Magic-Resource Generating Power of Quantum Channels

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In the stabilizer formalism of quantum error correction and fault-tolerant quantum computation, stabilizer states, as common eigenstates (with the eigenvalue 1) of operators in maximal Abelian subgroups of the Pauli group, play the role of classical states, while magic (non-stabilizer) states are quantum resources enabling universal quantum computation when injected into stabilizer circuits. The natural issue arises as how to characterize and quantify the ability of quantum channels (including quantum gates) in generating magic resource (non-stabilzerness) from stabilizer states. In this work, we introduce two intuitive quantities characterizing magic-resource generating power of quantum channels and reveal their basic properties. We evaluate these quantifies for several prototypical quantum channels and quantum gates. In particular, we demonstrate that the qubit T-gate (i.e., $\pi/8$-gate) and its qutrit generalization are optimal in generating magic resource in a class of diagonal unitaries. This highlights the significance of the T-gate and its extensions from a resource-theoretic perspective and provides a theoretical support of their wide applications in stabilizer quantum computation.

Keywords: magic- resource, stabilizer formalism, qutrit T-gate

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I. INTRODUCTION

In quantum information theory, various quantum resources have been studied and exploited for the purpose of surpassing classical means of information processing. Some prominent examples are entanglement [1–8], nonlocality [9–15], contextuality [16–22], quantum discord [23–28], optical nonclassicality [29–38], coherence [39–47], etc. All these resources have their origin in quantum superposition principle and noncommutativity for observables and states. There are extensive and intensive studies on characterization, detection, quantification and manipulation of these resources, which are active and important subfields of quantum theory.

For the purpose of quantum information processing, it is desirable to create quantum resources in some controllable and optimal ways. Along this line, entanglement-generating power of quantum channels has been widely explored [48–53]. Within the resource theory of coherence, cohering and decohering power of quantum channels have also been studied in recent years [54–60].

In the stabilizer formalism of quantum computation [61–64], another quantum resource called magic (non-stabilizerness) emerges naturally [65–82]. The celebrated Gottesman-Knill theorem states that quantum algorithms that utilize only stabilizer circuits can be efficiently simulated on a classical computer [8, 61]. Stabilizer circuits employ only gates from the Clifford group (i.e., normalizer of the Pauli group), Pauli measurements, conditioning and classical randomness. In order to promote stabilizer circuits to universal fault-tolerant quantum computation, magic states (i.e., non-stabilizer states) must be injected into the circuits. Consequently, it is desirable to quantify magic resource and generating power of quantum channels in creating magic resource from stabilizer states. Various interesting and significant measures of magic resource have been introduced, such as sum negativity and associated mana, thauma [72, 80, 81], stabilizer rank [74, 78], relative entropy of magic, max-relative entropy and min-relative entropy of magic, robustness of magic [75, 77], etc. Most of them are difficult to compute. The sum negativity and associated quantities are easy to compute, but they rely heavily on the discrete Wigner formalism in odd dimensions, which restricts their applications in general case.

Some previous studies of magic-resource aspects of quantum channels are Refs. [79, 80]. More specifically, a resource theory for characterizing and quantifying magic (non-stabilizerness) of noisy quantum circuits was established in Ref. [80], in which two efficiently computable magic measures for quantum channels, called the mana (related to Wigner negativity) [72], and thauma (max-thauma, min-thauma) [80, 81], were studied. In a framework of quantifying magic of general multi-qubit channels, channel robustness and magic capacity were introduced in Ref. [79]. In this context, some natural questions arise as how to quantify magic-resource generating power of various quantum gates and quantum channels, and how to identify optimal channels for generating magic resource under certain constraints. In particular, given the wide usage of the T-gate (i.e., $\pi/8$-gate) in magic state distillation and fault-tolerant quantum computation [68–73], it is of interest to investigate the magic-resource generating power of the T-gate and to illuminate in which sense the T-gate is optimal in this task.

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In this work, by us of the quantifier of magic resource recently introduced in Ref. [82], which is simple to calculate and possesses direct physical significance, we will quantify magic-resource generating power of quantum channels from both maximal and average perspectives. We further apply these quantifiers to analyze magic-resource generating power of various channels.

The remainder of the work is structured as follows. In Sec. I, we present a crash course of stabilizer formalism in which stabilizer states serve as classical objects, while magic (non-stabilizer) states serve as quantum resources. We present a careful and comprehensive analysis of the basic concept of stabilizer states, which may be of tutorial interest. In Sec. III, by employing the recently introduced quantifier of magic in Ref. [82], we study magic-resource generating power of quantum channels by introducing two intuitive and computable quantities and reveal their basic features. We evaluate the magic-resource generating power of various prototypical quantum channels in Sec. IV. In particular, we elucidate the optimal feature of the $\pi/8$-gate in generating magic resource. Finally, we conclude with a summary and discussion in Sec. V.

II. STABILIZER FORMALISM: STABILIZER STATES VERSUS MAGIC STATES

In this section we present a brief overview of the stabilizer formalism, which arises from quantum error correction and is now playing an increasingly important role in fault-tolerant quantum computation [8, 61–77]. The key idea is to represent certain special class of states by their stabilizer groups, which are Abelian subgroups of the fundamental Pauli group. We elaborate on the concept and properties of stabilizer states, which serves as “classical states” in this formalism. Deviation from which then represents magic resource (non-stabilizerness), which is necessary for promoting stabilizer circuits to fault-tolerant quantum computation.

For any natural number d, let $\mathbb{Z}_d = \{0, 1, \cdots, d-1\}$ be the ring of integers modulo d, which serves as the discrete configuration space of our d-dimensional quantum system with Hilbert space $L^2(\mathbb{Z}_d) = \mathbb{C}^d$. The following two unitary operators

$$X = \sum_{k=0}^{d-1} |k+1\rangle\langle k|, \qquad Z = \sum_{k=0}^{d-1} \omega^k |k\rangle\langle k|$$

are basic building blocks of finite dimensional quantum mechanics. For $(k, l) \in \mathbb{Z}_d \times \mathbb{Z}_d$ (discrete phase space), the discrete Heisenberg-Weyl operators are defined as

$$D_{k,l} = \tau^{kl} X^k Z^l = \tau^{kl} \sum_{j=0}^{d-1} \omega^{lj} |j+k\rangle\langle j|,$$

where $\tau = -e^{\pi i/d}$ and $\omega = e^{2\pi i/d}$. These unitary operators constitute a projective representation of the translation group $\mathbb{Z}_d \times \mathbb{Z}_d$ and satisfy

$$D_{k,l} D_{s,t} = \tau^{ls-kt} D_{k+s,l+t}, \qquad k, l, s, t \in \mathbb{Z}_d. \tag{1}$$

Here $ls - kt$ is the ordinary integer arithmetics, while $k+s$ and $l+t$ are the modular d arithmetics in $\mathbb{Z}_d$. In particular,

$$\mathrm{tr}(D_{k,l} D_{s,t}^\dagger) = d\delta_{k,s}\delta_{l,t}, \qquad k, l, s, t \in \mathbb{Z}_d \tag{2}$$

and the set $\{\frac{1}{\sqrt{d}} D_{k,l} : k, l \in \mathbb{Z}_d\}$ constitutes an orthonormal basis for the operator space $L(\mathbb{C}^d)$ (the set of all operators on $\mathbb{C}^d$) equipped with the Hilbert-Schmidt inner product $\langle A|B\rangle = \mathrm{tr} A^\dagger B$.

The augmented set

$$\mathcal{P}_d = \{\tau^j D_{k,l} : j \in \mathbb{Z}_{2d}, k, l \in \mathbb{Z}_d\}$$

is the so-called Pauli group (discrete Heisenberg-Weyl group), which plays a basic role in quantum information in finite dimensional systems. Its normalizer (natural symmetry group)

$$\mathcal{C}_d = \{V \in \mathcal{U}(\mathbb{C}^d) : V\mathcal{C}_d V^\dagger = \mathcal{C}_d\}$$

in the full unitary group $\mathcal{U}(\mathbb{C}^d)$ of the system Hilbert space $\mathbb{C}^d$ is called the Clifford group, which is a central object in fault-tolerant quantum computation and quantum error correction. In particular, for a qubit system ($d = 2$),

$$\mathcal{P}_2 = \{c\mathbf{1}, c\sigma_x, c\sigma_y, c\sigma_z : c = \pm 1, \pm i\}.$$

There are at least three equivalent definitions of (pure) stabilizer states, which highlight formally different yet intrinsically related aspects of the concept. We denote the set of (pure) stabilizer states as $\mathcal{S}_d$, which actually consists of $d(d+1)$ elements for a d-dimensional system [67].

