On (a,d)-hyperedge Antimagic Labeling of Certain Classes of Hypergraphs: A New Notion

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Abstract. By a hypergraph $\mathbb{G}$, we mean a generalization of a graph G in which an edge can join any number of vertices. In an ordinary graph, an edge connects exactly two vertices, but in hypergraph, an edge or *hyperedge* may connect more than two vertices. Let $\mathbb{G} = (V, \mathscr{E})$ be a hypergraph, thus V contains a finite set of vertices, and $\mathscr{E}$ contains a hyperedge of subset of V. Some vertices are said to be adjacent if they are elements of a hyperedge. A vertex v is said to be incident to an hyperedge e if $v \in e$. Similarly, a hyperedge e is said to be incident to vertex v if $v \in e$. Furthermore, a bijection f from $V(\mathbb{G})$ into $\{1,2,3,\ldots,|V|\}$ is called and (a,d)-hyperedge antimagic labeling of hypergraph $\mathbb{G}$ if the hyperedge weights $\mathbb{W}(e) = \sum_{v \in e} f(v)$ form an arithmetic progression starting from a and having common difference d. In this paper, we initiate to study hyperedge antimagic labeling of certain classes of hypergraphs, including analyze the properties of the antimagicness of any hypergraph.

Keywords: Hyperedge, antimagic labeling, hypergraph

INTRODUCTION

A hypergraph is a mathematical structure that generalizes the concept of a graph. In a traditional graph, the basic building blocks are vertices and edges, where edges connect pairs of verticess. In a hypergraph, edges can connect more than two vertices. In other words, a hypergraph is defined by a set of vertices and a set of hyperedges. A hyperedge is a set of vertices, and it can connect any number of vertices (including just one vertex). This generalization allows for a more flexible representation of relationships between elements. Let $\mathbb{G} = (V, \mathscr{E})$ be a hypergraph, thus V contains a finite set of vertices, and $\mathscr{E}$ contains a hyperedge of subset of V.

Some vertices are said to be adjacent if they are elements of a hyperedge. A vertex v is said to be incident to an hyperedge e if $v \in e$. Similarly, a hyperedge e is said to be incident to vertex v if $v \in e$, see Figure $1(a)$. The order of hypergraph is the number of vertices, and size of hypergraph is number of the hypergraph. A hypergraph e with no vertex is said to be empty hyperedge. A vertex is said to be an isolated vertex if it is not an element of any hyperedge e. Singleton is a hyperedge which is incident to exactly one vertex. Pendant vertex is a vertex which is incident to exactly one hyperedge. Included hyperedge is a hyperedge that is a subset of another hyperedge, while multiple hyperedge is a hyperedge that has the same set of vertices as another hyperedge. A simple hypergraph is hypergraph without included or multiple hyperedges.

Figure $1(a)$ shows that vertex v_1 and vertex v_7 are adjacent, as well as v_2 and v_3. The vertices v_1, v_2, v_7 are incident to hyperedge e_1. The order of hypergraph $\mathbb{G}$ is $|V| = 9$ and the size of $\mathbb{G}$ is $|\mathscr{E}| = 7$. Hyperedge e_7 is empty hyperedge, e_6 is singleton hyperedge, v_8 is pendant vertex, and v_9 is isolated vertex. Finally, hyperedges e_4 and e_5 are included hyperedge or multiple hyperedge. Furthermore, a 2-section graph (or clique graph, representing graph, primal graph, Gaifman graph) of a hypergraph $\mathbb{G}$ is a graph G with the same vertices of the hypergraph, and edges between all pairs of vertices contained in the same hyperedge, see Figure $1(b)$.

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There is much more research on $(a,d)-$antimagic labeling. But it is limited to graphs, and there is still not much research on hypergraphs, especially more on $(a,d)-$antimagic labeling on hypergraphs. Research on $(a,d)-$antimagic labeling has grown rapidly and has been studied frequently between 2009-2016. In 2009, Dafik et al researched about super $(a,d)-$edge antimagic total labeling of disconnected graphs [1]. Two years later, Arumugam and Nalliah conducted research on super $(a,d)-$edge antimagic total labelings of friendship graphs [2]. In 2014, there was also research on super $(a,d)-$edge antimagic total labeling of connected Disc Brake graph [3]. In the next year, Arumugam and Nalliah conducted another study on super $(a,d)-$edge antimagic total labelings of friendship and generalized friendship graphs [4]. That year, Dafik et al. did a lot of similar research, namely on super $(a,d)-$edge antimagic total labeling of silkworm graph [5], pentagonal chain graph [6], connected lampion graph [7], connected tribune graph [8], and connected Ferris wheel graph [9]. Dafik et al. continued their research in 2016, there was research on super $(a,d)-Fn-$antimagic total labeling for a connected and disconnected amalgamation of fan graphs [10]. In the same year, Agustin et al. studied the connected and disjoint union of semi Jahangir graphs admitting a cycle-super $(a,d)-$antimagic total labeling [11]. In the following years, the research on antimagic labeling did not really develop, but there was a mutation in the research on antimagic coloring, because the research was limited to graphs. Therefore, antimagic labeling was developed again, but on hypergraphs.

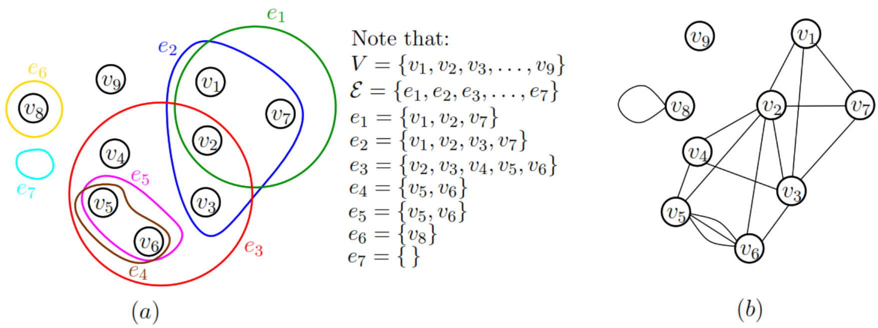

FIGURE 1. (a) The Illustration of hypergraph of order 9 and size 7, (b) The Illustration of 2-section graph of hypergraph $\mathbb{G}$.

