On Rainbow Antimagic Coloring of Graphs

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Abstract. Let G be a simple graph. For a bijective function $f : V(G) \to \{1,2,...,|V(G)|\}$, the edge weight for any $uv \in E(G)$ under f is $w_f(uv) = f(u) + f(v)$. A path P in a graph G is said to be a rainbow path, if for every two edges uv, $u'v' \in E(P)$, the edges weight satisfies $w_f(uv) \neq w_f(u'v')$. If for every two vertices u and v of G, there exists a rainbow path u-v, then f is called a rainbow antimagic coloring of graph G. The minimum number of colors induced by all edge weight required is called rainbow antimagic connection number, denoted by $(rac(G))$. In this research, we will find the rainbow antimagic connection number $rac(G)$ and new theorems on rainbow antimagic coloring of Double Quadrilateral Windmill Graph $DQ_{(n)}$, Double Quadrilateral Flower Graph $FDQ_{(m)}$, Split Graph of Star $Spl(K_{1,n})$ and Tunjung Graph Tj_n.
Keywords: rainbow antimagic coloring, rainbow antimagic connection number, graph coloring.

INTRODUCTION

Graph theory is one of the mathematical topics that first appeared in 1736. Leonhard Euler a mathematician from Switzerland, was the first to introduce this theory in his writings containing efforts to solve the Konigsberg bridge problem. In this study, the definition of a graph G is a set pair $(V(G),E(G))$, where $V(G)$ is a finite nonempty set of elements called vertex and $E(G)$ is a (possibly empty) set of unordered pairs (u,v) of vertex $u,v \in V(G)$ are referred to as edges [1] [2]. Then graph theory developed over time, until in 2008 Chartrand introduced a coloring called the rainbow connection. This rainbow connection is defined as coloring the edges of a graph such that no two edges in the path have the same color. The research goal of rainbow coloring is how to find the minimum number of colors called the rainbow connection number $rc(G)$ [3] [4] [5] [6]. Recently, the topic of rainbow connection has been combined with antimagic labeling. Antimagic labeling is defined as labeling (giving number) on vertices of graph such that there is no same weight for any two distinct edges. A graph is called an antimagic labeling if there exists a bijective function $f : E(G) \to \{1,2,...,|E(G)|\}$ such that all its vertex weights are different. The vertex weight of a vertex x denoted by $w(x)$ which is the number of labels of each edge associated with x, can be written as $w(x) = \sum e \in E(x) f(e)$. In this case, f is called antimagic labeling [7] [8] [9] [10].

Dafik et al. (2019), combined rainbow coloring and antimagic labeling to create a new topic called rainbow antimagic connection number. If for any two vertex u and v of a graph G, there exists a rainbow path uv, then the function f is called rainbow antimagic labeling of the graph G. If for every edge uv with weight color $w(uv)$, then it can be called rainbow antimagic coloring. The rainbow antimagic connection number is denoted by $rac(G)$, which is to find the minimum number of colors in a graph [11] [12] [13] [14]. The research in this article will find the rainbow antimagic connection number $rac(G)$ on Double Quadrilateral Windmill Graph $DQ_{(n)}$, Double Quadrilateral Flower Graph $FDQ_{(m)}$, Split Graph Of Star $Spl(K_{1,n})$ and Tunjung Graph Tj_n. The following are the Definitions, Lemma, and Theorem used in this research.

Definition 1 *[3] Let $c : E(G) \to 1,2,...,k$ be an edge k-coloring of a simple connected graph G where adjacent edges may have same color. If there is no two edges in a path of G that have same color, then the path is called a rainbow. The edge-colored graph G is rainbow connected if every two distinct vertices are connected by a rainbow path. The edge k-coloring in which G is rainbow-connected is called a rainbow k-coloring. The minimum number of color k to make G rainbow-connected is called the rainbow connection number of G and denoted by $rc(G)$.*

Definition 2 *[15] Let $G = (V,E)$ be simple connected graph. The graph G is called an antimagic if G has an antimagic labeling. A bijection $f : E \to 1,2,...,|E|$ is called an antimagic labeling if for every $u \in V(G)$ and weight $w(u) = \sum e \in E(u) f(e)$, where $E(u)$ is a set of incident edges to u, there is $w(u) \neq w(v)$ for two distenct vertices u and v.*

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Definition 3 *[13] Let G be a simple connected graph. For a bijection $f : V(G) \rightarrow 1,2,...,V(G)$, the associated weight of an edge $uv \in E(G)$ under f is $w_f(uv) \neq w_f(u'v')$. If for every two vertices u and v of G, there is a rainbow antimagic if G has a rainbow antimagic labeling.*

Lemma 1 *[3] Suppose G is a connected graph of size m, then $diam(G) \leq rc(G) \leq src(G) \leq m$, where $diam(G)$ is the diameter of G and m is the number of edges of G.*

Lemma 2 *[3] If G is a nontrivial connected graph of size m, then*

(a) src(G) = 1 if and only if G is a complete graph,

(b) rc(G) = 2 if and only if src(G) = 2,

(c) rc(G)=m if and only if G is a tree graph

Lemma 3 *[11] [12] [13] [14] Suppose G is an arbitrary connected graph. Suppose $rc(G)$ and $\Delta(G)$ respectively have the rainbow connection number of G and the maximum degree of G, such that $rac(G) \geq max(rc(G), \Delta(G))$.*

Lemma 4 *[3] Suppose G is a connected graph with $d(G) \geq 2$, then G is an interval graph, $diam(G) \leq rc(G) \leq diam(G) + 1$ while if G is an interval graph, $diam(G) = rc(G)$.*

Theorem 1 *[16] Let G be a Tunjung Graph, denoted by Tj_n. The rainbow connection number of tunjung graph, $rc(Tj_n) = 3$.*

THE RESULTS

In this section, we will show the rainbow antimagic coloring of double quadrilateral windmill graph $DQ_{(n)}$, Double Quadrilateral Flower Graph $FDQ_{(n)}$, Split Graph Of Star $Spl(K1,n)$, and Tunjung graph Tj_n.