Definition 1. For a d-dimensional quantum system with the Pauli group $\mathcal{P}_d$, a stabilizer state is defined as any eigenstate (with eigenvalue 1) of any nontrivial (i.e., other than $\mathbf{1}$) discrete Heisenberg-Weyl operator in $\mathcal{P}_d$.

Definition 2. For a d-dimensional quantum system with computational basis $\{|j\rangle : j \in \mathbb{Z}_d\}$ and the Clifford group $\mathcal{C}_d$, a stabilizer state is any state of the form $V|j\rangle$ with $|j\rangle$ any computational basis state and $V \in \mathcal{C}_d$ any Clifford operator.

Since $X^j|0\rangle = |j\rangle$ and the shift operator X is apparently a Clifford operator, we may simply restrict $|j\rangle$ to $|0\rangle$, that is,

$$\{V|j\rangle : V \in \mathcal{C}_d, j \in \mathbb{Z}_d\} = \{V|0\rangle : V \in \mathcal{C}_d\}.$$

In particular, we see that the Clifford group acts transitively on stabilizer states, i.e., for any two stabilizer states $|\psi\rangle$ and $|\phi\rangle$, there exists a unique Clifford operator $V \in \mathcal{C}_d$ such that $V|\psi\rangle = |\phi\rangle$.

Definition 3. For a d-dimensional quantum system with the Pauli group $\mathcal{P}_d$, a stabilizer state is defined as the common eigenstate (with common eigenvalue 1) of any maximal Abelian subgroup $\mathcal{A} \subset \mathcal{P}_d$ containing no element $c\mathbf{1}$ for any $c \neq 1$. Naturally, $\mathcal{A}$ is called the stabilizer group of the stabilizer state.

In fact, the stabilizer state associated with $\mathcal{A}$ can be constructed explicitly in a straightforward way as follows [67]. Let

$$\Pi_{\mathcal{A}} = \frac{1}{|\mathcal{A}|} \sum_{W \in \mathcal{A}} W \tag{3}$$

with $|\mathcal{A}|$ the number of elements in $\mathcal{A}$, then by the group property and noting that each group element is a unitary operator, we have

$$\Pi_{\mathcal{A}}^2 = \Pi_{\mathcal{A}}, \quad \Pi_{\mathcal{A}}^\dagger = \frac{1}{|\mathcal{A}|} \sum_{W \in \mathcal{A}} W^\dagger = \frac{1}{|\mathcal{A}|} \sum_{W \in \mathcal{A}} W^{-1} = \Pi_{\mathcal{A}}.$$

Consequently, $\Pi_{\mathcal{A}}$ is an orthogonal projection. From the orthogonality relation (2), we see that each group element except for the identity $\mathbf{1}$ is traceless. Now taking the trace of Eq. (3), we obtain

$$\mathrm{tr}\Pi_{\mathcal{A}} = \frac{1}{|\mathcal{A}|} \sum_{W \in \mathcal{A}} \mathrm{tr}W = \frac{1}{|\mathcal{A}|} \mathrm{tr}\mathbf{1} = \frac{d}{|\mathcal{A}|}. \tag{4}$$

Since $\mathrm{tr}\Pi_{\mathcal{A}} \leq d$, it follows that $1 \leq |\mathcal{A}| \leq d$, and the maximal possible number of $|\mathcal{A}|$ is d. Consequently, such a group $\mathcal{A}$ has $|\mathcal{A}| = d$ elements. Typical examples of $\mathcal{A}$ are $\{X^j : j \in \mathbb{Z}_d\}$ and $\{Z^j : j \in \mathbb{Z}_d\}$. Moreover, $\mathcal{A}_V = \{VAV^\dagger : A \in \mathcal{A}\}$ is also such a maximal Abelian subgroup for any Clifford operator $V \in \mathcal{C}_d$.

From $|\mathcal{A}| = d$ we conclude that $\Pi_{\mathcal{A}}$ is an orthogonal projection on a one-dimensional subspace of $\mathbb{C}^d$ and thus defines a pure state which is denoted as $\Pi_{\mathcal{A}} = |\mathcal{A}\rangle\langle\mathcal{A}|$ with $|\mathcal{A}\rangle \in \mathbb{C}^d$. This is exactly the stabilizer state associated with the maximal Abelian subgroup $\mathcal{A}$, i.e., it is the common eigenstates (with common eigenvalue 1) of operators in $\mathcal{A}$. Indeed, by the group property of $\mathcal{A}$, we have

$$W|\mathcal{A}\rangle\langle\mathcal{A}| = W\Pi_{\mathcal{A}} = \Pi_{\mathcal{A}} = |\mathcal{A}\rangle\langle\mathcal{A}|, \quad \forall\, W \in \mathcal{A},$$

which implies that $W|\mathcal{A}\rangle = |\mathcal{A}\rangle$, $\forall\, W \in \mathcal{A}$, that is, $|\mathcal{A}\rangle$, as a stabilizer state, is the common eigenstate (with common eigenvalue 1) of the operators in $\mathcal{A}$.

The above three definitions of stabilizer states are actually equivalent in the sense that they define the same class of states. For completeness and reader's convenience, we establish this statement via two steps:

(a) Definitions 1 and 2 are equivalent.

(b) Definitions 2 and 3 are equivalent.

To establish item (a), suppose that $|\psi\rangle$ is a stabilizer state according to Definition 1, then there exists a discrete Heisenberg-Weyl operator W such that $W|\psi\rangle = |\psi\rangle$. Since the Clifford group acts (by conjugation) transitively on the Pauli group $\mathcal{P}_d$, we know that there exists a Clifford operator $V \in \mathcal{C}_d$ such that $V^\dagger WV = Z$, or equivalently, $W = VZV^\dagger$. Consequently, $VZV^\dagger|\psi\rangle = W|\psi\rangle = |\psi\rangle$, from which we obtain $ZV^\dagger|\psi\rangle = V^\dagger|\psi\rangle$. However, $|0\rangle$ is the unique eigenstate of Z with eigenvalue 1. It follows that $V^\dagger|\psi\rangle = |0\rangle$, i.e., $|\psi\rangle = V|0\rangle$. This means that $|\psi\rangle$ is a stabilizer state according to Definition 2.

Conversely, suppose that $|\psi\rangle = V|0\rangle$ for $V \in \mathcal{C}_d$ is a stabilizer state according to Definition 2, then $V^\dagger|\psi\rangle = |0\rangle$. Let $W = VZV^\dagger$, then W is a discrete Heisenberg-Weyl operator and

$$W|\psi\rangle = VZV^\dagger|\psi\rangle = VZ|0\rangle = V|0\rangle = |\psi\rangle,$$

which implies that $|\psi\rangle$ is a stabilizer state according to Definition 1.

To establish item (b), suppose that $V|0\rangle$ is a stabilizer state according to Definition 2 with $V \in \mathcal{C}_d$ a Clifford operator, then $V|0\rangle$ is the common eigenstate (with the common eigenvalue 1) of the maximal Abelian subgroup $\{VZ^jV^\dagger : j \in \mathbb{Z}_d\}$ (containing no elements $c\mathbf{1}$ for $c \neq 1$), which can be directly verified as

$$(VZ^jV^\dagger)V|0\rangle = VZ^j|0\rangle = V|0\rangle, \qquad j \in \mathbb{Z}_d.$$

This means that $V|0\rangle$ is a stabilizer state according to Definition 3.