In 2000, several articles discussed hypergraphs, such as classes of hypergraphs with sum number one [11] and sum labelings of cycle hypergraphs [12]. Two years later, Martin Sonntag published a study entitled antimagic vertex labelings of hypergraphs [13]. Much later, in 2011 Toufiq Parag and Ahmed Elgammal redeveloped the hypergraph under the title of supervised hypergraph labeling [14]. This was followed by the publication of a book on the hypergraph by Muhammad Javaid in 2013, entitled labeling of graphs and hypergraphs [15]. Then, in 2019, research on hypergraphs was conducted again, which discussed on cordial labeling of hypertrees [16]. The following year, Swaminathan et al. did research on the unimodular hypergraph for DNA sequencing [17]. Two years later, Christopher Purcell was researching on exclusive sum labelings of hypergraphs [18]. Then, a book by Qionghai Dai and Yue Gao was published in 2023 entitled hypergraph computation [19].

The hypergraph has wide-ranging applications in various fields, including biology, machine learning, social network analysis, transportation, circuit design, and chemistry. In biology, hypergraph is used to model complex interactions between biomolecules such as proteins, genes, and metabolic pathways, aiding in understanding gene regulation and diseases [20]. In machine learning, representing data as a hypergraph allows models to comprehend more intricate relationships between entities, enhancing performance in predicting patterns within data [21]. In social network analysis, hypergraphs help identify more segmented groups and complex interaction patterns [22]. In transportation, hypergraph is used to plan optimal routes and optimize resource allocation in complex transportation networks [23]. In circuit design and VLSI manufacturing, hypergraph aids in analyzing and optimizing complex circuit designs [24]. Finally, in chemistry and materials science, hypergraph is used to represent complex molecular structures and aids in modeling chemical processes, designing new molecules, and developing innovative materials [25]. Here are some additional applications of hypergraphs: database design, knowledge representation, bio-informatics, image processing, constraint satisfaction problems, cryptography, secret sharing, and the other important applications. With further development, the hypergraph has great potential to become a valuable tool in solving complex problems in various fields of science and technology.

METHODS

This research employs deductive analytic methods to obtain the (a,d)-hyperedge antimagic labeling of hypergraphs. The procedure begins by determining the upper bound of $d(\mathbb{G})$, a critical parameter in understanding the properties of the hypergraph $\mathbb{G}$. This involves constructing a bijective function combined with an arithmetic sequence. Subsequently, the hypergraph $\mathbb{G}$ is defined, elucidating its structure and elements. The cardinality of $\mathbb{G}$ is then ascertained by calculating both its order and size, providing insights into the scale and complexity of the hypergraph. Further characterization of $\mathbb{G}$ entails establishing the vertex label function, assigning unique identifiers to each vertex within the hypergraph. If this labeling is bijective, the edge weight function of $\mathbb{G}$ is determined to quantify the significance or relevance of each hyperedge. Upon identifying that the edge weight set comprises identical elements or consecutive integers, it is concluded that $\mathbb{G}$ exhibits an (a,d)-hyperedge antimagic labeling. This finding is pivotal as it demonstrates a special property of $\mathbb{G}$, shedding light on its potential applications and theoretical implications. Finally, armed with the knowledge of the (a,d)-hyperedge antimagic labeling of $\mathbb{G}$, a new theorem can be formulated and its proof furnished, contributing novel insights to the realm of hypergraph theory.

RESULTS

In this paper, we explore novel findings concerning the (a,d)-hyperedge antimagic labeling of the path hypergraph $\mathscr{P}_n$ and triangular ladder hypergraph $\mathscr{T}l_n$, as well as introduce the concept of super (a,d)-hyperedge antimagic labeling. We demonstrate that both the path hypergraph $\mathscr{P}_n$ and triangular ladder hypergraph $\mathscr{T}l_n$, with $n \geq 2$, can exhibit an (a,d)-hyperedge antimagic labeling for values of d belonging to the set $\{0,1,2\}$.

Theorem 1. *If (p,q)-hypergraph is (a,d)-hyperedge antimagic labeling, then*

$$d \leq \frac{p_H(p_G - p_H)}{s-1}$$

for $p_G = |V|, q_G = |\mathscr{E}|, p_H = |V'|, q_H = |\mathscr{E}'|, s = |H_i|$.

Proof. If (p,q)-hypergraph has (a,d)-hyperedge antimagic labeling with $f(V) = \{1,2,3,...,p_G\}$, then the weight set of a hypergraph is $\{a, a+d, a+2d, ..., a(s-1)d\}$ where a is the smallest weight. Let A be the weight set, then it holds:

$$\begin{aligned}
\sum A = \sum_{V \in V(A)} f(V) &= \{a, a+d, a+2d, ..., a+(s-1)d\} \\
1+2+...+p_H &\leq a \\
\frac{p_H}{2}(1+p_H) &\leq a
\end{aligned}$$

While for the largest weight, it holds:

$$\begin{aligned}
a+(s-1)d &\le p_G+p_G-1+p_G-2+...+(p_G-(p_H-1))\\
&= \frac{p_H}{2}(p_G+(p_G-(p_H-1)))\\
&= \frac{p_H}{2}(p_G+p_G-p_H+1)\\
(s-1)d &\le \frac{p_H}{2}(2p_G-p_H+1)-a\\
(s-1)d &\le \frac{p_H}{2}(2p_G-p_H+1)-\left(\frac{p_H}{2}(1+p_H)\right)\\
&= \frac{p_H}{2}((2p_G-p_H+1)-(1+p_H))\\
&= \frac{p_H}{2}(2p_G-2p_H)\\
(s-1)d &\le p_H(p_G-p_H)\\
d &\le \frac{p_H(p_G-p_H)}{s-1}
\end{aligned}$$

So, for $p_G=|V|, q_G=|\mathscr{E}|, p_H=|V'|, q_H=|\mathscr{E}'|, s=|H_i|$, it concludes that $d\le \frac{p_H(p_G-p_H)}{s-1}$. □

Theorem 2. *For m is odd, $m\ge 1$, and $n\ge 2$, hypergraph $\mathscr{P}_n$ admits an $(a,0)$-hyperedge antimagic labeling.*