We will show the cardinality of the Double Quadrilateral Windmill Graph (DQ_n). Suppose $DQ_{(n)}$ is a double quadrilateral windmill graph with $n \geq 2$. The vertex set and edge set of $DQ_{(n)}$ with $n \geq 2$ are $DQ_{(n)}$ have $V(DQ_{(n)}) = \{u_0\} \cup \{v_{i,j} | 1 \leq i \leq n, j = 1,2,3\} \cup \{w_{i,j} | 1 \leq i \leq n, j = 1,2\}$ and $E(DQ_{(n)}) = \{u_0 v_{i,j} | 1 \leq i \leq n, j = 1,2,3\} \cup \{v_{i,2j-1} w_{i,j} | 1 \leq i \leq n, j = 1,2\} \cup \{w_{i,j} v_{i,2} | 1 \leq i \leq n, j = 1,2\}$ then obtained $|V(DQ_{(n)})| = 5k + 1$ and $|E(DQ_{(n)} = 7k)|$ [17].

Theorem 2 *Let G be a double quadrilateral windmill graph, denoted by $DQ_{(n)}$. The rainbow connection number of double quadrilateral windmill graph, $rc(DQ_{(n)}) = 4$.*

Suppose $DQ_{(n)}$ is a double quadrilateral windmill graph with $n \geq 2$. Firstly, will be shown the lower bound of double quadrilateral windmill graph $DQ_{(n)}$. Double quadrilateral windmill graph $DQ_{(n)}$ has $diam(DQ_{(n)}) = 4$. Based on Lemma 1, it is found $rc(DQ_n) \geq diam(DQ_n)$, then $rc(DQ_{(n)}) \geq 4$. Second, will be proven upper bound by defining a function $c : E(DQ_n) \rightarrow 1,2,...,k$ as follows:

$f(v_i^2 w_i^j) = 1$, for $1 \leq i \leq n$, $1 \leq j \leq 2$
$f(v_i^1 w_i^1) = 2$, for $1 \leq i \leq n$
$f(v_i^3 w_i^2) = 2$, for $1 \leq i \leq n$
$f(u_0 v_i^j) = 3$, for $j = 1$ and $j = 3$, $1 \leq i \leq n$
$f(u_0 v_i^2) = 4$, for $1 \leq i \leq n$

Based on the coloring function above, we get $|f(V)| = 4$ then based on the upper bound and lower bound we get $4 \leq rc(DQ_n) \leq 4$ then $rc(DQ_n) = 4$. the rainbow path of $u - v$ is found which can be seen in table 1. The illustration of rainbow coloring can be seen in Figure 1 which is an example of rainbow coloring in the double quadrilateral windmill graph $DQ_{(n)}$. In Figure 1, there is also a rainbow coloring from the double quadrilateral windmill graph $rc(DQ_{(4)}) = 4$.

TABLE 1. Rainbow connection from u to v in $DQ_{(n)}$

Case	u	v	Rainbow Connection	Condition
1	v_i^2	v_k^2	u_0, v_k^1, w_k^1, v_k^2	$i \neq k, 1 \leq i \leq n, 1 \leq k \leq n$
2	w_i^1	w_i^2	v_i^2, u_0, v_i^3, w_i^2	$1 \leq i \leq n$
3	w_i^1	w_k^1	v_i^2, u_0, v_k^1, w_k^1	$i \neq k, 1 \leq i \leq n, 1 \leq k \leq n$
4	w_i^1	v_k^2	v_i^1, u_0, v_k^2	$i \neq k, 1 \leq i \leq n, 1 \leq k \leq n$
5	w_i^2	v_k^2	v_k^3, u_0, v_k^2	$i \neq k, 1 \leq i \leq n, 1 \leq k \leq n$

Theorem 3 *Let G be a double quadrilateral windmill graph, denoted by $DQ_{(n)}$. The rainbow antimagic connection number of double quadrilateral windmill graph, $rac(DQ_{(n)}) = 3n$.*

Suppose $DQ_{(n)}$ is a double quadrilateral windmill graph with $n \geq 2$. Firstly, will be shown the lower bound of double quadrilateral windmill graph $DQ_{(n)}$. Double quadrilateral windmill graph $DQ_{(n)}$ has $\Delta(DQ_{(n)}) = 3n$ and has rainbow connection number $rc(DQ_{(n)}) = 4$. Based on Lemma 3, it is found $rac(DQ_{(n)}) \geq max(rc(DQ_{(n)}), \Delta(DQ_{(n)}))$, then $rac(DQ_{(n)}) \geq max(4, 3n)$. Second, there will be shown the upper bound of Double quadrilateral windmill graph $rac(DQ_{(n)}) \geq 3n$. Next we will prove that $rac \leq 3n$ by defining the vertex label function $f(V(G)) \to 1, 2, 3, ...|V(G)|$ as follows:

$f(u_0) = 2n$
$f(w_1^1) = 2n + i$
$f(w_i^2) = 2n - ij + 1$, for $2 \leq i \leq n$
$f(v_i^3) = 2n + 3i + 1$, for $1 \leq i \leq n$
$f(v_i^2) = 2n + 3i$, for $1 \leq i \leq n$
$f(w_i^1) = 2n - ij - (i - 2)$, for $2 \leq i \leq n$
$f(v_1^j) = 2n + 3i - 1$, for $1 \leq i \leq n$

Based on a vertex label function, the edge weight of double quadrilateral windmill graph is as follows:

$w(u_0 v_i^1) = 4n + 3i - 1$, for $1 \leq i \leq n$
$w(u_0 v_i^3) = 4n + 3i + 1$, for $1 \leq i \leq n$
$w(v_i^2 w_i^2) = 4n + 3i - ij + 1$, for $2 \leq i \leq n$
$w(v_i^1 w_i^1) = 4n + i + 1$, for $2 \leq i \leq n$
$w(v_i^3 w_i^2) = 4n + i + 2$, for $2 \leq i \leq n$
$w(u_0 v_i^2) = 4n + 3i$, for $1 \leq i \leq n$
$w(v_1^2 w_1^1) = 4n + 3i + 1$
$w(v_1^1 w_1^1) = 4n + 4i - 1$

