Computing VL Status Index of Composite Graphs

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Abstract. In this study, we delve into the versatile application of the Neighborhood M-Polynomial (NM) in predicting a wide array of material characteristics. Our research investigates the capability of the neighborhood M-polynomial to discern neighborhood degree sum-based topological indices when analyzing synthetic polymers. These indices serve as pivotal tools, enabling us to accurately predict the diverse physical, chemical, and biological properties inherent in the materials under scrutiny.

Keywords: Status index, Degree, Distance, Connected graph, Topological coindices

INTRODUCTION

Topological indices are mathematical metrics that offer insights into the structural characteristics of finite and uncomplicated graphs [1]. These indices possess the remarkable property of remaining invariant even when graphs exhibit identical structural patterns, rendering them invaluable in the domains of Quantitative Structure-Activity Relationship ($QSAR$) and Quantitative Structure-Property Relationship ($QSPR$) research [2, 3, 4]. These structural descriptors, often referred to as topological indices, find wide-ranging applications in theoretical chemistry, enabling the simulation of diverse properties of chemical compounds. These properties encompass physicochemical, pharmacological, toxicological, nanoscale, biological attributes, and more [5, 6].

The origin of distance-based topological indices can be traced back to Wiener's Wiener index, which was initially employed to characterize the physical properties of a group of alkanes commonly known as paraffin [7, 8].

This article discusses VL status index and several types of composite graphs for the VLS index, such as Cartesian products, join graphs, and compositions of two linked graphs [9, 10, 11, 12, 13]. In this study, we direct our focus to a connected graph denoted as G, with its vertex set represented by $V(G)$ and its edge set by $E(G)$. The order of G is defined as $|V(G)| = n$, while the size of G is given by $|E(G)| = m$ [14]. An edge connecting two vertices, denoted as u and v, is typically represented as uv, and the degree of a vertex within G is denoted as $\Gamma_G(v)$.

Definition 1. *[9] The status of a vertex v in a graph G is the sum of the distances from v to every other vertex in G and it is denoted by $\xi_G(v)$, that is,*

$$\xi_G(v) = \sum_{u \in V(G)} \Gamma_G(u,v)$$

where, $\Gamma_G(u,v)$ is the distance between u and v in G. The status of a vertex is also called the "transmission" of the vertex.

Definition 2. *[7] The Wiener index $W(G)$ of a connected graph G is defined as the sum of the distances between all pairs of vertices of G, that is,*

$$W(G) = \frac{1}{2} \sum_{u,v \in V(G)} \Gamma_G(u,v) = \frac{1}{2} \sum_{u \in V(G)} \xi_G(u)$$

Definition 3. *[15] The first Zagreb index is defined as*

$$M_1(G) = \sum_{u \in V(G)} (\Gamma_G(u))^2 = \sum_{uv \in E(G)} (\Gamma_G(u) + \Gamma_G(v))$$

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Definition 4. *[15] The second Zagreb index is defined as*

$$M_2(G) = \sum_{uv\in E(G)} (\Gamma_G(u)\Gamma_G(v))$$

Definition 5. *[16] The VL index is defined as*

$$VL(G) = \frac{1}{2}\sum_{uv\in E(G)} [\Gamma_G(u)+\Gamma_G(v)+\Gamma_G(u).\Gamma_G(v)].$$

Inspired by invariants such as the Zagreb indices, Ramane et al. introduced the inaugural status connectivity index, denoted as $S_1(G)$ and the second Status connectivity index, denoted as $S_2(G)$, for a connected graph G.