Conversely, suppose that $|\psi\rangle$ is a stabilizer state according to Definition 3, i.e., $|\psi\rangle$ is a common eigenstate (with the common eigenvalues 1) of a maximal Abelian subgroup $\mathcal{A} \subset \mathcal{P}_d$ containing no element $c\mathbf{1}$ for $c \neq 1$, we need to show that $|\psi\rangle = V|0\rangle$ for some Clifford operator $V \in \mathcal{C}_d$. Now since $\mathcal{A}$ is isomorphic to $\{Z^j : j \in \mathbb{Z}_d\}$ and the Clifford operators, by definition, permute the elements of the Pauli group, it follows that there exists a Clifford operator V implementing the isomorphism between $\mathcal{A}$ and $\{Z^j : j \in \mathbb{Z}_d\}$, that is, $\mathcal{A} = \{VZ^jV^\dagger : j \in \mathbb{Z}_d\}$. Accordingly, $VZ^jV^\dagger|\psi\rangle = |\psi\rangle, \forall\, j \in \mathbb{Z}_d$, which implies that $Z^jV^\dagger|\psi\rangle = V^\dagger|\psi\rangle, \forall j \in \mathbb{Z}_d$. But $|0\rangle$ is the unique common eigenstate (with common eigenvalue 1) of $Z^j, j \in \mathbb{Z}_d$. This means that $V^\dagger|\psi\rangle = |0\rangle$, i.e., $V|0\rangle = |\psi\rangle$.

Probabilistic mixtures of pure stabilizer states are called mixed stabilizer states, which together with the pure stabilizer states constitute the set of stabilizer polytope [67], and all other states are called magic states or non-stabilizer states. Thus the set of stabilizer states is convex with the pure stabilizer states as the extreme points.

For composite systems, the corresponding Pauli group is defined as the group generated by the Heisenberg-Weyl operators on each component systems, and the corresponding stabilizer states and Clifford group are defined similarly.

For a qubit system with computational basis $\{|0\rangle, |1\rangle\}$, there are 16 discrete Heisenberg-Weyl operators, i.e., $|\mathcal{P}_2| = 16$. If we ignore the phase, there are essentially 4 operators $\mathbf{1}, \sigma_x, \sigma_y, \sigma_z$. For this system, there are six pure stabilizer states

$$\begin{aligned}
&|+\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle + |1\rangle\big), \quad |-\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle - |1\rangle\big),\\
&|+i\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle + i|1\rangle\big), \quad |-i\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle - i|1\rangle\big)\\
&|0\rangle, \quad |1\rangle,
\end{aligned}$$

which are the eigenstates of the Pauli operators $\sigma_x, \sigma_y, \sigma_z$, respectively, that is, $\sigma_x|\pm\rangle = \pm|+\rangle$, $\sigma_y|\pm i\rangle = \pm|\pm i\rangle$, $\sigma_z|0\rangle = |0\rangle$, $\sigma_z|1\rangle = -|1\rangle$, or in terms of eigenstates (with eigenvalue 1) of discrete Heisenberg-Weyl operators (stabilizer operators),

$$\begin{aligned}
&\sigma_x|+\rangle = |+\rangle, \qquad -\sigma_x|-\rangle = |-\rangle,\\
&\sigma_y|+i\rangle = |+i\rangle, \quad -\sigma_y|-i\rangle = |-i\rangle,\\
&\sigma_z|0\rangle = |0\rangle, \qquad -\sigma_z|1\rangle = |1\rangle.
\end{aligned}$$

These six stabilizer states constitute the vertex of the stabilizer octahedron inscribed in the qubit Bloch sphere. We list the corresponding stabilizer groups in Table I for latter convenience.

TABLE I: Qubit stabilizer states and the corresponding stabilizer groups

Stabilizer state	$\|+\rangle$	$\|-\rangle$	$\|+i\rangle$	$\|-i\rangle$	$\|0\rangle$	$\|1\rangle$
stabilizer group	$\{\mathbf{1}, \sigma_x\}$	$\{\mathbf{1}, -\sigma_x\}$	$\{\mathbf{1}, \sigma_y\}$	$\{\mathbf{1}, -\sigma_y\}$	$\{\mathbf{1}, \sigma_z\}$	$\{\mathbf{1}, -\sigma_z\}$

For a qutrit system (i.e., $d = 3$) with computational basis $\{|0\rangle, |1\rangle, |2\rangle\}$, there are $3(3+1) = 12$ pure stabilizer states, which are listed in Table II together with the corresponding stabilizer groups. It is remarkable that these 12 stabilizer states can be partitioned into 4 mutually unbiased bases of $\mathbb{C}^3$ as

$$B_\mu = \{|\phi_{i+3\mu}\rangle : i = 1, 2, 3\}, \qquad \mu = 0, 1, 2, 3.$$

TABLE II: Qutrit stabilizer states $|\phi_i\rangle, i = 1, 2, \cdots, 12$, and the corresponding stabilizer groups. Noting that $XX^\dagger = ZZ^\dagger = \mathbf{1}$, $X^3 = Z^3 = \mathbf{1}$, $XZ = \omega^{-1}ZX$, $\omega = e^{2\pi i/3}$.

stabilizer state	stabilizer group
$\|\phi_1\rangle = \|0\rangle$	$\{\mathbf{1}, Z, Z^\dagger\}$
$\|\phi_2\rangle = \|1\rangle$	$\{\mathbf{1}, \omega^{-1}Z, \omega Z^\dagger\}$
$\|\phi_3\rangle = \|2\rangle$	$\{\mathbf{1}, \omega Z, \omega^{-1}Z^\dagger\}$
$\|\phi_4\rangle = \big(\|0\rangle + \|1\rangle + \|2\rangle\big)/\sqrt{3}$	$\{\mathbf{1}, X, X^\dagger\}$
$\|\phi_5\rangle = \big(\|0\rangle + \omega^{-1}\|1\rangle + \omega\|2\rangle\big)/\sqrt{3}$	$\{\mathbf{1}, \omega^{-1}X, \omega X^\dagger\}$
$\|\phi_6\rangle = \big(\|0\rangle + \omega\|1\rangle + \omega^{-1}\|2\rangle\big)/\sqrt{3}$	$\{\mathbf{1}, \omega X, \omega^{-1}X^\dagger\}$
$\|\phi_7\rangle = \big(\|0\rangle + \|1\rangle + \omega\|2\rangle\big)/\sqrt{3}$	$\{\mathbf{1}, XZ, \omega X^\dagger Z^\dagger\}$
$\|\phi_8\rangle = \big(\|0\rangle + \omega\|1\rangle + \|2\rangle\big)/\sqrt{3}$	$\{\mathbf{1}, \omega XZ, X^\dagger Z^\dagger\}$
$\|\phi_9\rangle = \big(\omega\|0\rangle + \|1\rangle + \|2\rangle\big)/\sqrt{3}$	$\{\mathbf{1}, \omega^{-1}XZ, \omega^{-1}X^\dagger Z^\dagger\}$
$\|\phi_{10}\rangle = \big(\|0\rangle + \|1\rangle + \omega^{-1}\|2\rangle\big)/\sqrt{3}$	$\{\mathbf{1}, XZ^\dagger, \omega^{-1}X^\dagger Z\}$
$\|\phi_{11}\rangle = \big(\|0\rangle + \omega^{-1}\|1\rangle + \|2\rangle\big)/\sqrt{3}$	$\{\mathbf{1}, \omega^{-1}XZ^\dagger, X^\dagger Z\}$
$\|\phi_{12}\rangle = \big(\omega^{-1}\|0\rangle + \|1\rangle + \|2\rangle\big)/\sqrt{3}$	$\{\mathbf{1}, \omega XZ^\dagger, \omega X^\dagger Z\}$

III. MAGIC-RESOURCE GENERATING POWER OF QUANTUM CHANNELS

In order to quantify magic-resource generating power of quantum channels, we need quantifiers of magic. As a quantum resource, magic (non-stabilizerness) has attracted much attention in fault-tolerant quantum computation [61–82]. Although several significant quantifiers of magic have been introduced in the literature, most of them are not easy to compute.