The next step is to see that the edge weights induce rainbow antimagic coloring on the double quadrilateral windmill graph (DQ_n). Let $W(DQ_n)$ be the set of the edge weights of graph. To determine the upper bound, we should find the cardinality of the set $W(DQ_n)$. In this step, we use an arithmetic sequence to know the cardinality of the set. To show the number of edge weights in the double quadrilateral windmill graph (DQ_n) we can classify the edges into two types namely edges connected to vertex u_0 which we denote by W_1 and edges not connected to vertex u_0 which we denoted by $W_2 = W_3, W_4, W_5, W_6, W_7$. First we will calculate the sum of the weights of $|W_1|$, the weights of $|W_1|$ can be described through the following arithmetic sequence $\{4n+2, 4n+3, 4n+4, ..., 7n+1\}$.

$$a + (n-1)b = U_n$$
$$4n + 2 + (|W_1| - 1)1 = 7n + 1$$
$$|W_1| = 3n$$

From the first step we get the number of $|W_1|$ is $3n$. Second, we will show that $W_2 \subset W_1$. Based on the edge weight function we will show that $W_3 \subset W_1$, $W_4 \subset W_1$, $W_5 \subset W_1$, $W_6 \subset W_1$, and $W_7 \subset W_1$. $W_3 = W(v_1^2 w_1^1) = 4n + 3i + 1$, then we get the edge weight $W_3 = \{4n+5\}$. $W_4 = W(v_i^2 w_i^2) = 4n + 3i - ij + 1$, for $2 \leq i \leq n$, then we get the set of edge weight $W_4 = \{4n+3, 4n+4, 4n+5, ..., 5n+1\}$. $W_5 = W(v_1^1 w_1^1) = 4n + 4i - 1$, then we get the edge weight $W_5 = \{4n - 3\}$. $W_6 = W(v_i^1 w_i^1) = 4n + i + 1$, for $2 \leq i \leq n$, then we get the set of edge weight $W_6 = \{4n+3, 4n+4, 4n+5, ..., 5n+1\}$. $W_7 = W(v_i^3 w_i^2) = 4n + i + 2$, for $2 \leq i \leq n$, then we get the set of edge weight $W_7 = \{4n+4, 4n+5, 4n+6, ..., 5n+2\}$. From the second step we get that $W_3 \subset W_1$, $W_4 \subset W_1$, $W_5 \subset W_1$, $W_6 \subset W_1$, and $W_7 \subset W_1$. Based on the set of edge weights, the number of edge weights is $3n$, so $rac(DQ_n) \leq 3n$. We know from the upper bound and lower bound that $3n \leq rac(DQ_n) \leq 3n$ we get that the rainbow antimagic connection number of double quadrilateral windmill graph is $rac(DQ_n) = 3n$.

Based on the edge weight above, the rainbow path of u-v is found which can be seen in table 2, and illustration of rainbow antimagic coloring of graph DQ_n is in Figure 2, with $rac(DQ_4) = 3n = 3.4 = 12$.

TABLE 2. Rainbow connection from u to v in $DQ_{(n)}$

Case	u	v	Rainbow Connection	Condition
1	v_i^2	v_k^2	u_0, v_k^2	$i \neq k, 1 \leq i \leq n, 1 \leq k \leq n$
2	w_i^1	w_k^2	v_i^2, w_i^2	$1 \leq i \leq n, 1 \leq k \leq n, i = k$
3	w_i^1	w_k^2	v_i^1, u_0, v_k^3, w_k^2	$1 \leq i \leq n, 1 \leq k \leq n, i \neq k$
4	w_i^1	w_k^1	v_i^1, u_0, v_k^2, w_k^1	$1 \leq i \leq n, 1 \leq k \leq n$
5	w_i^2	w_k^2	v_i^2, u_0, v_k^3, w_k^2	$1 \leq i \leq n, 1 \leq k \leq n, i \neq k$

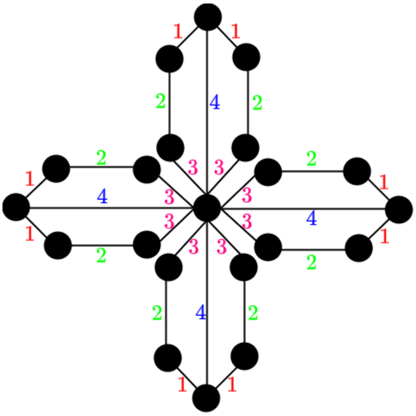

FIGURE 1. Rainbow coloring DQ_4

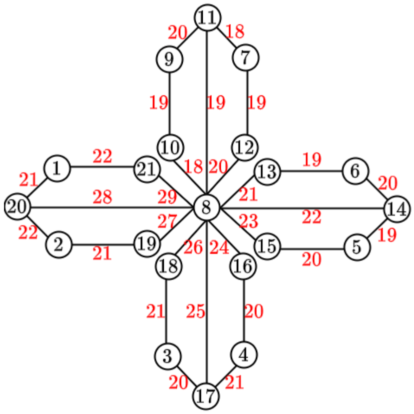

FIGURE 2. Rainbow antimagic coloring DQ_4

We will show the cardinality of the Double Quadrilateral Flower Graph (FDQ_n). Suppose $FDQ_{(n)}$ is a double quadrilateral flower graph with $n \geq 2$. The vertex set and edge set of $FDQ_{(n)}$ with $n \geq 2$ are $FDQ_{(n)}$ have $FDQ_{(n)}$ have $V(FDQ(n)) = \{u_0\} \cup \{u_i | 1 \leq i \leq 2n+1\} \cup \{v_i^j | 1 \leq i \leq n, j = 1,2\}$ and $E(FDQ_{(n)}) = \{u_0 u_i | 1 \leq i \leq 2n+1\} \cup \{u_{2i-1} v_i^j | 1 \leq i \leq n, j = 1\} \cup \{v_i^j u_{2i} | 1 \leq i \leq n, j = 1,2\} \cup \{u_{2i+1} v_i^j | 1 \leq i \leq n, j = 2\}$ then obtained $|V(FDQ_{(n)})| = 4n+2$ and $|E(FDQ_{(n)} = 6n+1)|$ [18].

