact, assume $y \in \Downarrow_{\Delta_L} x$. Then for any net $(x_i)_{i\in I}$ with $(x_i)_{i\in I} \xrightarrow{\Delta_L} x$, we have that $(d)_{d\in D_i} \xrightarrow{\Delta_L} x_i$ for all $i \in I$. From Lemma 2.1, we have that $(x_{(i,f(i))})_{(i,f)\in I\times M} \xrightarrow{\Delta_L} x$, where $M = \prod_{i\in I} D_{x_i}$ and $x_{(i,f(i))} = f(i)$ for all $(i,f) \in I \times M$. Meanwhile, $(x_{(i,f(i))})_{(i,f)\in I\times M} \in \uparrow y$ eventually. Thus there exists $(i_0, f_0) \in I \times M$ such that $x_{(i,f(i))} \in \uparrow y$ for all $(i,f) \geq (i_0, f_0)$. So all $i \geq i_0$, $y \leq x_{(i,f)} \ll_{\Delta_L} x_i$; hence, $y \ll_{\Delta_L} x_i$ for all $i \geq i_0$. We conclude that $x_i \in \Uparrow_{\Delta_L} y$ eventually and $y \in \Downarrow_{\Delta} x$. Since X is Δ_L-continuous, there exists a directed subset D_x such that $D_x \subseteq \Downarrow_{\Delta_L} x \subseteq \Downarrow_{\Delta} x$ and $\vee D_x = x$. Therefore, X is weak continuous by Proposition 5.19.

Corollary 5.7 *Let X be a Δ-space. Then X is weak continuous if and only if the following statements hold:*

(i) X is Δ_L-continuous.

(ii) $\Uparrow_{\Delta_L} x \in \tau_{\Delta_L}$ for all $x \in X$.

Acknowledgement

This work was supported by the National Natural Science Foundation of China (No. 12231007). Additionally, the authors express their gratitude towards the reviewers for their meticulous feedback.

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