In terms of characteristic functions of quantum states, which are always well defined, a convenient quantifier of magic on a d-dimensional quantum system was introduced as

$$M(\rho) = \sum_{k,l} |\mathrm{tr}(\rho D_{k,l})| \tag{5}$$

in Ref. [82], which has the following properties, and in particular is easy to compute [82].

(1) (*Bound*) $1 \leq M(\rho) \leq 1 + (d-1)\sqrt{d+1}$.

(2) (*Clifford invariance*) $M(\rho)$ is invariant under the Clifford operations in the sense that

$$M(V\rho V^\dagger) = M(\rho), \qquad \forall\ V \in \mathcal{C}_d.$$

(3) (*Convexity*) $M(\rho)$ is convex in ρ.

(4) (*Minimal value for mixed states*) $M(\rho) \geq 1$ with the minimum value achieved if and only if $\rho = \mathbf{1}/d$ is the maximally mixed state.

(5) (*Minimal value for pure states*) $M(|\psi\rangle\langle\psi|) \geq d$ for any pure state $|\psi\rangle$ with the low bound achieved if and only if $|\psi\rangle$ is a stabilizer state.

(6) (*Criterion for magic states*) If $M(\rho) > d$, then the state ρ is magic. It should be noticed that this is only a sufficient, but not necessary, condition for a state to be magic.

(7) (*Maximal value*) The maximal value of $M(\rho)$ is achieve by any SIC-POVM fiducial state (assuming its existence, which has been proved in many dimensions).

Recall that a SIC-POVM (symmetric informationally complete positive operator valued measure) in $\mathbb{C}^d$ is a POVM $\{E_\alpha : \alpha = 1, 2, \cdots, d^2\}$ (i.e., $E_\alpha \geq 0, \sum_{\alpha=1}^{d^2} E_\alpha = \mathbf{1}$) which consisting of d^2 rank-one operators with equal trace and equal overlap, and span the whole state space [83, 84]. These conditions entail that $\mathrm{tr}(E_\alpha E_\beta) = 1/(d^2(d+1))$ for $\alpha \neq \beta$, and $\mathrm{tr}E_\alpha = 1/d$ for any α. Despite of the above naive conditions, SIC-POVMs are surprisingly deep in mathematics, and their existence in any dimension, although widely believed and supported by vast evidence, remains an outstanding open issue (Zauner's conjecture) [83–94]. A pure state $|f\rangle \in \mathbb{C}^d$ is called a SIC-POVM fiducial state if its orbit $\{E_{k,l} = \frac{1}{d}D_{k,l}|f\rangle\langle f|D_{k,l}^\dagger : k, l \in \mathbb{Z}_d\}$ under the action of the discrete Heisenberg operators is a SIC-POVM. Most known SIC-POVMs are constructed in this fashion, and fiducial states have been explicitly constructed in many dimensions, although the general issue remains open [83, 84].

To simplify notation, for any pure state $\rho = |\psi\rangle\langle\psi|$, we denote $M(|\psi\rangle\langle\psi|) = M(|\psi\rangle)$. Similarly, for any quantum channel $\mathcal{E}$, we denote $\mathcal{E}(|\psi\rangle\langle\psi|) = \mathcal{E}(|\psi\rangle)$.

We proceed to employ $M(\cdot)$ defined by Eq. (5) to study magic-resource generating power of a general quantum channel

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger$$

on a d-dimensional quantum system $\mathbb{C}^d$. For this purpose, we introduce the following quantities

$$M_{\max}(\mathcal{E}) = \max_{|\psi\rangle \in \mathcal{S}_d} M(\mathcal{E}(|\psi\rangle)), \tag{6}$$

$$M_{\text{ave}}(\mathcal{E}) = \frac{1}{|\mathcal{S}_d|} \sum_{|\psi\rangle \in \mathcal{S}_d} M(\mathcal{E}(|\psi\rangle)), \tag{7}$$

which characterize the maximal and average magic-resource generating power of $\mathcal{E}$, respectively. Here $|\mathcal{S}_d| = d(d+1)$ is the number of pure stabilizer states in $\mathbb{C}^d$ [67].

Proposition 1. $M_{\max}(\mathcal{E})$ has the following properties.

(i) (*Clifford invariance*) $M_{\max}(\mathcal{E})$ is invariant under Clifford conjugation in the sense that

$$M_{\max}(\mathcal{E} \circ \mathcal{V}) = M_{\max}(\mathcal{V} \circ \mathcal{E}) = M_{\max}(\mathcal{E}),$$

where $\mathcal{V}(\rho) = V \rho V^\dagger$ for $V \in \mathcal{C}_d$, and $\circ$ is the composition of maps.

(ii) (*Convexity*) $M_{\max}(\mathcal{E})$ is convex in $\mathcal{E}$ in the sense that

$$M_{\max}\Big(\sum_i p_i \mathcal{E}_i\Big) \leq \sum_i p_i M_{\max}(\mathcal{E}_i)$$

for quantum channels $\mathcal{E}_i$ and probabilities p_i with $p_i \geq 0, \sum_i p_i = 1$.

(iii) (*Super-multiplicativity*) $M_{\max}(\mathcal{E})$ is super-multiplicative under tensor product in the sense that

$$M_{\max}(\mathcal{E}_1 \otimes \mathcal{E}_2) \geq M_{\max}(\mathcal{E}_1) M_{\max}(\mathcal{E}_2).$$

for quantum channels $\mathcal{E}_1$ and $\mathcal{E}_2$ on two quantum systems $\mathbb{C}^{d_1}$ and $\mathbb{C}^{d_2}$, respectively.

We sketch a proof of the above properties.

For item (i), the equality $M_{\max}(\mathcal{E} \circ \mathcal{V}) = M_{\max}(\mathcal{E})$ follows from the fact that Clifford operators permute stabilizer states, while the equality $M_{\max}(\mathcal{V} \circ \mathcal{E}) = M_{\max}(\mathcal{E})$ follows from the corresponding invariance of the quantifier of magic $M(\cdot)$.

Noting that $M(\cdot)$ defined by Eq. (5) is convex, item (ii) follows from

$$\begin{aligned} M_{\max}\Big(\sum_i p_i \mathcal{E}_i\Big) &= \max_{|\psi\rangle \in \mathcal{S}} M\Big(\sum_i p_i \mathcal{E}_i(|\psi\rangle)\Big) \\ &\leq \max_{|\psi\rangle \in \mathcal{S}} \sum_i p_i M(\mathcal{E}_i(|\psi\rangle)) \\ &\leq \sum_i p_i \max_{|\psi\rangle \in \mathcal{S}} M\big(\mathcal{E}_i(|\psi\rangle)\big) \\ &= \sum_i p_i M_{\max}(\mathcal{E}_i) \end{aligned}$$

For item (iii), when $|\psi\rangle$ is a product stabilizer state $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$, we have

$$\begin{aligned} & M\big((\mathcal{E}_1 \otimes \mathcal{E}_2)(|\psi\rangle)\big) \\ =& M\big((\mathcal{E}_1 \otimes \mathcal{E}_2)(|\psi_1\rangle \otimes |\psi_2\rangle)\big) \\ =& M\big(\mathcal{E}_1(|\psi_1\rangle)| \otimes \mathcal{E}_2(|\psi_2\rangle)\big) \\ =& \sum_{k_1,l_1,k_2,l_2} \big|\mathrm{tr}\big(\mathcal{E}_1(|\psi_1\rangle) \otimes \mathcal{E}_2(|\psi_2\rangle) D_{k_1,l_1} \otimes D_{k_2,l_2}\big)\big| \\ =& \sum_{k_1,l_1} \big|\mathrm{tr}\big(\mathcal{E}_1(|\psi_1\rangle) D_{k_1,l_1}\big)\big| \cdot \sum_{k_2,l_2} \big|\mathrm{tr}\big(\mathcal{E}_2(|\psi_2\rangle|) D_{k_2,l_2}\big)\big| \\ =& M\big(\mathcal{E}_1\big(|\psi_1\rangle\big)\big) M\big(\mathcal{E}_2(|\psi_2\rangle)\big). \end{aligned}$$

By taking the maximum over all stabilizer states, we obtain

$$\begin{aligned}
&M_{\max}(\mathcal{E}_1 \otimes \mathcal{E}_2) \\
&= \max_{|\psi\rangle \in \mathcal{S}_{d_1 d_2}} M\big(\mathcal{E}_1 \otimes \mathcal{E}_2(|\psi\rangle)\big) \\
&\geq \max_{|\psi_1\rangle \otimes |\psi_2\rangle \in \mathcal{S}_{d_1 d_2}} M\big(\mathcal{E}_1 \otimes \mathcal{E}_2(|\psi_1\rangle \otimes |\psi_2\rangle)\big) \\
&= \max_{|\psi_1\rangle \in \mathcal{S}_{d_1}} M\big(\mathcal{E}_1(|\psi_1\rangle)\big) \max_{|\psi_2\rangle \in \mathcal{S}_{d_2}} M\big(\mathcal{E}_2(|\psi_2\rangle)\big) \\
&= M_{\max}(\mathcal{E}_1) M_{\max}(\mathcal{E}_2),
\end{aligned}$$

which is the desired result.