Proof. $\mathscr{P}_n$ is a specific family of hypergraphs with vertex set $V=\{x_{i,j};1\le i\le n-1, 1\le j\le m\}\cup\{x_i;1\le i\le n\}$ and hyperedge set $\mathscr{E}=\{e_i;1\le i\le n-1\}$ where $e_i=\{x_i,x_{i,j},x_{i+1};1\le i\le n-1, 1\le j\le m\}$. Thus, we have $|V(\mathscr{P}_n)|=mn-m+n$ and $|\mathscr{E}(\mathscr{P}_n)|=n-1$. For m is odd, $m\ge 1$, and $n\ge 2$, define a bijection f_1 from $V(\mathbb{G})\longrightarrow\{1,2,3,\dots,mn-m+n\}$ as follows:

$$f_1(x_i)=\begin{cases}\frac{i+1}{2}, & \text{if } i\equiv 1 \pmod 2, 1\le i\le n\\ \frac{n}{2}+\frac{i}{2}, & \text{if } i\equiv 0 \pmod 2, 1\le i\le n\end{cases}$$

$$f_1(x_{i,j})=\begin{cases}nj+n-i-j+1, & \text{if } j\equiv 1 \pmod 2, 1\le i\le n, 1\le j\le m\\ nj+i-j+1, & \text{if } j\equiv 0 \pmod 2, 1\le i\le n, 1\le j\le m\end{cases}$$

Clearly, the labeling f_1 from $V(\mathbb{G})\longrightarrow\{1,2,3,\dots,mn-m+n\}$ is a bijection. The edge weight under the labeling f_1 constitute the sets

$$\begin{aligned}
W^1_{f_1}(e_i) &= \frac{m}{4}(4n+2mn-2m+2)+\frac{n}{2}+2, \text{if } n\equiv 1 \pmod 2\\
W^2_{f_1}(e_i) &= \frac{m}{4}(4n+2mn-2m+2)+\frac{n}{2}+3, \text{if } n\equiv 0 \pmod 2
\end{aligned}$$

The edge weight set $W^1_{f_1}=\{\frac{m}{4}(4n+2mn-2m+2)+\frac{n}{2}+2,...,\frac{m}{4}(4n+2mn-2m+2)+\frac{n}{2}+2\}$ consists of same elements, it implies that hypergraph $\mathscr{P}_n$ admits an $(\frac{m}{4}(4n+2mn-2m+2)+\frac{n}{2}+2,0)$-hyperedge antimagic labeling for n is even. And the edge set $W^2_{f_1}=\{\frac{m}{4}(4n+2mn-2m+2)+\frac{n}{2}+3,...,\frac{m}{4}(4n+2mn-2m+2)+\frac{n}{2}+3\}$ consists of same elements, it implies that hypergraph $\mathscr{P}_n$ admits an $(\frac{m}{4}(4n+2mn-2m+2)+\frac{n}{2}+3,0)$-hyperedge antimagic labeling for n is odd. □

For having more detail illustration of the existence of $(4n+2mn-2m,0)$-hyperedge antimagic labeling of $\mathscr{P}_n$, we depict the graph in Figure 2.

Theorem 3. *For m is even, $m\ge 2$, and $n\ge 2$, hypergraph $\mathscr{P}_n$ admits an $(a,1)$-hyperedge antimagic labeling.*

Proof. $\mathscr{P}_n$ is a specific family of hypergraphs with vertex set $V=\left\{x_{i,j};1\le i\le n-1, 1\le j\le m\right\}\cup\{x_i;1\le i\le n\}$, and hyperedge set $\mathscr{E}=\{e_i;1\le i\le n-1\}$ where $e_i=\{x_ix_{i,j}x_{i+1};1\le i\le n-1, 1\le j\le m$. Thus, we have $|V(\mathscr{P}_n)|=mn-m+n$ and $|\mathscr{E}(\mathscr{P}_n)|=n-1$. For m is even, $m\ge 2$, and $n\ge 2$, define a bijection f_2 from $V(\mathbb{G})\longrightarrow\{1,2,3,\dots,mn-m+n\}$ as follows:

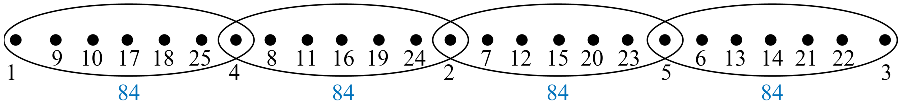

FIGURE 2. $(84,0)$-hyperedge antimagic labeling of $\mathscr{P}_5$

$$f_2(x_i)=\begin{cases} \frac{i+1}{2}, & \text{if } i \equiv 1 \pmod 2, 1 \le i \le n \\ \frac{n}{2}+\frac{i}{2}, & \text{if } i \equiv 0 \pmod 2, 1 \le i \le n \end{cases}$$

$$f_2(x_{i,j})=\begin{cases} nj+n-i-j+1, & \text{if } j \equiv 1 \pmod 2, 1 \le i \le n, 1 \le j \le m \\ nj+i-j+1, & \text{if } j \equiv 0 \pmod 2, 1 \le i \le n, 1 \le j \le m \end{cases}$$

Clearly, the labeling f_2 from $V(\mathbb{G}) \longrightarrow \{1,2,3,\dots,mn-m+n\}$ is a bijection. The edge weight under the labeling f_2 constitute the sets

$$W_{f_2}(e_i) = \frac{m}{4}(4n+2mn-2m+2)+\left\lceil \frac{n}{2} \right\rceil + i + 1, \text{if } 1 \le i \le n$$

The edge weight set $W_{f_2} = \{\frac{m}{4}(4n+2mn-2m+2)+\left\lceil \frac{n}{2} \right\rceil+2, \frac{m}{4}(4n+2mn-2m+2)+\left\lceil \frac{n}{2} \right\rceil+3, \frac{m}{4}(4n+2mn-2m+2)+\left\lceil \frac{n}{2} \right\rceil+4, \dots\}$ consists of consecutive integers, it implies that hypergraph $\mathscr{P}_n$ admits an $(\frac{m}{4}(4n+2mn-2m+2)+\left\lceil \frac{n}{2} \right\rceil+2, 1)$-hyperedge antimagic labeling. □

For having more detail illustration of the existence of $(\frac{m}{4}(4n+2mn-2m+2)+\left\lceil \frac{n}{2} \right\rceil+2, 1)$-hyperedge antimagic labeling of $\mathscr{P}_n$, we depict the graph in Figure 3.