Theorem 4 *Let G be a Double Quadrilateral Flower Graph, denoted by $FDQ_{(n)}$. The rainbow connection number of Double Quadrilateral Flower Graph, $rc(FDQ_{(n)}) = 4$.*

Suppose $FDQ_{(n)}$ is a double quadrilateral flower graph with $n \geq 2$. Firstly, will be shown the lower bound of double quadrilateral flower graph $FDQ_{(n)}$. Double quadrilateral flower graph $FDQ_{(n)}$ has $diam(FDQ_{(n)}) = 4$. Based on Lemma 1, it is found $rc(FDQ_{(n)}) \geq diam(FDQ_{(n)})$, then $rc(FDQ_{(n)}) \geq 4$. Second, will be proven upper bound by defining a function $c : E(FDQ_n) \rightarrow 1, 2, ..., k$ as follows:

$f(u_{2i} v_i^j) = 1$, for $1 \leq i \leq n$, $j = 1,2$
$f(u_{2i+1} v_i^2) = 2$, for $1 \leq i \leq n$, $j = 2$
$f(u_0 u_i) = 4$, for $i = even$
$f(u_{2i-1} v_i^1) = 2$, for $1 \leq i \leq n$
$f(u_0 u_i) = 3$, for $i = odd$

Based on the rainbow connection function above, we get $|f(V)| = 4$ then based on the upper bound and lower bound we get $4 \leq rc(FDQ_n) \leq 4$ then $rc(FDQ_n) = 4$. The rainbow path of $u - v$ is found which can be seen in table 3. The illustration of rainbow coloring can be seen in figure 3 which is an example of rainbow coloring in the double quadrilateral flower graph $FDQ_{(n)}$. In figure 3, there is also a rainbow coloring from the double quadrilateral flower graph $rc(FDQ_{(4)}) = 4$.

TABLE 3. Rainbow connection from u to v in $FDQ_{(n)}$

Case	u	v	Rainbow Connection	Condition
1	u_2i	u_2k	$u_0, u_{2k+1}, v_k^j, u_{2k}$	$i \neq k, 1 \leq i \leq n, 1 \leq k \leq n, j = 1,2$
2	v_i^j	v_k^j	$u_{2i-1}, u_0, u_{2k}, v_k^j$	$i \neq k, 1 \leq i \leq n, 1 \leq k \leq n, j = 1$
3	v_i^j	v_k^j	$u_{2i+1}, u_0, u_{2k}, v_k^j$	$i \neq k, 1 \leq i \leq n, 1 \leq k \leq n, j = 2$
4	v_i^j	v_k^l	$u_{2i}, u_0, u_{2k+1}, v_k^l$	$i \neq k, 1 \leq i \leq n, 1 \leq k \leq n, j = 1, l = 2$
5	u_{2i}	v_k^j	u_0, u_{2k-1}, v_k^j	$i \neq k, 1 \leq i \leq n, 1 \leq k \leq n, j = 1$
6	u_{2i}	v_k^j	u_0, u_{2k+1}, v_k^j	$i \neq k, 1 \leq i \leq n, 1 \leq k \leq n, j = 2$

Theorem 5 *Let G be a Double Quadrilateral Flower Graph, denoted by $FDQ_{(n)}$. The rainbow antimagic connection number of Double Quadrilateral Flower Graph, $rac(FDQ_{(n)}) = 2n+1$.*

Suppose $FDQ_{(n)}$ is a double quadrilateral flower graph with $n \geq 2$. Firstly, will be shown the lower bound of double quadrilateral flower graph $FDQ_{(n)}$. Double quadrilateral flower graph $FDQ_{(n)}$ has $\Delta(FDQ_{(n)}) = 2n+1$ and has rainbow connection number $rc(FDQ_{(n)}) = 4$. Based on Lemma 3, it is found $rac(FDQ_{(n)}) \geq max(rc(FDQ_{(n)}), \Delta(FDQ_{(n)}))$, then $rac(FDQ_{(n)}) \geq max(4, 2n+1)$. Second, there will be shown the upper bound of Double quadrilateral flower graph $FDQ_{(n)} \geq 2n+1$. Next we will prove that $rac \leq 2n+1$ by defining the vertex label function $f(V(G)) \rightarrow 1,2,3,...,|V(G)|$ as follows:

$f(u_0) = 2n$
$f(v_i^2) = 2i$, for $1 \leq i \leq n-1$
$f(u_i) = 4n+3-i$, for $1 \leq i \leq 2n-2$
$f(v_i^1) = 2i-1$, for $1 \leq i \leq n$
$f(v_n^2) = 2i+4$
$f(u_i) = 4n+2-i$, for $2n-1 \leq i \leq 2n+1$

Based on a vertex label function, the edge weight of Double quadrilateral flower graph is as follows:

$w(u_i v_k^1) = 4n+2k-i+2$, for $1 \leq i \leq 2n-3$, and $1 \leq k \leq n-1$
$w(u_i v_k^1) = 4n+2k-i+2$, for $i = even$ and $1 \leq k \leq n-1$
$w(u_i v_k^2) = 4n+2k-i+3$, for $i = even$ and $1 \leq k \leq n-1$
$w(u_i v_k^2) = 4n+2k-i+3$, for $1 \leq i \leq 2n-3$ and $1 \leq k \leq n-2$
$w(u_0 u_i) = 6n-i+3$, for $1 \leq i \leq 2n-2$
$w(u_0 u_i) = 6n-i+2$, for $2n-1 \leq i \leq 2n+1$
$w(u_{2n+1} v_n^2) = 4n+5$,
$w(u_{2n} v_n^2) = 4n+6$,
$w(u_{2n} v_n^1) = 4n+1$

The next step is to see that the edge weights induce rainbow antimagic coloring on the double quadrilateral flower graph (FDQ_n). Let $W(FDQ_n)$ be the set of the edge weights of graph. To determine the upper bound, we should find the cardinality of the set $W(FDQ_n)$. In this step, we use an arithmetic sequence to know the cardinality of the set. To show the number of edge weights in the double quadrilateral flower graph (FDQ_n) we can classify the edges into two types namely edges connected to vertex u_0 which we denoted by $W_1 = \{W_2, W_3\}$ and edges not connected to vertex u_0 which we denoted by $W_4 = \{W_5, W_6, W_7, W_8, W_9, W_{10}, W_{11}\}$. First we will calculate the sum of the weights of $|W_1|$, The weights of the $W_1 = \{W_2, W_3\}$ can be described through the following arithmetic sequences $W_2 = \{6n+2, 6n+1, 6n, 6n-1, ..., 4n+5\}$ for $1 \leq i \leq 2n-2$ and $W_3 = \{4n+3, 4n+2, 4n+1\}$ for $2n-1 \leq i \leq 2n+1$.