For the average magic-resource generating power $M_{\mathrm{ave}}(\mathcal{E})$, similar properties as items (i)-(ii) also hold.

In magic state distillation and gate synthesis, the non-Clifford T-gate (i.e., $\pi/8$-gate, the name seems a little confusing but is clear from the generator expression)

$$T_{\pi/4} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} = e^{i\pi/8} e^{-i\sigma_z \pi/8}$$

has been widely used as a benchmark for non-stabilizerness. It is natural to ask whether this gate is optimal in generating magic resource in some sense. To address this issue, consider the family of qubit quantum channels

$$\mathcal{E}_\theta(\rho) = T_\theta \rho T_\theta^\dagger$$

with

$$T_\theta = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}, \qquad \theta \in [0, 2\pi)$$

then we have the following satisfying result, which show that indeed the T-gate is in some sense optimal in generating magic resource.

Proposition 2. In the qubit gate set $\{T_\theta : \theta \in [0, 2\pi)\}$, the T-gate $T_{\pi/4}$ has the maximal magic-resource generating power, i.e.,

$$\max_\theta M_{\max}(\mathcal{E}_\theta) = M_{\max}(\mathcal{E}_{\pi/4}).$$

Moreover, the T-gate $T_{\pi/4}$ is equivalent (under Clifford conjugation) to the rotation gate $R_{\pi/4}$ with

$$R_\gamma = \begin{pmatrix} \cos\frac{\gamma}{2} & -\sin\frac{\gamma}{2} \\ \sin\frac{\gamma}{2} & \cos\frac{\gamma}{2} \end{pmatrix}, \qquad \gamma \in [0, 2\pi).$$

In fact, $T_{\pi/4} = e^{i\pi/8} H S^\dagger R_{\pi/4} S H$ with

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad S = Z^{1/2} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{pmatrix}$$

being the Hadarmad gate and S-gate (phase gate), respectively, both of which are Clifford unitaries.

The proof is relegated to the next section.

Similarly, for a qutrit system ($d = 3$), we have the following optimal characterization of the qutrit version of the T-gate.

Proposition 3. In the qutrit gate set $\{\Lambda_\theta : \theta \in [0, 2\pi)\}$, the qutrit T-gate $\Lambda_{2\pi/9}$ has the maximal magic-resource generating power, i.e.,

$$\max_\theta M_{\max}(\mathcal{E}_{\Lambda_\theta}) = M_{\max}(\mathcal{E}_{\Lambda_{2\pi/9}}),$$

where $\mathcal{E}_{\Lambda_\theta} = \Lambda_\theta \rho \Lambda_\theta^\dagger$ with

$$\Lambda_\theta = \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\theta} \end{pmatrix}, \qquad \theta \in [0, 2\pi).$$

being qutrit diagonal unitary operators.

The proof is also relegated to the next section.

IV. EVALUATING MAGIC-RESOURCE GENERATING POWER OF CHANNELS

In this section, we evaluate explicitly the magic-resource generating power for various prototypical quantum channels. In particular, we illuminate some optimal features of the T-gate. For simplicity, we only treat the qubit and qutrit cases.

A. Qubit channels

Recall that for a qubit system ($d = 2$), there are 6 pure stabilizer states

$$|+\rangle, \quad |-\rangle, \quad |+i\rangle, \quad |-i\rangle, \quad |0\rangle, \quad |1\rangle$$

as listed in Table I.

(1) *Unitary channel*

For a general unitary channel

$$\mathcal{E}_U = U\rho U^\dagger$$

on a qubit system associated with the unitary operator

$$U = \begin{pmatrix} \cos\frac{\gamma}{2} & -e^{i\alpha}\sin\frac{\gamma}{2} \\ e^{i\beta}\sin\frac{\gamma}{2} & e^{i(\alpha+\beta)}\cos\frac{\gamma}{2} \end{pmatrix},$$

we have

$$\begin{aligned} M(\mathcal{E}_U(|\pm\rangle)) &= 1 + |\sin\alpha\sin\beta - \cos\alpha\cos\beta\cos\gamma| \\ &\quad + |\sin\alpha\cos\beta + \cos\alpha\sin\beta\cos\gamma| + |\cos\alpha\sin\gamma|, \\ M(\mathcal{E}_U(|\pm i\rangle)) &= 1 + |\cos\alpha\sin\beta + \sin\alpha\cos\beta\cos\gamma| \\ &\quad + |\cos\alpha\cos\beta - \sin\alpha\sin\beta\cos\gamma| + |\sin\alpha\sin\gamma|, \\ M(\mathcal{E}_U(|0\rangle)) &= M(\mathcal{E}_U(|1\rangle)) \\ &= 1 + (|\cos\beta| + |\sin\beta|)\sin\gamma + |\cos\gamma|. \end{aligned}$$

Clearly, the magic-resource generating power depends on the three parameters α, β, γ in a rather complicated way. To simply matters, we consider several special cases.

First, when $\alpha = \beta = 0$, we have

$$U = R_\gamma = \begin{pmatrix} \cos\frac{\gamma}{2} & -\sin\frac{\gamma}{2} \\ \sin\frac{\gamma}{2} & \cos\frac{\gamma}{2} \end{pmatrix} = e^{-i\gamma\sigma_y/2}$$

and

$$\begin{aligned} M(\mathcal{E}_{R_\gamma}(|\pm\rangle)) &= M(\mathcal{E}_{R_\gamma}(|0\rangle)) = M(\mathcal{E}_{R_\gamma}(|1\rangle)) \\ &= 1 + |\cos\gamma| + |\sin\gamma|, \\ M(\mathcal{E}_{R_\gamma}(|\pm i\rangle)) &= 2. \end{aligned}$$

Noting that $|\cos\gamma| + |\sin\gamma| \geq |\cos\gamma|^2 + |\sin\gamma|^2 = 1$, it follows that

$$\begin{aligned} M_{\max}(\mathcal{E}_{R_\gamma}) &= 1 + |\cos\gamma| + |\sin\gamma|, \\ M_{\text{ave}}(\mathcal{E}_{R_\gamma}) &= \frac{4}{3} + \frac{2}{3}\big(|\cos\gamma| + |\sin\gamma|\big), \end{aligned}$$

from which we see that

$$2 \leq M_{\max}(\mathcal{E}_{R_\gamma}) \leq 1 + \sqrt{2}.$$

Moreover, the lower bound 2 is achieved (i.e., $M_{\max}(\mathcal{E}_{R_\gamma}) = 2$) if and only if $\gamma = k\pi/2$, $k = 0, 1, 2, 3$, and the upper bound is achieved (i.e., $M_{\max}(\mathcal{E}_{R_\gamma}) = 1 + \sqrt{2}$) if and only if $\gamma = (2k+1)\pi/4$, $k = 0, 1, 2, 3$. Consequently,

$$M_{\max}(\mathcal{E}_{R_{\pi/4}}) = 1 + \sqrt{2}, \tag{8}$$

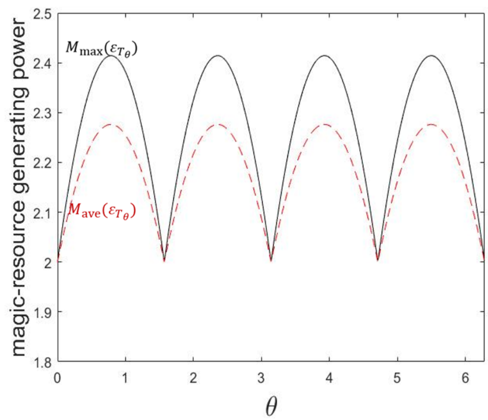

FIG. 1: The maximal and average magic-resource generating power $M_{\max}(\mathcal{E}_{T_\theta})$ and $M_{\rm ave}(\mathcal{E}_{T_\theta})$ as functions of the parameter $\theta \in [0, 2\pi)$. Here $\mathcal{E}_{T_\theta}(\rho) = T_\theta \rho T_\theta^\dagger$ with T_θ defined by Eq. (10).