Definition 6. *[17] The first status connectivity index is defined as*

$$S_1(G) = \sum_{uv\in E(G)} (\xi_G(u)+\xi_G(v))$$

Definition 7. *[17] The second status connectivity index is defined as*

$$S_2(G) = \sum_{uv\in E(G)} (\xi_G(u)\xi_G(v))$$

Definition 8. *[10] The VL status index is defined as*

$$VLS(G) = \frac{1}{2}\sum_{uv\in E(G)} [\xi_G(u)+\xi_G(v)+\xi_G(u)\cdot\xi_G(v)]$$

Definition 9. *The Cartesian product, $G\otimes H$, of the graphs G and H has the vertex set $V(G\otimes H) = V(G)\times V(H)$ and $(u,x)(v,y)$ is an edge of $G\otimes H$ if $u = v$ and $xy \in E(H)$ or, $uv \in E(G)$ and $x = y$. To each vertex $u \in V(G)$, there is an isomorphic copy of H in $G\otimes H$ and to each vertex $v \in V(H)$, there is an isomorphic copy of G in $G\otimes H$.*

METHODOLOGY

This study thoroughly investigates the *VLS* index within composite graphs, including Cartesian products, join graphs and compositions of two connected graphs. Our method entails defining and computing the *VLS* index for connected graphs and extending its application to composite structures, incorporating considerations of distances and degrees. The findings eloquently demonstrate the effectiveness of the *VLS* index in understanding the intricacies of connectivity and transmission within composite graphs, underscoring its potential significance, particularly in the realms of *QSAR* and *QSPR* research.

RESULTS ON CARTESIAN PRODUCTS OF GRAPHS

The Cartesian product of G and H is denoted by $G\otimes H$ and it as vertex set $V(G\otimes H) = V(G)\times V(H)$ and (u,v) and $(u\prime,v\prime)\in E(G\otimes H)$ if $u = u\prime$ and $vv\prime \in E(H)$ or $v = v\prime$ and $uu\prime \in E(G)$ [18].

Lemma 10. *[19][kandan] Let G and H be two connected graph with n_1 and n_2 vertices, respectively. Then the status of any vertex $x_r \in V(G\otimes H)$ is $n_2\xi_G(u_r)+n_1\xi_H(v_r)$.*

Theorem 11. *If G and H are connected graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively, then*

$$\begin{aligned} VLS(G\otimes H) = \frac{1}{2}\Bigg[& n_2(n_2+2n_1W(H))S_1(G)+2n_2(2m_2+n_1S_1(H))W(G) \\ &+ n_2^3S_2(G)+4n_1m_1W(H)+n_1^2(S_1(H)+n_1S_2(H))+n_1^2m_1 \\ &\sum_{v_r\in V(H)} \xi_H^2(v_r)+n_2^2m_2\sum_{u_r\in V(G)} \xi_G^2(u_r)\Bigg]. \end{aligned}$$

Proof. From the definition of VLS status index, we have

$$
\begin{aligned}
VLS(G\otimes H) &= \frac{1}{2}\sum_{x_r y_r\in E(G\otimes H)} \xi_{G\otimes H}(x_r)+\xi_{G\otimes H}(y_r)+\xi_{G\otimes H}(x_r).\xi_{G\otimes H}(y_r)\\
&= \frac{1}{2}\Bigg[\sum_{x_r y_r\in E(G\otimes H)} \xi_{G\otimes H}(x_r)+\xi_{G\otimes H}(y_r)\\
&\quad +\sum_{x_r y_r\in E(G\otimes H)} \xi_{G\otimes H}(x_r).\xi_{G\otimes H}(y_r)\Bigg]\\
&= \frac{1}{2}\Bigg[\sum_{x_r\in V(G\otimes H)} d(x_r)\xi_{G\otimes H}(x_r)+\sum_{\substack{k=1\\ u_iu_j\in E(H)\\ v_k\in V(H)}}^{n_2}(n_2\xi_G(u_i)+n_1\xi_H(v_k))\\
&\quad (n_2\xi_G(u_j)+n_1\xi_H(v_k))+\sum_{\substack{s=1\\ v_iv_j\in E(H)\\ u_s\in V(G)}}^{n_1}(n_2\xi_G(u_s)+n_1\xi_H(v_i))\\
&+(n_2\xi_G(u_s)+n_1\xi_H(v_j))\Bigg]\\
&= \frac{1}{2}[A_1+A_2] \qquad (1)
\end{aligned}
$$