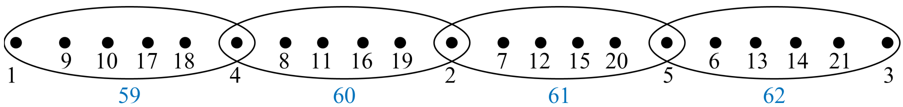

FIGURE 3. $(59,1)$-hyperedge antimagic labeling of $\mathscr{P}_5$

Theorem 4. *For m is odd, $m \ge 1$, and $n \ge 2$, the hypergraph $\mathscr{P}_n$ admits an $(a,2)$-hyperedge antimagic labeling.*

Proof. $\mathscr{P}_n$ is a specific family of hypergraphs with vertex set $V = \{x_{i,j}; 1 \le i \le n-1, 1 \le j \le m\} \cup \{x_i; 1 \le i \le n\}$, and hyperedge set $\mathscr{E} = \{e_i; 1 \le i \le n-1\}$ where $e_i = \{x_i x_{i,j} x_{i+1}; 1 \le i \le n-1, 1 \le j \le m$. Thus, we have $|V(\mathscr{P}_n)| = mn-m+n$ and $|\mathscr{E}(\mathscr{P}_n)| = n-1$. For m is odd, $m \ge 1$, and $n \ge 2$, define a bijection f_3 from $V(\mathbb{G}) \longrightarrow \{1,2,3,\dots,mn-m+n\}$ as follows:

$$f_3(x_i)=\begin{cases} \frac{i+1}{2}, & \text{if } i \equiv 1 \pmod 2, 1 \le i \le n \\ \frac{n}{2}+\frac{i}{2}, & \text{if } i \equiv 0 \pmod 2, 1 \le i \le n \end{cases}$$

$$f_3(x_{i,j})=\begin{cases} nj+i-j+1, & \text{if } j \equiv 1 \pmod 2, 1 \le i \le n, 1 \le j \le m \\ nj+n-i-j+1, & \text{if } j \equiv 0 \pmod 2, 1 \le i \le n, 1 \le j \le m \end{cases}$$

Clearly, the labeling f_3 from $V(\mathbb{G}) \longrightarrow \{1,2,3,\dots,mn-m+n\}$ is a bijection. The edge weight under the labeling f_3 constitute the sets

$$W_{f_3}(e_i) = \frac{m}{4}(4n+2mn-2m+2)+\left\lceil \frac{n}{2} \right\rceil + 2i, \text{if } 1 \le i \le n$$

The edge weight set $W_{f_3} = \{\frac{m}{4}(4n+2mn-2m+2)+\left\lceil\frac{n}{2}\right\rceil+2, \frac{m}{4}(4n+2mn-2m+2)+\left\lceil\frac{n}{2}\right\rceil+4, \frac{m}{4}(4n+2mn-2m+2)+\left\lceil\frac{n}{2}\right\rceil+6, \ldots\}$ consists of consecutive integers, it implies that hypergraph $\mathscr{P}_n$ admits an $(\frac{m}{4}(4n+2mn-2m+2)+\left\lceil\frac{n}{2}\right\rceil+2, 2)$-hyperedge antimagic labeling. $\square$

For having more detail illustration of the existence of $(\frac{m}{4}(4n+2mn-2m+2)+\left\lceil\frac{n}{2}\right\rceil+2, 2)$-hyperedge antimagic labeling of $\mathscr{P}_n$, we depict the graph in Figure 4.

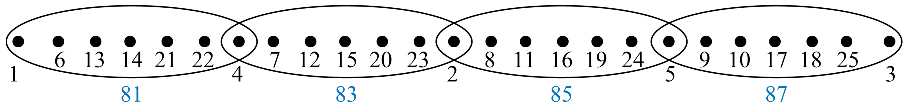

FIGURE 4. $(81,2)$-hyperedge antimagic labeling of $\mathscr{P}_5$

Theorem 5. *For m is odd, $m \geq 1$, and $n \geq 2$, hypergraph $\mathscr{T}l_n$ admits an $(a,0)$-hyperedge antimagic labeling.*

Proof. $\mathscr{T}l_n$ is a specific family of hypergraphs with vertex set $V = \{x_i; 1 \leq i \leq n\} \cup \{y_i; 1 \leq i \leq n\} \cup \{x_{i,j}; 1 \leq i \leq n-1, 1 \leq j \leq m\} \cup \{y_{i,j}; 1 \leq i \leq n-1, 1 \leq j \leq m\} \cup \{p_{i,j}; 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{q_{i,j}; 1 \leq i \leq n-1, 1 \leq j \leq m\}$ and hyperedge set $\mathscr{E} = \{e_{1,i}; 1 \leq i \leq n\} \cup \{e_{2,i}; 1 \leq i \leq n-1\} \cup \{e_{3,i}; 1 \leq i \leq n-1\} \cup \{e_{4,i}; 1 \leq i \leq n-1\}$ where $e_{1,i} = \{x_i, p_{i,j}, y_i; 1 \leq i \leq n, 1 \leq j \leq m\} \cup e_{2,i} = \{x_i, x_{i,j}, x_{i+1}; 1 \leq i \leq n-1, 1 \leq j \leq m\} \cup e_{3,i} = \{x_i, q_{i,j}, y_{i+1}; 1 \leq i \leq n-1, 1 \leq j \leq m\} \cup e_{4,i} = \{y_i, y_{i,j}, y_{i+1}; 1 \leq i \leq n-1, 1 \leq j \leq m\}$. Thus, we have $|V(\mathscr{T}l_n)| = 4mn-3m+2n$ and $|\mathscr{E}(\mathscr{T}l_n)| = 4n-3$. For m is odd, $m \geq 1$, and $n \geq 2$, define a bijection f_4 from $V(\mathbb{G}) \longrightarrow \{1,2,3,\ldots,4mn-3m+2n\}$ as follows:
$f_4(x_i) = 2i-1, \text{if } 1 \leq i \leq n$
$f_4(y_i) = 2i, \text{if } 1 \leq i \leq n$

$$f_4(p_{i,j}) = \begin{cases} 4nj+2n-4i-3j+4, & \text{if } j \equiv 1 \pmod 2, 1 \leq i \leq n, 1 \leq j \leq m \\ 4nj-2n+4i-3j, & \text{if } j \equiv 0 \pmod 2, 1 \leq i \leq n, 1 \leq j \leq m \end{cases}$$

$$f_4(x_{i,j}) = \begin{cases} 4nj+2n-4i-3j+3, & \text{if } j \equiv 1 \pmod 2, 1 \leq i \leq n, 1 \leq j \leq m \\ 4nj-2n+4i-3j+1, & \text{if } j \equiv 0 \pmod 2, 1 \leq i \leq n, 1 \leq j \leq m \end{cases}$$

$$f_4(q_{i,j}) = \begin{cases} 4nj+2n-4i-3j+2, & \text{if } j \equiv 1 \pmod 2, 1 \leq i \leq n, 1 \leq j \leq m \\ 4nj-2n+4i-3j+2, & \text{if } j \equiv 0 \pmod 2, 1 \leq i \leq n, 1 \leq j \leq m \end{cases}$$

$$f_4(y_{i,j}) = \begin{cases} 4nj+2n-4i-3j+1, & \text{if } j \equiv 1 \pmod 2, 1 \leq i \leq n, 1 \leq j \leq m \\ 4nj-2n+4i-3j+3, & \text{if } j \equiv 0 \pmod 2, 1 \leq i \leq n, 1 \leq j \leq m \end{cases}$$