Similarly,

$$2 \le M_{\rm ave}(\mathcal{E}_{R_\gamma}) \le \frac{1}{3}(4 + 2\sqrt{2}),$$

and the lower bound 2 is achieved (i.e., $M_{\rm ave}(\mathcal{E}_{R_\gamma}) = 2$) if and only if $\gamma = k\pi/2$, $k = 0, 1, 2, 3$, the upper bound is achieved (i.e., $M_{\rm ave}(\mathcal{E}_{R_\gamma}) = (4 + 2\sqrt{2})/3$) if and only if $\gamma = (2k + 1)\pi/4$, $k = 0, 1, 2, 3$. Thus

$$M_{\rm ave}(\mathcal{E}_{R_{\pi/4}}) = \frac{1}{3}(4 + 2\sqrt{2}). \tag{9}$$

We see that the maximal magic-resource generating power $M_{\max}(\mathcal{E}_{R_\gamma})$ and the average magic-resource generating power $M_{\rm ave}(\mathcal{E}_{R_\gamma})$ share similar and consistent behaviors.

Second, when $\alpha = \gamma = 0$, and following the convention let $\beta = \theta$, then we have

$$U = T_\theta = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix} = e^{i\theta/2} e^{-i\sigma_z/2}, \tag{10}$$

and

$$M(\mathcal{E}_{T_\theta}(|\pm\rangle)) = M(\mathcal{E}_{T_\theta}(|\pm i\rangle)) = 1 + |\cos\theta| + |\sin\theta|,$$
$$M(\mathcal{E}_{T_\theta}(|0\rangle)) = M(\mathcal{E}_{T_\theta}(|1\rangle)) = 2.$$

Consequently,

$$M_{\max}(\mathcal{E}_{T_\theta}) = 1 + |\cos\theta| + |\sin\theta|,$$
$$M_{\rm ave}(\mathcal{E}_{T_\theta}) = \frac{4}{3} + \frac{2}{3}(|\cos\theta| + |\sin\theta|).$$

In particular, for $\theta = \pi/2$, $T_{\pi/2}$ is the S-gate, or $\pi/4$-gate

$$T_{\pi/2} = S = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{pmatrix} = e^{i\pi/4} \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix},$$

which is actually a Clifford unitary and thus has no magic-resource generating power. Indeed,

$$M_{\max}(\mathcal{E}_{T_{\pi/2}}) = M_{\rm ave}(\mathcal{E}_{T_{\pi/2}}) = 2,$$

which shows that the S-gate has the minimal magic-resource generating power among the class of unitary gates $T_\theta, \theta \in [0, 2\pi)$. In contrast, for $\theta = \pi/4$, $T_{\pi/4}$ is the celebrated T-gate, or $\pi/8$-gate

$$T_{\pi/4} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} = e^{i\pi/8} \begin{pmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{pmatrix},$$

which belongs to the third level of the Clifford hierarchy [65], and we have

$$M_{\max}(\mathcal{E}_{T_{\pi/4}}) = 1 + \sqrt{2}, \tag{11}$$

$$M_{\text{ave}}(\mathcal{E}_{T_{\pi/4}}) = \frac{1}{3}(4 + 2\sqrt{2}), \tag{12}$$

which show that $\pi/8$-gate has the maximal magic-resource generating power among the class of unitary gates $T_\theta, \theta \in [0, 2\pi)$. This appealing result has been summarized as Proposition 2 in the previous section (i.e., Sec. III).

Comparing Eqs. (8) and (9) with Eqs. (11) and (12), the coincidence is not accidental, since it can be directly checked that $T_{\pi/4} = e^{i\pi/8} H S^\dagger R_{\pi/4} S H$ with H the Hadamard gate and S the conventional phase gate. Now by item (i) of Proposition 1, it must holds $M_{\max}(\mathcal{E}_{R_{\pi/4}}) = M_{\max}(\mathcal{E}_{T_{\pi/4}})$, and similarly $M_{\text{ave}}(\mathcal{E}_{R_{\pi/4}}) = M_{\text{ave}}(\mathcal{E}_{T_{\pi/4}})$. We also note that geometrically, $R_{\pi/4} = e^{-i\pi\sigma_y/8}, T_{\pi/4} = e^{i\pi/8} e^{-i\pi\sigma_z/8}$.

To visualize the behaviors of the magic-resource generating power of the quantum channels associated with the gates T_θ, we depict the graphs of $M_{\max}(\mathcal{E}_{T_\theta})$ and $M_{\text{ave}}(\mathcal{E}_{T_\theta})$ in Fig. 1

(2) *Amplitude damping channel*

For the amplitude damping channel

$$\mathcal{E}_{\text{AD}}(\rho) = \sum_i K_i \rho K_i^\dagger$$

on a qubit system with the Kraus operators

$$K_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}, \qquad 0 \le p \le 1,$$

we have

$$M(\mathcal{E}_{\text{AD}}(|\pm\rangle)) = M(\mathcal{E}_{\text{AD}}(|\pm i\rangle)) = 1 + p + \sqrt{1-p},$$
$$M(\mathcal{E}_{\text{AD}}(|0\rangle)) = 2, \qquad M(\mathcal{E}_{\text{AD}}(|1\rangle)) = 1 + |1 - 2p|.$$

Consequently,

$$M_{\max}(\mathcal{E}_{\text{AD}}) = 1 + p + \sqrt{1-p},$$
$$M_{\text{ave}}(\mathcal{E}_{\text{AD}}) = \frac{1}{6}\big(7 + 4\sqrt{1-p} + 4p + |1-2p|\big).$$

It follows that

$$2 \le M_{\max}(\mathcal{E}_{\text{AD}}) \le \frac{9}{4},$$

and the lower bound is achieved (i.e., $M_{\max}(\mathcal{E}_{\text{AD}}) = 2$) if and only if $p = 0$ or 1, while the upper bound is achieved (i.e., $M_{\max}(\mathcal{E}_{\text{AD}}) = 9/4$) if and only if $p = 3/4$. For the average magic-resource generating power,

$$\frac{3}{2} + \frac{\sqrt{2}}{3} \le M_{\text{ave}}(\mathcal{E}_{\text{AD}}) \le \frac{19}{9},$$

and the lower bound is achieved if and only if $p = 1/2$, while the upper bound is achieved if and only if $p = 8/9$.

The behaviors of magic-resource generating power of the amplitude damping channel are depicted in Fig. 2

(3) *Phase damping channel*

For the phase damping channel

$$\mathcal{E}_{\text{PD}}(\rho) = \sum_i K_i \rho K_i^\dagger$$

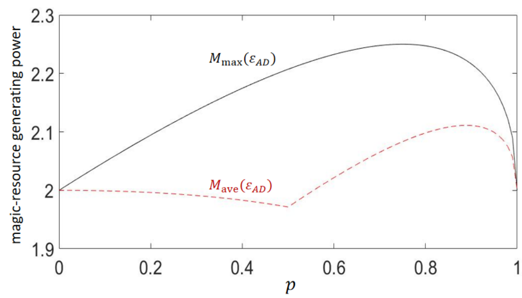

FIG. 2: The maximal and average magic-resource generating power $M_{\max}(\mathcal{E}_{\mathrm{AD}})$ and $M_{\mathrm{ave}}(\mathcal{E}_{\mathrm{AD}})$ as functions of the parameter $p \in [0, 1]$.