$U_n = a+(n-1)b$
$4n+5 = 6n+2+(|W_2|-1)(-1)$
$|W_2| = 2n-2$

$U_n = a+(n-1)b$
$4n+1 = 4n+3+(|W_3|-1)(-1)$
$|W_3| = 3$

From the first step we get the number of edge weights $|W_1|$ is $|W_2|+|W_3| = 2n+1$. Second, we will show that $W_4 \subset W_1$. Based on the edge weight function we will show that $W_5 \subset W_1$, $W_6 \subset W_1$, $W_7 \subset W_1$, $W_8 \subset W_1$, $W_9 \subset W_1$, $W_{10} \subset W_1$, and $W_{11} \subset W_1$. $W_5 = W(u_i v_k^1) = 4n+2k-i+2$, for $1 \leq i \leq 2n-3$, and $1 \leq k \leq n-1$, then we get the edge weight $W_5 = \{4n+3\}$. $W_6 = W(u_i v_k^2) = 4n+2k-i+3$, for $1 \leq i \leq 2n-3$ and $1 \leq k \leq n-2$, then we get the edge weight $W_6 = \{4n+2\}$. $W_7 = W(u_{2n+1} v_n^2) = 4n+5$, then we get the edge weight $W_7 = \{4n+5\}$. $W_8 = W(u_i v_k^1) = 4n+2k-i+2$, for $i = even$ and $1 \leq k \leq n-1$ then we get the edge weight $W_8 = \{4n+3\}$, $W_9 = W(u_i v_k^2) = 4n+2k-i+3$, for $i = even$ and $1 \leq k \leq n-1$ then we get the edge weight $W_9 = \{4n+2\}$. $W_{10} = W(u_{2n}v_n^1) = 4n+1$, then we get the edge weight $W_{10} = \{4n+1\}$. $W_{11} = W(u_{2n}v_n^2) = 4n+6$, then we get the edge weight $W_{11} = \{4n+6\}$. From the second step we get that $W_5 \subset W_1$, $W_6 \subset W_1$, $W_7 \subset W_1$, $W_8 \subset W_1$, $W_9 \subset W_1$, $W_{10} \subset W_1$, and $W_{11} \subset W_1$. Based on the set of edge weights, the number of edge weights is $2n+1$, so $rac(FDQ_n) \leq 2n+1$. We know from the upper bound and lower bound that $2n+1 \leq rac(FDQ_n) \leq 2n+1$ we get that the rainbow antimagic connection number of double quadrilateral flower graph is $rac(FDQ_n) = 2n+1$.

Based on the edge weight above, the rainbow path of u-v is found which can be seen in table 4 and for illustration of rainbow antimagic coloring of graph FDQ_n is in Figure 4, with $rac(FDQ_5) = 2n+1 = 2.5+1 = 11$.

TABLE 4. Rainbow connection from u to v in $FDQ_{(n)}$

Case	u	v	Rainbow Connection	Condition
1	u_i	u_k	u_0, u_k	$i = even, k = even$ $1 \leq i \leq n, 1 \leq k \leq n,$
2	u_i	v_k^j	u_0, u_{2i-1}, v_k^j	$i = even, 1 \leq i \leq n, 1 \leq k \leq n, j = 1$
3	u_i	v_k^j	u_0, u_{2i+1}, v_k^j	$i = even, 1 \leq i \leq n, 1 \leq k \leq n, j = 2$
4	v_i^j	v_i^k	u_{2i}	$1 \leq i \leq n, j = 1, k = 2$
5	v_i^j	v_l^k	$u_{2i}, u_0, u_{2l}, v_l^k$	$1 \leq i \leq n, j = 1, 1 \leq l \leq n, k = 2$
6	v_i^j	v_l^j	$u_{2i-1}, u_0, u_{2l}, v_l^j$	$1 \leq i \leq n, j = 1, 1 \leq l \leq n$
7	v_i^j	v_l^k	$u_{2i}, u_0, u_{2l}, v_l^k$	$1 \leq i \leq n, j = 2, 1 \leq l \leq n, k = 1$
8	v_i^j	v_l^j	$u_{2i+1}, u_0, u_{2l}, v_l^j$	$1 \leq i \leq n, j = 2, 1 \leq l \leq n$

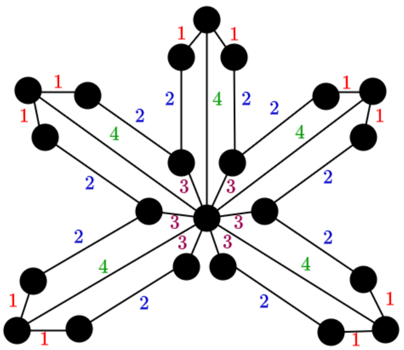

FIGURE 3. Rainbow coloring FDQ_5

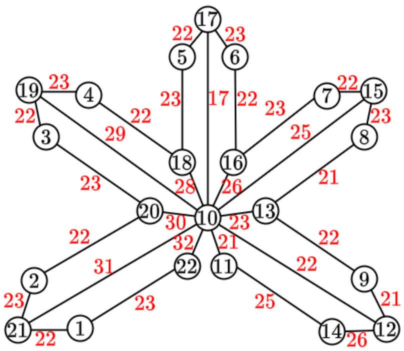

FIGURE 4. Rainbow antimagic coloring FDQ_5

We will show the cardinality of the split graph of star $(Spl(K_{1,n}))$. Suppose $(Spl(K_{1,n}))$ is a split graph of star with $n \geq 3$. The vertex set and edge set of $Spl(K_{1,n})$ with $n \geq 3$ are $Spl(K_{1,n})$ have $V(Spl(K_{1,n})) = \{v\} \cup \{v_i\} \cup \{v'\} \cup \{v'_i\}$ for $1 \leq i \leq n$ and $E(Spl(K_{1,n})) = \{vv_i | 1 \leq i \leq n\} \cup \{v_i v' | 1 \leq i \leq n\} \cup \{v'v'_i | 1 \leq i \leq n\}$ then obtained $|V(Spl(K_{1,n})| = 2n+2$ and $|E(Spl(K_{1,n})) = 3n|$ [19].