Clearly, the labeling f_4 from $V(\mathbb{G}) \longrightarrow \{1,2,3,\ldots,4mn-3m+2n\}$ is a bijection. The edge weight under the labeling f_4 constitute the sets

$$W^1_{f_4}(e_{p_{i,j}}) = \frac{m}{2}(4mn-3m+4n+1)+2n+1$$
$$W^2_{f_4}(e_{x_{i,j}}) = \frac{m}{2}(4mn-3m+4n+1)+2n+1$$
$$W^3_{f_4}(e_{q_{i,j}}) = \frac{m}{2}(4mn-3m+4n+1)+2n+1$$
$$W^4_{f_4}(e_{y_{i,j}}) = \frac{m}{2}(4mn-3m+4n+1)+2n+1$$

The edge weight set $\bigcup_{r=1}^{4} W^r_{f_4} = \{\frac{m}{2}(4mn-3m+4n+1)+2n+1, \ldots, \frac{m}{2}(4mn-3m+4n+1)+2n+1\}$ consists of same elements, it implies that hypergraph $\mathscr{T}l_n$ admits an $(\frac{m}{2}(4mn-3m+4n+1)+2n+1, 0)$-hyperedge antimagic labeling. $\square$

For having more detail illustration of the existence of $(\frac{m}{2}(4mn-3m+4n+1)+2n+1,0)$-hyperedge antimagic labeling of $\mathscr{T}l_n$, we depict the graph in Figure 5.

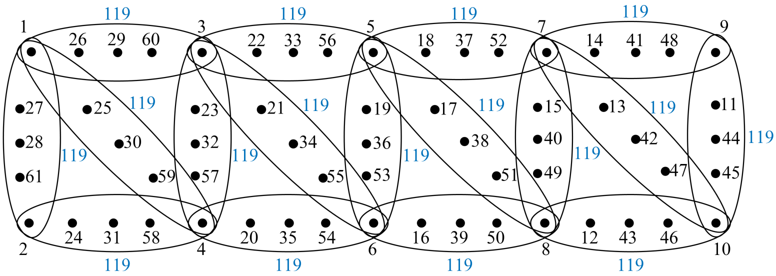

FIGURE 5. $(119,0)$-hyperedge antimagic labeling of $\mathscr{T}l_5$

Theorem 6. *For m is even, $m \geq 2$, and $n \geq 2$, hypergraph $\mathscr{T}l_n$ admits an $(a,1)$-hyperedge antimagic labeling.*

Proof. $\mathscr{T}l_n$ is a specific family of hypergraphs with vertex set $V = \{x_i; 1 \leq i \leq n\} \cup \{y_i; 1 \leq i \leq n\} \cup \{x_{i,j}; 1 \leq i \leq n-1, 1 \leq j \leq m\} \cup \{y_{i,j}; 1 \leq i \leq n-1, 1 \leq j \leq m\} \cup \{p_{i,j}; 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{q_{i,j}; 1 \leq i \leq n-1, 1 \leq j \leq m\}$ and hyperedge set $\mathscr{E} = \{e_{1,i}; 1 \leq i \leq n\} \cup \{e_{2,i}; 1 \leq i \leq n-1\} \cup \{e_{3,i}; 1 \leq i \leq n-1\} \cup \{e_{4,i}; 1 \leq i \leq n-1\}$ where $e_{1,i} = \{x_i, p_{i,j}, y_i; 1 \leq i \leq n, 1 \leq j \leq m\} \cup e_{2,i} = \{x_i, x_{i,j}, x_{i+1}; 1 \leq i \leq n-1, 1 \leq j \leq m\} \cup e_{3,i} = \{x_i, q_{i,j}, y_{i+1}; 1 \leq i \leq n-1, 1 \leq j \leq m\} \cup e_{4,i} = \{y_i, y_{i,j}, y_{i+1}; 1 \leq i \leq n-1, 1 \leq j \leq m\}$. Thus, we have $|V(\mathscr{T}l_n)| = 4mn-3m+2n$ and $|\mathscr{E}(\mathscr{T}l_n)| = 4n-3$. For m is even, $m \geq 2$, and $n \geq 2$, define a bijection f_5 from $V(\mathbb{G}) \longrightarrow \{1,2,3,\ldots,4mn-3m+2n\}$ as follows:

$f_5(x_i) = 2i-1, \text{if } 1 \leq i \leq n$

$f_5(y_i) = 2i, \text{if } 1 \leq i \leq n$

$$f_5(p_{i,j})=\begin{cases} 4nj+2n-4i-3j+4, & \text{if } j \equiv 1 \pmod 2, 1 \leq i \leq n, 1 \leq j \leq m \\ 4nj-2n+4i-3j, & \text{if } j \equiv 0 \pmod 2, 1 \leq i \leq n, 1 \leq j \leq m \end{cases}$$

$$f_5(x_{i,j})=\begin{cases} 4nj+2n-4i-3j+3, & \text{if } j \equiv 1 \pmod 2, 1 \leq i \leq n, 1 \leq j \leq m \\ 4nj-2n+4i-3j+1, & \text{if } j \equiv 0 \pmod 2, 1 \leq i \leq n, 1 \leq j \leq m \end{cases}$$

$$f_5(q_{i,j})=\begin{cases} 4nj+2n-4i-3j+2, & \text{if } j \equiv 1 \pmod 2, 1 \leq i \leq n, 1 \leq j \leq m \\ 4nj-2n+4i-3j+2, & \text{if } j \equiv 0 \pmod 2, 1 \leq i \leq n, 1 \leq j \leq m \end{cases}$$

$$f_5(y_{i,j})=\begin{cases} 4nj+2n-4i-3j+1, & \text{if } j \equiv 1 \pmod 2, 1 \leq i \leq n, 1 \leq j \leq m \\ 4nj-2n+4i-3j+3, & \text{if } j \equiv 0 \pmod 2, 1 \leq i \leq n, 1 \leq j \leq m \end{cases}$$