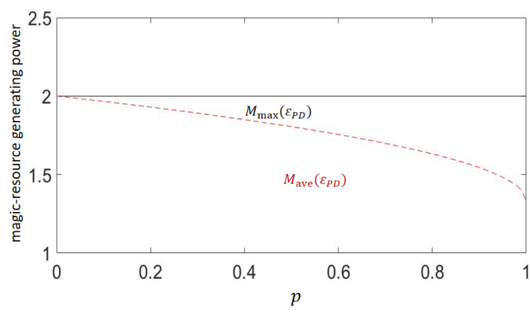

FIG. 3: The maximal and average magic-resource generating power $M_{\max}(\mathcal{E}_{\mathrm{PD}})$ and $M_{\mathrm{ave}}(\mathcal{E}_{\mathrm{PD}})$ as functions of the parameter $p \in [0, 1]$.

on a qubit system with the Kraus operators

$$K_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{p} \end{pmatrix}, \qquad 0 \le p \le 1,$$

we have

$$M(\mathcal{E}_{\mathrm{PD}}(|\pm\rangle)) = M(\mathcal{E}_{\mathrm{PD}}(|\pm i\rangle)) = 1 + \sqrt{1-p},$$
$$M(\mathcal{E}_{\mathrm{PD}}(|0\rangle)) = M(\mathcal{E}_{\mathrm{PD}}(|1\rangle)) = 2.$$

Consequently,

$$M_{\max}(\mathcal{E}_{\mathrm{PD}}) = 2,$$
$$M_{\mathrm{ave}}(\mathcal{E}_{\mathrm{PD}}) = \frac{1}{3}\big(4 + 2\sqrt{1-p}\big).$$

The behaviors of magic-resource generating power of the phase damping channel are depicted in Fig. 3.

(5) *Completely decoherent channel*
The completely decoherent channel on a qubit system is defined as

$$\mathcal{E}_{\mathrm{Cd}}(\rho) = M * \rho$$

with M a non-negative definite matrix with all diagonal elements being 1 and the notation $*$ denoting the Hadamard (entry-wise) product of matrices. For simplicity, we take

$$M = \begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix}, \qquad -1 \le c \le 1.$$

The channel can be equivalently expressed as $\mathcal{E}_{\rm CD}(\rho) = \sum_i K_i \rho K_i^\dagger$ with the Kraus operators

$$K_1 = \begin{pmatrix} \sqrt{1-|c|} & 0 \\ 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{1-|c|} \end{pmatrix},$$

$$K_3 = \sqrt{|c|} \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{sgn} c \end{pmatrix}.$$

By direct calculations, we have

$$M(\mathcal{E}_{\rm Cd}(|\pm\rangle)) = M(\mathcal{E}_{\rm Cd}(|\pm i\rangle|)) = 1 + |c|,$$
$$M(\mathcal{E}_{\rm Cd}(|0\rangle)) = M(\mathcal{E}_{\rm Cd}(|1\rangle)) = 2,$$

from which we obtain

$$M_{\max}(\mathcal{E}_{\rm Cd}) = 2,$$
$$M_{\rm ave}(\mathcal{E}_{\rm Cd}) = \frac{1}{3}\big(4 + 2|c|\big).$$

(6) *Werner-Holevo channel*
The Werner-Holevo channel

$$\mathcal{E}_{\rm WH}(\rho) = \frac{1}{d-1}(\mathbf{1} - \rho^T)$$

provides a counterexample to an additivity conjecture for output purity of channels. Here ρ^T is the transpose of ρ in an orthonormal basis $\{|k\rangle : k \in \mathbb{Z}^d\}$ of $\mathbb{C}^d$. In particular, when $d = 2$, the Werner-Holevo channel reduces to the unitary channel

$$\mathcal{E}_{\rm WH}(\rho) = \mathrm{tr}(\rho)\mathbf{1} - \rho^T = \sigma_y \rho \sigma_y$$

with σ_y the second Pauli matrix. It is known that a Kraus representation of $\mathcal{E}$ is

$$\mathcal{E}_{\rm WH}(\rho) = \frac{1}{2(d-1)} \sum_{k,l} (|k\rangle\langle l| - |l\rangle\langle k|)\rho(|k\rangle\langle l| - |l\rangle\langle k|)^\dagger.$$

Direct calculations show that

$$M(\mathcal{E}_{\rm WH}(|\psi\rangle) = \frac{d}{d-1}$$

for any stabilizer state $|\psi\rangle$. Thus

$$M_{\max}(\mathcal{E}_{\rm WH}) = M_{\rm ave}(\mathcal{E}_{\rm WH}) = \frac{d}{d-1}.$$

(7) *Channels induced by weak measurements*
For fixed $x \in [0, 1/2]$, consider the channel

$$\mathcal{E}_{\rm W}(\rho) = K_x \rho K_x^\dagger + K_{1-x} \rho K_{1-x}^\dagger$$

induced by the qubit weak measurement with $K_x = \sqrt{1-x}|0\rangle\langle 0| + \sqrt{x}|1\rangle\langle 1|$. The case $x = 0$ reduces to the von Neumann measurement $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ corresponding to the computational basis $\{|0\rangle, |1\rangle\}$. Direct calculations show that

$$M(\mathcal{E}_{\rm W}(|\pm\rangle)) = M(\mathcal{E}_{\rm W}(\pm i\rangle)) = 1 + \sqrt{x(1-x)},$$
$$M(\mathcal{E}_{\rm W}(|0\rangle)) = M(\mathcal{E}_{\rm W}(|1\rangle)) = 2.$$

Consequently,

$$M_{\max}(\mathcal{E}_{\rm W}) = 2,$$
$$M_{\rm ave}(\mathcal{E}_{\rm W}) = \frac{4}{3}\big(1 + \sqrt{x(1-x)}\big).$$

Clearly,

$$\frac{4}{3} \le M_{\rm ave}(\mathcal{E}_{\rm W}) \le 2,$$

and $M_{\rm ave}(\mathcal{E}_{\rm W}) = 4/3$ when $x = 0$, and $M_{\rm ave}(\mathcal{E}_{\rm W}) = 2$ when $x = 1/2$.

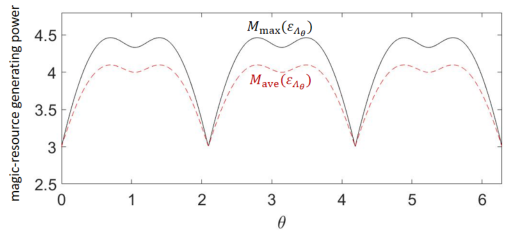

FIG. 4: The maximal and average magic-resource generating power $M_{\max}(\mathcal{E}_{\Lambda_\theta})$ and $M_{\text{ave}}(\mathcal{E}_{\Lambda_\theta})$ as functions of the parameter $\theta \in [0, 2\pi)$.

B. Qutrit channels

Apart from the qubit channels, the next important ones are qutrit channels, which are also under active study [95–101].

For a qutrit system ($d = 3$), there are 12 pure stabilizer states as listed in Table II.

(1) Consider the unitary channel

$$\mathcal{E}_{\Lambda_\theta} = \Lambda_\theta \rho \Lambda_\theta^\dagger$$

on a qutrit system associated with the diagonal unitary operator

$$\Lambda_\theta = \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\theta} \end{pmatrix}, \qquad \theta \in [0, 2\pi). \tag{13}$$

Note that the qutrit T-gate is a special case with $\theta = 2\pi/9$ [68], and the qutrit T-state

$$\begin{aligned} |T\rangle &= \frac{1}{\sqrt{3}}\left(e^{2\pi i/9}|0\rangle + |1\rangle + e^{-2\pi i/9}\right) \\ &= \Lambda_{2\pi/9}\frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle), \end{aligned}$$

is an important non-stabilizer state analogous to the corresponding qubit T-state in qubit magic state distillation [66, 69].