Now,

$$
\begin{aligned}
A_1 &= \sum_{x_r\in V(G\otimes H)} d(x_r)\xi_{G\otimes H}(x_r)\\
&= \sum_{\substack{u_r\in V(G)\\ v_r\in V(H)}}(d_G(u_r)+d_H(v_r))(n_2\xi_G(u_r)+n_1\xi_H(v_r))\\
&\quad \textit{Since } d_{G\otimes H}(.)=d_G(.)+d_H(.) \textit{ and using lemma } 1\\
&= n_2\sum_{\substack{u_r\in V(G)\\ v_r\in V(H)}} d_G(u_r)\xi_G(u_r)+n_1\sum_{\substack{u_r\in V(G)\\ v_r\in V(H)}} d_G(u_r)\xi_H(v_r)\\
&\quad +n_2\sum_{\substack{u_r\in V(G)\\ v_r\in V(H)}} d_H(v_r)\xi_G(u_r)+n_1\sum_{\substack{u_r\in V(G)\\ v_r\in V(H)}} d_H(v_r)\xi_H(v_r)\\
&= n_2^2\sum_{u_r\in V(G)} d_G(u_r)\xi_G(u_r)+n_1\sum_{u_r\in V(G)} d_G(u_r)\sum_{u_r\in V(G)}\xi_H(v_r)+n_2\sum_{v_r\in V(H)} d_H(v_r)\\
&\quad \sum_{u_r\in V(G)}\xi_G(u_r)+n_1^2\sum_{v_r\in V(H)} d_H(v_r)\xi_H(v_r)\\
&= n_2^2S_1(G)+n_1(2m_1)\sum_{v_r\in V(H)}\xi_H(v_r)+n_2(2m_2)\sum_{u_r\in V(G)}\xi_G(u_r)+n_1^2S_1(H)\\
&= n_2^2S_1(G)+4n_1m_1W(H)+4n_2m_2W(G)+n_1^2S_1(H)
\end{aligned}
$$

and

$$A_2 = \sum_{\substack{k=1\\ u_iu_j\in E(H)\\ v_k\in V(H)}}^{n_2} (n_2\xi_G(u_i)+n_1\xi_H(v_k))(n_2\xi_G(u_j)+n_1\xi_H(v_k))$$

$$+ \sum_{\substack{s=1\\ v_iv_j\in E(H)\\ u_s\in V(G)}}^{n_1} (n_2\xi_G(u_s)+n_1\xi_H(v_i))+(n_2\xi_G(u_s)+n_1\xi_H(v_j))\Bigg]$$

Since by lemma 1

$$= \sum_{\substack{k=1\\ u_iu_j\in E(H)\\ v_k\in V(H)}}^{n_2} n_2^2\xi_G(u_i)\xi_G(u_j) + \sum_{\substack{k=1\\ u_iu_j\in E(H)\\ v_k\in V(H)}}^{n_2} n_1n_2\xi_G(u_i)\xi_H(v_k)$$

$$+ \sum_{\substack{k=1\\ u_iu_j\in E(H)\\ v_k\in V(H)}}^{n_2} n_1n_2\xi_G(u_j)\xi_H(v_k) + \sum_{\substack{k=1\\ u_iu_j\in E(H)\\ v_k\in V(H)}}^{n_2} n_1^2\xi_H^2(v_k)$$

$$+ \sum_{\substack{s=1\\ v_iv_j\in E(H)\\ u_s\in V(G)}}^{n_1} n_1^2\xi_H(v_i)\xi_H(v_j) + \sum_{\substack{s=1\\ v_iv_j\in E(H)\\ u_s\in V(G)}}^{n_1} n_1n_2\xi_G(u_s)\xi_H(v_i)$$