Theorem 6 *Let G be a Split Graph Of Star, denoted by $Spl(K_{1,n})$. The rainbow connection number of Split Graph Of Star, $rc(Spl(K_{1,n})) = n$. For every integer $n \geq 3$.*

Suppose $Spl(K_{1,n})$ is a split graph of star with $n \geq 3$. Firstly, will be shown the lower bound of split graph of star $Spl(K_{1,n})$. Split graph of star $Spl(K_{1,n})$ has $diam(Spl(K_{1,n})) = 3$. Based on Lemma 1, it is found $diam(Spl(K_{1,n})) \geq rc(Spl(K_{1,n}))$, then $rc(Spl(K_{1,n})) \geq 3$. Secondly, this graph will not be able to satisfy the upper bound in Lemma 1 which is $Spl(K_{1,n}) \geq 3$, because the split star graph has n vertices of degree one, then like the tree graph in Lemma 2, all vertices of degree one will have different colors. So a split graph of star will have n different colors. Will be proven upper bound by defining a function $c : E(Spl(K_{1,n})) \to 1,2,...,k$ as follows:

$f(v'v_i) = 1$ for $1 \leq i \leq n$
$f(v'v_i) = 3$ for $i = \lceil \frac{n}{2} \rceil$
$f(v'v'_i) = i$ for $1 \leq i \leq n$
$f(vv_i) = 2$ for $1 \leq i \leq n$
$f(vv_i) = 4$ for $i = \lceil \frac{n}{2} \rceil$

From the above function we get a rainbow connection number of n because the set of edges $f(v'v_i) = 1$, $f(vv_i) = 2$, $f(v'v_i) = 3$, $f(vv_i) = 4$ will always equal the set of edges $f(v'v'_i) = i$ when $i = 1,2,3,4$. Based on the rainbow connection function above, we get $|f(V)| = n$ then based on the upper bound and lower bound we get $n \leq rc(spl(K_{1,n})) \leq n$ then $rc(spl(K_{1,n})) = n$. The rainbow path of u-v is found which can be seen in table 5. The illustration of rainbow coloring can be seen in Figure 5 which is an example of rainbow coloring in the split graph of star $spl(K_{1,n})$. In Figure 5, there is also a rainbow coloring from the split graph of star $rc(spl(K_{1,n})) = n$.

TABLE 5. Rainbow connection from u to v in $spl(K_{1,n})$

Case	u	v	Rainbow Connection	Condition
1	v	v'_i	v_i, v', v'_i	$1 \leq i \leq n$, $i \neq 1,2,3,4$
2	v	v'_i	$v_{\lceil \frac{n}{2} \rceil}, v', v'_i$	$1 \leq i \leq n$, $i = 1,2$
3	v	v'_i	$v_{i \neq \lceil \frac{n}{2} \rceil}, v', v'_i$	$1 \leq i \leq n$, $i = 3,4$
4	v'_i	v'_k	v', v'_i	$i \neq k, 1 \leq i \leq n, 1 \leq k \leq n$
5	v_i	v_k	$v', v_{\lceil \frac{n}{2} \rceil}, v, v_k$	$i \neq k, 1 \leq i \leq n, 1 \leq k \leq n$

Theorem 7 *Let G be a Split Graph Of Star, denoted by $Spl(K_{1,n})$. The rainbow antimagic connection number of Split Graph Of Star, $rac(Spl(K_{1,n})) = 2n$. For every integer $n \geq 3$.*

Suppose $Spl(K_{1,n})$ is a split graph of star with $n \geq 3$. Firstly, will be shown the lower bound of split graph of star $Spl(K_{1,n})$. Split graph of star $Spl(K_{1,n})$ has $\Delta(Spl(K_{1,n})) = 2n$ and has rainbow connection number $rc(Spl(K_{1,n})) = n$. Based on Lemma 3, it is found $rac(Spl(K_{1,n})) \geq max(rc(Spl(K_{1,n})), \Delta(Spl(K_{1,n})))$, then $rac(Spl(K_{1,n})) \geq max(n, 2n)$. Second, there will be shown the upper bound of split graph of star $Spl(K_{1,n}) \geq 2n$. Next we will prove that $rac(spl(K_{1,n})) \leq 2n$ by defining the vertex label function $f(V(G)) \to 1,2,3,...,|V(G)|$ as follows:

$f(v) = 2$
$f(v_i) = 2i+1$ for $1 \leq i \leq n$
$f(v') = 1$
$f(v'_i) = 2i+2$ for $1 \leq i \leq n$

Based on a vertex label function, the edge weight of split graph of star is as follows:

$w(vv_i) = 2i+3$ for $1 \leq i \leq n$
$w(v'v'_i) = 2i+3$ for $1 \leq i \leq n$
$w(v'v_i) = 2i+2$ for $1 \leq i \leq n$

The next step is to see that the edge weights induce rainbow antimagic coloring on the split graph of star $Spl(K_{1,n})$. Let $W(Spl(K_{1,n}))$ be the set of the edge weights of graph. To determine the upper bound, we should find the cardinality of the set $W(Spl(K_{1,n}))$. Based on the edge weight function of the split graph of star, we get that $W(vv_i)$ and $W(v'v'_i)$ have the same weight as n. While for the weight function $W(v'v_i)$ has a different edge weight with another n. So the number of edge weights on the split graph of star $|W(Spl(K_{1,n}))| = 2n$. Based on the set of edge weights, the number of edge weights is $2n$, so $rac(Spl(K_{1,n})) \leq 2n$. We know from the upper bound and lower bound that $2n \leq rac(Spl(K_{1,n})) \leq 2n$ we get that the rainbow antimagic connection number of split graph of star is $rac(Spl(K_{1,n})) = 2n$.

Based on the edge weight above, the rainbow path of u-v is found which can be seen in table 6 and for illustration of rainbow antimagic coloring of graph $Spl(K_{1,n})$ is in Figure 4, with $rac(Spl(K_{1,7})) = 2n = 14$.