Clearly, the labeling f_5 from $V(\mathbb{G}) \longrightarrow \{1,2,3,\ldots,4mn-3m+2n\}$ is a bijection. The edge weight under the labeling f_5 constitute the sets

$$\begin{aligned} W^1_{f_5}(e_{p_{i,j}}) &= \frac{m}{2}(4mn-3m+4n+1)+4i-1, \text{if } 1 \leq i \leq n \\ W^2_{f_5}(e_{x_{i,j}}) &= \frac{m}{2}(4mn-3m+4n+1)+4i, \text{if } 1 \leq i \leq n \\ W^3_{f_5}(e_{q_{i,j}}) &= \frac{m}{2}(4mn-3m+4n+1)+4i+1, \text{if } 1 \leq i \leq n \\ W^4_{f_5}(e_{y_{i,j}}) &= \frac{m}{2}(4mn-3m+4n+1)+4i+2, \text{if } 1 \leq i \leq n \end{aligned}$$

The edge weight set $\bigcup_{r=1}^{4} W^r_{f_5} = \{\frac{m}{2}(4mn-3m+4n+1)+3, \frac{m}{2}(4mn-3m+4n+1)+4, \frac{m}{2}(4mn-3m+4n+1)+$

$5, \ldots\}$ consists of consecutive integers, it implies that hypergraph $\mathscr{T}l_n$ admits an $(\frac{m}{2}(4mn-3m+4n+1)+4i-1,1)$-hyperedge antimagic labeling. $\square$

For having more detail illustration of the existence of $(\frac{m}{2}(4mn-3m+4n+1)+4i-1,1)$-hyperedge antimagic labeling of $\mathscr{T}l_n$, we depict the graph in Figure 6.

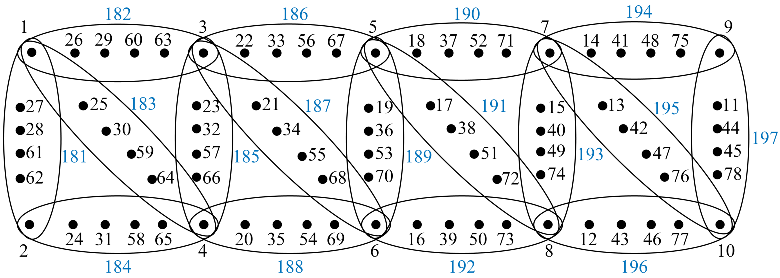

FIGURE 6. $(181,1)$-hyperedge antimagic labeling of $\mathscr{T}l_5$

Theorem 7. *For m is odd, $m \geq 1$, and $n \geq 2$, hypergraph $\mathscr{T}l_n$ admits an $(a,2)$-hyperedge antimagic labeling.*

Proof. $\mathscr{T}l_n$ is a specific family of hypergraphs with vertex set $V = \{x_i; 1 \leq i \leq n\} \cup \{y_i; 1 \leq i \leq n\} \cup \{x_{i,j}; 1 \leq i \leq n-1, 1 \leq j \leq m\} \cup \{y_{i,j}; 1 \leq i \leq n-1, 1 \leq j \leq m\} \cup \{p_{i,j}; 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{q_{i,j}; 1 \leq i \leq n-1, 1 \leq j \leq m\}$ and hyperedge set $\mathscr{E} = \{e_{1,i}; 1 \leq i \leq n\} \cup \{e_{2,i}; 1 \leq i \leq n-1\} \cup \{e_{3,i}; 1 \leq i \leq n-1\} \cup \{e_{4,i}; 1 \leq i \leq n-1\}$ where $e_{1,i} = \{x_i, p_{i,j}, y_i; 1 \leq i \leq n, 1 \leq j \leq m\} \cup e_{2,i} = \{x_i, x_{i,j}, x_{i+1}; 1 \leq i \leq n-1, 1 \leq j \leq m\} \cup e_{3,i} = \{x_i, q_{i,j}, y_{i+1}; 1 \leq i \leq n-1, 1 \leq j \leq m\} \cup e_{4,i} = \{y_i, y_{i,j}, y_{i+1}; 1 \leq i \leq n-1, 1 \leq j \leq m\}$. Thus, we have $|V(\mathscr{T}l_n)| = 4mn-3m+2n$ and $|\mathscr{E}(\mathscr{T}l_n)| = 4n-3$. For m is odd, $m \geq 1$, and $n \geq 2$, define a bijection f_6 from $V(\mathbb{G}) \longrightarrow \{1,2,3,\ldots,4mn-3m+2n\}$ as follows:

$f_6(x_i) = 2i-1, \text{if } 1 \leq i \leq n$

$f_6(y_i) = 2i, \text{if } 1 \leq i \leq n$

$$f_6(p_{i,j}) = \begin{cases} 4nj-2n+4i-3j, & \text{if } j \equiv 1 \pmod 2, 1 \leq i \leq n, 1 \leq j \leq m \\ 4nj+2n-4i-3j+4, & \text{if } j \equiv 0 \pmod 2, 1 \leq i \leq n, 1 \leq j \leq m \end{cases}$$

$$f_6(x_{i,j}) = \begin{cases} 4nj-2n+4i-3j+1, & \text{if } j \equiv 1 \pmod 2, 1 \leq i \leq n, 1 \leq j \leq m \\ 4nj+2n-4i-3j+3, & \text{if } j \equiv 0 \pmod 2, 1 \leq i \leq n, 1 \leq j \leq m \end{cases}$$

$$f_6(q_{i,j}) = \begin{cases} 4nj-2n+4i-3j+2, & \text{if } j \equiv 1 \pmod 2, 1 \leq i \leq n, 1 \leq j \leq m \\ 4nj+2n-4i-3j+2, & \text{if } j \equiv 0 \pmod 2, 1 \leq i \leq n, 1 \leq j \leq m \end{cases}$$

$$f_6(y_{i,j}) = \begin{cases} 4nj-2n+4i-3j+3, & \text{if } j \equiv 1 \pmod 2, 1 \leq i \leq n, 1 \leq j \leq m \\ 4nj+2n-4i-3j+1, & \text{if } j \equiv 0 \pmod 2, 1 \leq i \leq n, 1 \leq j \leq m \end{cases}$$