After direct calculations, we obtain

$$M(\mathcal{E}_{\Lambda_\theta}(|\phi_i\rangle)) = 3, \qquad i = 1, 2, 3,$$

and

$$M(\mathcal{E}_{\Lambda_\theta}(|\phi_i\rangle)) = 1 + \frac{2}{3}\sqrt{5 + 4\cos 3\theta} + \frac{4}{3}\sqrt{2 - 2\cos 3\theta}$$

for $i = 4, 5, \cdots, 12$. Consequently,

$$\begin{aligned} M_{\max}(\mathcal{E}_{\Lambda_\theta}) &= 1 + \frac{2}{3}\sqrt{5 + 4\cos 3\theta} + \frac{4}{3}\sqrt{2 - 2\cos 3\theta}, \\ M_{\text{ave}}(\mathcal{E}_{\Lambda_\theta}) &= \frac{3}{2} + \frac{1}{2}\sqrt{5 + 4\cos 3\theta} + \sqrt{2 - 2\cos 3\theta}. \end{aligned}$$

Direct calculations show that $M_{\max}(\mathcal{E}_{\Lambda_\theta})$ achieves its minimal value 4.0971 when $\theta = \pi/3$, and achieves its maximal value 4.4641 when $\theta = 2\pi/9$. Similarly, $M_{\text{ave}}(\mathcal{E}_{\Lambda_\theta})$ achieves its minimal value 3.3660 when $\theta = \pi/3$, and achieves its maximal value 4.0981 when $\theta = 2\pi/9$. We thus have established the optimality of the qutrit T-gate $\Lambda_{2\pi/9}$ in generating magic-resource in this context.

The graphs of $M_{\max}(\mathcal{E}_{\Lambda_\theta})$ and $M_{\text{ave}}(\mathcal{E}_{\Lambda_\theta})$ as functions of θ are depicted in Fig. 4.

(2) Consider the N-gate (also called reflection gate, metaplectic gate, R-gate)

$$N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{14}$$

first introduced in Ref. [69], which admits a magic state distillation and injection protocol. It was further proved that it achieves approximate universality when combined with the Clifford gate set [99, 101]. As a competing gate of the qutrit T-gate $\Lambda_{2\pi/9}$ defined by Eq. (13) with $\theta = 2\pi/9$, it is desirable to compare the magic resource generating power between the qutrit N-gate and qutrit T-gate $\Lambda_{2\pi/9}$.

Let $\mathcal{E}_N(\rho) = N\rho N^\dagger$ be the corresponding unitary channel. After straightforward calculations, we obtain

$$M(\mathcal{E}_N(|\phi_i\rangle)) = \begin{cases} 3, & i = 1, 2, 3 \\ \frac{13}{3}, & i = 4, 5, \cdots, 12 \end{cases}$$

from which we readily obtain

$$M_{\max}(\mathcal{E}_N) = \frac{13}{3} \approx 4.3333,$$
$$M_{\text{ave}}(\mathcal{E}_N) = 4.$$

We see that

$$M_{\max}(\mathcal{E}_N) \approx 4.3333 < M_{\max}(\mathcal{E}_{\Lambda_{2\pi/9}}) \approx 4.4641,$$
$$M_{\text{ave}}(\mathcal{E}_N) = 4 < M_{\text{ave}}(\mathcal{E}_{\Lambda_{2\pi/9}}) \approx 4.0981.$$

Consequently, the qutrit T-gate $\Lambda_{2\pi/9}$ is more powerful in generating magic resource than the N-gate, which is consistent with the observation that the qutrit N-gate is less versatile than the qutrit T-gate in the sense that the qutrit Clifford+N unitaries form a strict subset of the Clifford+T unitaries in Ref. [101].

(3) Consider the unitary channel

$$\mathcal{E}_{H_3} = H_3 \rho H_3^\dagger$$

on a qutrit system associated with the diagonal unitary operator

$$H_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}, \tag{15}$$

which is a kind of simple ternary extension of the qubit Hadamard gate [95]. Direct calculations show that

$$M(\mathcal{E}_{H_3}(|\phi_i\rangle)) = \begin{cases} 5, & i = 1, 2 \\ 3, & i = 3 \\ 1 + \frac{2\sqrt{3}}{3} + 2\sqrt{2} \approx 4.9831, & i = 4, 7, 10 \end{cases}$$

and for all other i,

$$\begin{aligned} M(\mathcal{E}_{H_3}(|\phi_i\rangle)) &= 1 + \frac{1}{3}\left(\sqrt{3} + \sqrt{5}\right) \\ &\quad + \frac{1}{3}\left(\sqrt{17 - 6\sqrt{2}} + \sqrt{11 + 6\sqrt{2}}\right) \\ &\approx 4.7767. \end{aligned}$$

Consequently,

$$M_{\max}(\mathcal{E}_{H_3}) = 5,$$
$$M_{\text{ave}}(\mathcal{E}_{H_3}) \approx 4.7171.$$

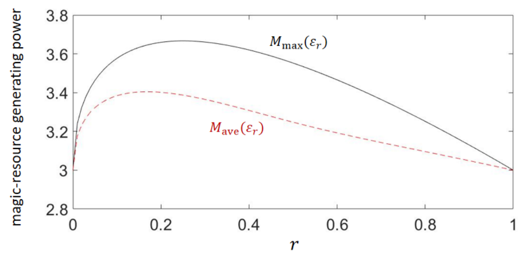

FIG. 5: The maximal and average magic-resource generating power $M_{\max}(\mathcal{E}_r)$ and $M_{\text{ave}}(\mathcal{E}_r)$ as functions of the parameter $r \in [0, 1]$.

(4) Consider the qutrit amplitude channel [96, 98]

$$\mathcal{E}_r = \sum_{i=0}^{2} K_i \rho K_i^\dagger$$

on a qutrit system with the Kraus operators

$$\begin{aligned} K_0 &= |0\rangle\langle 0| + \sqrt{r}\big(|1\rangle\langle 1| + |2\rangle\langle 2|\big), \\ K_1 &= \sqrt{1-r}|0\rangle\langle 1|, \qquad r \in [0,1] \\ K_2 &= \sqrt{1-r}|0\rangle\langle 2|. \end{aligned}$$

Direct calculations show that

$$M(\mathcal{E}_r(|\phi_i\rangle)) = \begin{cases} 3, & i = 1 \\ 1 + 2\sqrt{3r^2 - 3r + 1}, & i = 2, 3 \\ 3 - \frac{8}{3}\big(r - \sqrt{r}\big), & i = 4, \cdots, 12 \end{cases}$$

and consequently,

$$\begin{aligned} M_{\max}(\mathcal{E}_r) &= 3 - \frac{8}{3}\big(r - \sqrt{r}\big) \\ M_{\text{ave}}(\mathcal{E}_r) &= \frac{1}{3}\sqrt{3r^2 - 3r + 1} - 2\big(r - \sqrt{r}\big) + \frac{8}{3}. \end{aligned}$$

The behaviors of magic-resource generating power of the qutrit amplitude damping channel $\mathcal{E}_r$ are depicted in Fig. 5.

V. SUMMARY

In the stabilizer formalism, magic states constitute an important resource for fault-tolerant quantum computation. Consequently, it is necessary to generate magic states from stabilizer states via some methods. Quantum channels represent the most general physical transformations of quantum states. Different channels may have different effects in generating or destroying magic resource, and it is desirable to study this issue from a quantitative perspective in order to manipulate magic resource for quantum computation.

We have quantified magic-resource generating power of quantum channels by use of the quantifier of magic resource via characteristic functions. We have discussed the maximal generating power and the average generating power, and have evaluated these quantities for various quantum channels. In particular, we have shown that both the qubit T-gate and its qutrit extension are optimal in generating magic resource form stabilizer states in some classes of diagonal unitaries.

A related and pressing question arises naturally: Is there any operational significance for the maximal and average magic-resource generating power? This important issue is under further studies.

We have mainly treated single-partite systems, and it will be more interesting to study multi-partite systems and extend the results to the general case. Complete classification and catalogue of magic-resource generating power for all unitary gates is also a basic issue awaiting for investigations.

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