TABLE 6. Rainbow connection from u to v in $spl(K_{1,n})$

Case	u	v	Rainbow Connection	Condition
1	v	v'_i	v_i, v', v'_i	$1 \leq i \leq n$
2	v'_i	v'_k	v', v'_i	$i \neq k, 1 \leq i \leq n, 1 \leq k \leq n$

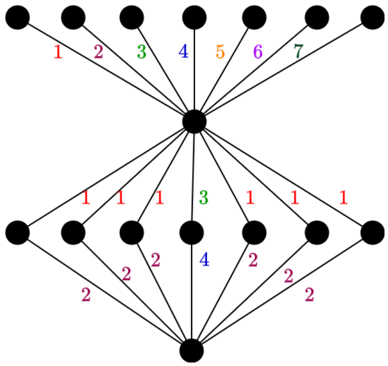

FIGURE 5. Rainbow coloring $Spl(K_{1,n})$

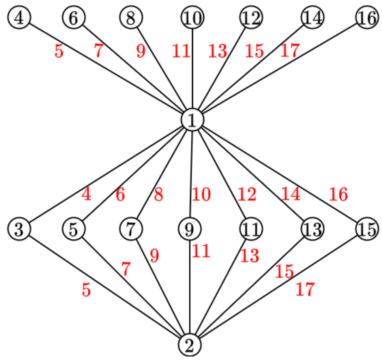

FIGURE 6. Rainbow antimagic coloring $Spl(K_{1,n})$

We will show the cardinality of the Tunjung Graph Tj_n. Suppose (Tj_n) is a tunjung graph with $n \geq 3$. The vertex set and edge set of (Tj_n) with $n \geq 3$ are (Tj_n) have $V(Tj_n) = \{c\} \cup \{x_i | 1 \leq i \leq n\} \cup \{y_i | 1 \leq i \leq n\} \cup \{z_i | 1 \leq i \leq n\}$ and $E(Tj_n) = \{cx_i | 1 \leq i \leq n\} \cup \{cy_i | 1 \leq i \leq n\} \cup \{cz_i | 1 \leq i \leq n\} \cup \{x_i y_i | 1 \leq i \leq n\} \cup \{y_i z_i | 1 \leq i \leq n\} \cup \{x_i x_{i+1} | 1 \leq i \leq n-1\} \cup \{x_n x_1\} \cup \{y_i y_{i+1} | 1 \leq i \leq n-1\} \cup \{y_n y_1\}$then obtained $|V(Tj_n)| = 3n+1$ and $|E(Tj_n)| = 7n$ [20].

Theorem 8 *Let G be a Tunjung Graph, denoted by Tj_n. The rainbow antimagic connection number of Tunjung Graph, $rac(Tj_n) = 3n$.*

Suppose Tj_n is a tunjung graph with $n \geq 4$. Firstly, will be shown the lower bound of tunjung graph Tj_n. Tunjung graph Tj_n has $\Delta(Tj_n) = 3n$ and based thoerema 1 has rainbow connection number $rc(Tj_n) = 3$. Based on Lemma 3, it is found $rac(Tj_n) \geq max(rc(Tj_n), \Delta(Tj_n)$, then $rac(Tj_n) \geq max(3, 3n)$. Second, there will be shown the upper bound of tunjung graph $Tj_n \geq 3n$. Next we will prove that $rac \leq 3n$ by defining the vertex label finction $f(V(G)) \to 1, 2, 3, ..., |V(G)|$ as follows:

$$f(c) = 3,\ f(x_i) = \begin{cases} \frac{3i+1}{2}, i = odd \\ \frac{6n-3i+4}{2}, i = even \end{cases},\ f(y_i) = \begin{cases} \frac{6n-3i-1}{2}, i = odd, i \neq n \\ \frac{3i+2}{2}, i = even, i \neq n \\ 1, i = n \end{cases},\ f(z_i) = \begin{cases} \frac{3i+9}{2}, i = odd, i \neq n \\ \frac{6n-3i+6}{2}, i = even, i \neq n \\ 3n+1, i = n \end{cases}$$

Based on a vertex label function above, then it will be shown the edge weight of tunjung graph is as follows:

$$W(cx_i) = \begin{cases} \frac{3i+7}{2}, & i = odd \\ \frac{6n-3i+10}{2}, & i = even \end{cases} \qquad W(cy_i) = \begin{cases} \frac{6n-3i+5}{2}, & i = odd, i \neq n \\ \frac{3i+8}{2}, & i = even, i \neq n \\ 4, & i = n \end{cases}$$

$$W(cz_i) = \begin{cases} \frac{3i+15}{2}, & i = odd, i \neq n \\ \frac{6n-3i+12}{2}, & i = even, i \neq n \\ 3n+4, & i = n \end{cases} \qquad W(x_i y_i) = \begin{cases} 3n, & i = odd, i \neq n \\ 3n+3, & i = even, i \neq n \\ \frac{3n+3}{2}, & i = n, n = odd \\ \frac{3n+6}{2}, & i = n, n = even \end{cases}$$

$$W(y_i z_i) = \begin{cases} 3n+4, & i \neq n \\ 3n+2, & i = n \end{cases} \qquad W(x_i x_{i+1}) = \begin{cases} 3n+1, & i = odd, i \neq n \\ 3n+4, & i = even, i \neq n \end{cases}$$

$$W(x_n x_1) = \begin{cases} \frac{3n+5}{2}, & n = odd \\ \frac{3n+8}{2}, & n = even \end{cases} \qquad W(y_i y_{i+1}) = \begin{cases} 3n+2, & i = odd, i \neq n \\ 3n-1, & i = even, i \neq n \end{cases}$$

$$W(y_n y_1) = \begin{cases} \frac{3n+1}{2}, & n = odd \\ \frac{3n+4}{2}, & n = even \end{cases}$$