Clearly, the labeling f_6 from $V(\mathbb{G}) \longrightarrow \{1,2,3,\ldots,4mn-3m+2n\}$ is a bijection. The edge weight under the labeling f_6 constitute the sets

$$W^1_{f_6}(e_{p_{i,j}}) = \frac{m}{2}(4mn-3m+4n+1)+8i-2n-3, \text{if } 1 \leq i \leq n$$
$$W^2_{f_6}(e_{x_{i,j}}) = \frac{m}{2}(4mn-3m+4n+1)+8i-2n-1, \text{if } 1 \leq i \leq n$$
$$W^3_{f_6}(e_{q_{i,j}}) = \frac{m}{2}(4mn-3m+4n+1)+8i-2n+1, \text{if } 1 \leq i \leq n$$
$$W^4_{f_6}(e_{y_{i,j}}) = \frac{m}{2}(4mn-3m+4n+1)+8i-2n+3, \text{if } 1 \leq i \leq n$$

The edge weight set $\bigcup_{r=1}^{4} W_{f_6}^{r} = \{\frac{m}{2}(4mn-3m+4n+1)-2n+5, \frac{m}{2}(4mn-3m+4n+1)-2n+7, \frac{m}{2}(4mn-3m+4n+1)-2n+9, \ldots\}$ consists of consecutive integers, it implies that hypergraph $\mathscr{T}l_n$ admits an $(\frac{m}{2}(4mn-3m+4n+1)+8i-2n-3, 2)$-hyperedge antimagic labeling. $\square$

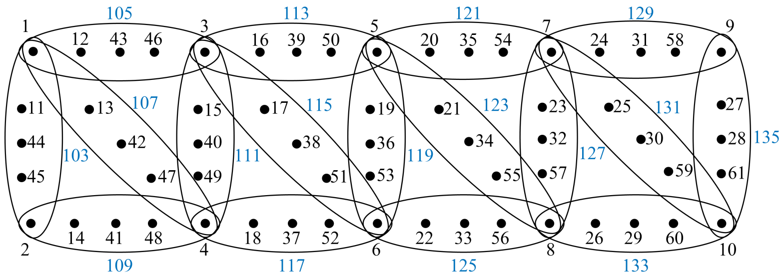

FIGURE 7. $(103,2)$-hyperedge antimagic labeling of $\mathscr{T}l_5$

For having more detail illustration of the existence of $(\frac{m}{2}(4mn-3m+4n+1)+8i-2n-3, 2)$-hyperedge antimagic labeling of $\mathscr{T}l_n$, we depict the graph in Figure 7.

DISCUSSION

We have studied (a,d)-hyperedge antimagic labeling on path hypergraph $\mathscr{P}_n$ and triangular ladder hypergraph $\mathscr{T}l_n$. We found the existence of both graphs on various a and d. Base on this type of antimagic labeling, we can extend for future study, namely super (a,d)-hyperedge antimagic labeling. The definition of super (a,d)-hyperedge antimagic labeling of hypergraph can be seen in Defintion 1. To determine the super antimagicness of hypergraph is considered to be a hard problem, even it will be a NP-Hard problem. Thus, we initiate to give an upper bound of super (a,d)-hyperedge antimagic labeling to give insight in which specific d the super (a,d)-hyperedge antimagic exists.

Definition 1. *Let $\mathbb{G} = (V,E)$ be simple connected hypergraph. The hypergraph $\mathbb{G}$ is called a super (a,d)-hyperedge antimagic labeling if $\mathbb{G}$ has an antimagic labeling. A vertex label and edge label functions $f : V(\mathbb{G}) \to \{1,2,3,\ldots,V(\mathbb{G})\}$ and $f : E(\mathbb{G}) \to \{V(\mathbb{G})+1,\ldots,V(\mathbb{G})+E(\mathbb{G})\}$ and weight $w(e_i) = \sum f(e_i) + \sum f(V_{i,j})$, where i is the number of hyperedges, j is the number of vertices in a hyperedge, and e_i is a set of vertices and edges on a hyperedge, there is a distinct $w(e_i)$ for each hyperedges.*

Theorem 8. *If (p,q)-hypergraph is super (a,d)-hyperedge antimagic labeling, then*

$$d \leq \frac{(p_G - p_H)p_H + (q_G - q_H)q_H}{s-1}$$

for $p_G = |V|, q_G = |\mathscr{E}|, p_H = |V'|, q_H = |\mathscr{E}'|, s = |H_i|$.

Proof. $f(V') = \{1,2,3,\ldots,p_H\}$ and $f(\mathscr{E}') = \{p_G+1, p_G+2, \ldots, p_G+q_H\}$. If (p,q)-hypergraph has super (a,d)-hyperedge antimagic labeling with $f(V \cup \mathscr{E}) = \{1,2,3,\ldots,p_G+q_G\}$, then the weight set of a hypergraph is $\{a, a+d, a+2d, \ldots, a(s-1)d\}$ where a is the smallest weight. Let A be the weight set, then it holds:

$$\sum A = \sum_{V \in V(A)} f(V) + \sum_{\mathscr{E} \in \mathscr{E}(A)} f(\mathscr{E}) = \{a, a+d, a+2d, \ldots, a+(s-1)d\}$$

$$\begin{aligned} 1+2+\ldots+p_H+(p_G+1)+(p_G+2)+\ldots+(p_G+q_H) &\leq a \\ \frac{p_H}{2}(1+p_H)+q_Hp_G+\frac{q_H}{2}(1+q_H) &= \\ \frac{p_H}{2}+\frac{p_H^2}{2}+q_Hp_G+\frac{q_H}{2}+\frac{q_H^2}{2} &\leq a \end{aligned}$$

While for the largest weight, it holds:

$$
\begin{aligned}
a+(s-1)d &\leq p_H p_G - \frac{p_H-1}{2}(1+(p_H-1)) + q_H p_G + q_H q_G - \frac{q_H-1}{2}(1+(q_H-1))\\
&= p_H p_G - \frac{p_H-1}{2}(p_H) + q_H p_G + q_H q_G - \frac{q_H-1}{2}(q_H)\\
(s-1)d &\leq p_H p_G - \frac{p_H-1}{2}p_H + q_H p_G + q_H q_G - \frac{q_H-1}{2}q_H - a\\
(s-1)d &\leq p_H p_G - \frac{p_H-1}{2}p_H + q_H p_G + q_H q_G - \frac{q_H-1}{2}q_H - \left(\frac{p_H}{2}+\frac{p_H^2}{2}+q_H p_G+\frac{q_H}{2}+\frac{q_H^2}{2}\right)\\
&= p_H p_G - \frac{p_H^2}{2}+\frac{p_H}{2}+q_H p_G + q_H q_G - \frac{q_H^2}{2}+\frac{q_H}{2}-\frac{p_H}{2}-\frac{p_H^2}{2}-q_H p_G-\frac{q_H}{2}-\frac{q_H^2}{2}\\
&= p_H p_G + q_H q_G - p_H^2 - q_H^2\\
&= p_H p_G - p_H^2 + q_H q_G - q_H^2\\
(s-1)d &\leq (p_G-p_H)p_H + (q_G-q_H)q_H\\
d &\leq \frac{(p_G-p_H)p_H + (q_G-q_H)q_H}{s-1}
\end{aligned}
$$