The next step is to see that the edge weights induce rainbow antimagic coloring on the tunjung graph (Tj_n). Let $W(Tj_n)$ be the set of the edge weights of graph. To determine the upper bound, we should find the cardinality of the set $W(Tj_n)$. In this step, we use an arithmetic sequence to know the cardinality of the set. To show the number of edge weights in the tunjung graph (Tj_n) we can classify the edges into two types namely edges connected to vertex c which we denoted by $|W_1|$ and edges not connected to vertex c which we denoted by $W_2 = \{W_3, W_4, W_5, W_6, W_7, W_8, W_9, W_{10}, W_{11}, W_{12}, W_{13}, W_{14}, W_{15}, W_{16}\}$. First we will calculate the sum of the weights of the $|W_1|$. Based on the cardinality of the edge set, it is known that all vertices in the tunjung graph are adjacent to vertex c and $f(c) = 3$ with labeling using a bijective function then the $|W_1|$ will always start from 4 to $3n+4$ and it is not possible to have an edge weight of 6 for every edge connected to vertex c. From this analysis, the number of $|W_1|$ can be found using the following arithmetic sequence $W_1 = \{4,5,6,7,...,3n,3n+1,3n+2,3n+3,3n+4\}$.

$$U_n = a+(n-1)b$$
$$3n+4 = 4+(|W_1|-1)1$$
$$3n+1 = |W_1|$$

From the first step we get the number of $|W_1|$ is $3n+1$, but the result obtained in the first step must be minus 1 because the edge weight 6 is not possible. So the sum of $|W_1|$ is $3n$. Second, we will show that $W_2 \subset W_1$. $W_3 = W(x_iy_i) = 3n$ for $i = odd$ and $i \neq n$ then we get the edge weight $W_3 = \{3n\}$. $W_4 = W(x_iy_i) = 3n+3$ for $i = even$ and $i \neq n$ then we get the edge weight $W_4 = \{3n+3\}$. $W_5 = W(x_ny_n) = \frac{3n+3}{2}$ for $n = odd$ then we get the edge weight $W_5 = \{\frac{3n+3}{2}\}$. $W_6 = W(x_ny_n) = \frac{3n+6}{2}$ for $n = even$ then we get the edge weight $W_6 = \{\frac{3n+6}{2}\}$. $W_7 = W(y_iz_i) = 3n+4$ for $i \neq n$ then we get the edge weight $W_7 = \{3n+4\}$. $W_8 = W(y_nz_n) = 3n+2$ for $i = n$ then we get the edge weight $W_8 = \{3n+2\}$. $W_9 = W(x_ix_{i+1}) = 3n+1$ for $i = odd$ and $i \neq n$ then we get the edge weight $W_9 = \{3n+1\}$. $W_{10} = W(x_ix_{i+1}) = 3n+4$ for $i = even$ and $i \neq n$ then we get the edge weight $W_{10} = \{3n+4\}$. $W_{11} = W(x_nx_1) = \frac{3n+5}{2}$ for $n = odd$ then we get the edge weight $W_{11} = \{\frac{3n+5}{2}\}$. $W_{12} = W(x_nx_1) = \frac{3n+8}{2}$ for $n = even$ then we get the edge weight $W_{12} = \{\frac{3n+8}{2}\}$. $W_{13} = W(y_iy_{i+1}) = 3n+2$ for $i = odd$ and $i \neq n$ then we get the edge weight $W_{13} = \{3n+2\}$. $W_{14} = W(y_iy_{i+1}) = 3n-1$ for $i = even$ and $i \neq n$ then we get the edge weight $W_{14} = \{3n-1\}$. $W_{15} = W(y_ny_1) = \frac{3n+1}{2}$ for $n = odd$ then we get the edge weight $W_{15} = \{\frac{3n+1}{2}\}$. $W_{16} = W(y_ny_1) = \frac{3n+4}{2}$ for $n = even$ then we get the edge weight $W_{16} = \{\frac{3n+4}{2}\}$. From the second step we get that $W_3 \subset W_1$, $W_4 \subset W_1$, $W_5 \subset W_1$, $W_6 \subset W_1$, $W_7 \subset W_1$, $W_8 \subset W_1$, $W_9 \subset W_1$, $W_{10} \subset W_1$, $W_{11} \subset W_1$, $W_{12} \subset W_1$, $W_{13} \subset W_1$, $W_{14} \subset W_1$, $W_{15} \subset W_1$ and $W_{16} \subset W_1$. Based on the set of edge weights, the number of edge weights is $3n$, so $rac(Tj_n) \leq 3n$. We know from the upper bound and lower bound that $3n \leq rac(Tj_n) \leq 3n$ we get that the rainbow antimagic connection number of tunjung graph is $rac(Tj_n) = 3n$.

Based on the edge weight above, the rainbow path of u-v is found which can be seen in table 7 and for illustration of rainbow antimagic coloring of graph Tj_n is in Figure 7, with $rac(Tj_8) = 3n = 24$.

TABLE 7. Rainbow connection from u to v in Tj_n

Case	u	v	Rainbow Connection	Condition
1	x_i	x_j	c	$i \neq j, 1 \leq i \leq n, 1 \leq j \leq n$
2	y_i	y_j	c	$i \neq j, 1 \leq i \leq n, 1 \leq j \leq n$
3	z_i	z_j	c	$i \neq j, 1 \leq i \leq n, 1 \leq j \leq n$
4	x_i	z_j	c	$1 \leq i \leq n, 1 \leq j \leq n$
5	x_i	y_j	c	$1 \leq i \leq n, 1 \leq j \leq n$
6	y_i	z_j	c	$1 \leq i \leq n, 1 \leq j \leq n$

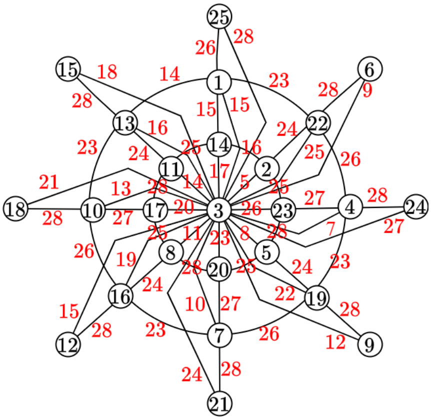

FIGURE 7. Rainbow antimagic coloring Tj_8

CONCLUSION

In this paper, we have studied 7 theorems about rainbow coloring and rainbow antimagic coloring of graph. There is 3 theorems about rainbow coloring and 4 theorems about rainbow antimagic coloring of double quadrilateral windmill graph (DQ_n), double quadrilateral flower graph (FDQ_n), split graph of star $(Spl(K_{1,n})$, and tunjung graph (Tj_n).

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