So, for $p_G = |V|, q_G = |\mathscr{E}|, p_H = |V'|, q_H = |\mathscr{E}'|, s = |H_i|$, it concludes $d \leq \frac{(p_G-p_H)p_H+(q_G-q_H)q_H}{s-1}$. □

CONCLUSION

In this paper, we have shown eight theorems on antimagicness of hypergraph study. Seven theorems studies for $(a,d)-$hyperedge antimagic labeling, namely path hypergraph $\mathscr{P}_n$ and triangular ladder hypergraph $\mathscr{T}l_n$, and one theorem for super $(a,d)-$hyperedge antimagic labeling. For the second type of hypergraph labeling, we just initiate to give an upper bound of super (a,d)-hyperedge antimagic labeling. Thus, we propose an open problems for future research. Determine the existence for super (a,d)-hyperedge antimagic labeling of specific families of hypergraph.

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REFERENCES

1. Dafik, M. Miller, J. Ryan, and B. M., "On super $(a,d)-$edge antimagic total labeling of disconnected graphs," Discrete Mathematics **309**, 4909–4915 (2009).
2. S. Arumugam and M. Nalliah, "Super $(a,d)-$edge antimagic total labelings of friendship graphs," Australas J Comb **53**, 237–244 (2012).
3. I. Y. Arianti, Dafik, and Slamin, "uper $(a,d)-$edge antimagic total labeling of connected disc brake graph," Prosiding Seminar Matematika dan Pendidikan Matematik **1** (2014).
4. S. Arumugam and M. Nalliah, "Super $(a,d)-$edge antimagic total labelings of friendship and generalized friendship graphs," Electronic Notes in Discrete Mathematics **48**, 103–110 (2015).
5. M. Mahmudah, Dafik, and Slamin, "Super $(a,d)-$edge antimagic total labeling of silkworm graph," Kadikma **6** (2015).
6. R. Adawiyah, Dafik, and Slamin, "Super $(a,d)-$edge antimagic total labeling of pentagonal chain graph," Kadikma **6** (2015).
7. E. R. Albirri, Dafik, and Slamin, "Super $(a,d)-$edge antimagic total labeling of connected lampion graph," Kadikma **6** (2015).
8. D. A. Hadi, Dafik, and Slamin, "Super $(a,d)-$edge antimagic total labeling of connected tribune graph," Kadikma **6** (2015).
9. D. B. Sumarno, Dafik, and K. A. Santoso, "Super $(a,d)-$edge antimagic total labeling of connected ferris wheel graph," Jurnal Ilmu Dasar **15**, 123–130 (2015).
10. I. H. Agustin and D. Hardiyantik, "The connected and disjoint union of semi jahangir graphs admit a cycle-super $(a,d)-$antimagic total labeling," Journal of Physics: Conference Series **693**, 12006 (2016).
11. Teichert, "Classes of hypergraphs with sum number one," Discussiones Mathematicae Graph Theory **20**, 93–103 (2000).

12. Teichert, "Sum labelings of cycle hypergraphs," Discussiones Mathematicae Graph Theory **20**, 255–265 (2000).
13. M. Sonntag, "Antimagic vertex labelings of hypergraphs," Discrete Mathematics **247**, 187–199 (2002).
14. T. Parag and A. Elgammal, "Supervised hypergraph labeling," Research Gate , 2289–2296 (2011).
15. M. Javaid, "Labeling of graphs and hypergraphs," Lahore (2013).
16. M. Tuczynski, P. Wenus, and K. Wesek, "On cordial labeling of hypertrees," Discrete Mathematics and Theoretical Computer Science **21**, 1–14 (2019).
17. S. Venkatraman, G. Rajaram, and K. Krithivasan, "Unimodular hypergraph for dna sequencing: A polynomial time algorithm," Proceedings of the National Academy of Sciences, India Section A: Physical Sciences **90**, 49–56 (2020).
18. C. Purcell, J. Ryan, Z. Ryjacek, and M. Skyvova, "On exclusive sum labellings of hypergraphs," Graphs and Combinatorics **38**, 1–16 (2022).
19. Q. Dai and Y. Gao, "Hypergraph computation artificial intelligence: Foundations, theory, and algorithms," Singapore (2023).
20. S. Jin, Y. Hong, L. Zeng, Y. Jiang, Y. Lin, L. Wei, Z. Yu, X. Zeng, and X. Liu, "A general hypergraph learning algorithm for drug multi-task predictions in micro-to-macro biomedical networks," PLOS Computational Biology **19**, 1011597 (2023).
21. V. S. Cola, D. Chiaro, E. Prezioso, S. Izzo, and F. Giampaolo, "Insight extraction from e-health bookings by means of hypergraph and machine learning," IEEE Journal of Biomedical and Health Informatics (2023).
22. X. Sun, H. Cheng, B. Liu, J. Li, H. Chen, G. Xu, and H. Yin, "Self-supervised hypergraph representation learning for sociological analysis," IEEE Transactions on Knowledge and Data Engineering (2023).
23. J. Huang, S. Zhang, F. Yang, T. Yu, L. N. Prasad, M. Guduri, and K. Yu, "Hypergraph-based interference avoidance resource management in customer-centric communication for intelligent cyber-physical transportation systems," IEEE Transactions on Consumer Electronics (2023).
24. K. Grzesiak-Kopeć, P. Oramus, and M. Ogorzałek, "Hypergraphs and extremal optimization in 3d integrated circuit design automation," Advanced Engineering Informatics **33**, 491–501 (2017).
25. L. Xia, P. Zheng, X. Huang, and C. Liu, "A novel hypergraph convolution network-based approach for predicting the material removal rate in chemical mechanical planarization," Journal of Intelligent Manufacturing **33**, 2295–2306 (2022).
26. Dafik and I. H. Agustin, "Super $(a,d)-fn-$antimagic total labeling for a connected and disconnected amalgamation of fan graphs," AIP Conference Proceedings **1707** (2016).