$$+ \sum_{\substack{s=1\\ v_iv_j\in E(H)\\ u_s\in V(G)}}^{n_1} n_1n_2\xi_G(u_s)\xi_H(v_j) + \sum_{\substack{s=1\\ v_iv_j\in E(H)\\ u_s\in V(G)}}^{n_1} n_2^2\xi_G^2(u_s)$$

$$= n_2^3S_2(G)+2n_1n_2S_1(G)W(H)+n_1^2m_1\sum_{v_r\in V(H)}\xi_H^2(v_r)+n_1^3S_2(H)$$

$$+ 2n_1n_2S_1(H)W(G)+n_2^2m_2\sum_{u_r\in V(G)}\xi_G^2(u_r).$$

Hence, by equation (1), we get

$$VLS(G\otimes H) = \frac{1}{2}\Bigg[n_2^2S_1(G)+4n_1m_1W(H)+4n_2m_2W(G)+n_1^2S_1(H)+n_2^3S_2(G)$$

$$+ 2n_1n_2S_1(G)W(H)+n_1^2m_1\sum_{v_r\in V(H)}\xi_H^2(v_r)+n_1^3S_2(H)$$

$$+ 2n_1n_2S_1(H)W(G)+n_2^2m_2\sum_{u_r\in V(H)}\xi_G^2(u_r)\Bigg]$$

$$= \frac{1}{2}\Bigg[n_2(n_2+2n_1W(H))S_1(G)+2n_2(2m_2+n_1S_1(H))W(G)$$

$$+ n_2^3S_2(G)+4n_1m_1W(H)+n_1^2(S_1(H)+n_1S_2(H))+n_1^2m_1$$

$$\sum_{v_r\in V(H)}\xi_H^2(v_r)+n_2^2m_2\sum_{u_r\in V(G)}\xi_G^2(u_r)\Bigg].$$

□

Theorem 12. *[17][Ramane] Let G be a connected graph with n vertices and m edges and $diam(G)\le 2$. Then $S_1(G)=4m(n-1)-M_1(G)$ and $S_2(G)=4m(n-1)^2-2(n-1)M_1(G)+M_2(G)$.*

The proof of the following corollaries are the direct consequence of Theorem 1 and Theorem 2. Let G and H be a connected graph on n_1 and n_2 vertices and m_1 and m_2 edges respectively. Let $diam(G) \leq 2$ and $diam(H) \leq 2$. Then

$$VLS(G \otimes H) = \frac{1}{2}\Big[4m_1n_2^2(n_1-1) - n_2^2M_1(G) + 4n_1m_1W(H) + 4n_2m_2W(G) + 4 m_2n_1^2(n_2-1) - n_1^2M_1(H) + n_2^3(4m_1(n_1-1)^2 + M_2(G)) + n_1^3(4m_2 (n_2-1)^2 + M_2(H)) + 8n_1n_2(m_1(n_1-1)W(H) + m_2(n_2-1)W(G)) + n_1^2m_1 \sum_{v_r \in V(H)} \xi_H^2(v_r) + n_2^2m_2 \sum_{u_r \in V(G)} \xi_G^2(u_r) - (2n_1^3(n_2-1) + 2n_1 n_2W(G))M_1(H) - (2n_2^3(n_1-1) + 2n_1n_2W(H))M_1(G)\Big].$$

Let G and H be a connected $r-regular$ graph with n_1 and n_2 vertices and m_1 and m_2 edges, respectively. Let $diam(G) \leq 2$ and $diam(H) \leq 2$. Then

$$VLS(G \otimes H) = \frac{1}{2}\Big[2m_1n_2^2(2(n_1-1) - r_1) + 4n_1m_1W(H) + 4n_2m_2W(G) + 2m_2 n_1^2(2(n_2-1) - r_2) + n_2^3(4m_1(n_1-1)^2 + m_1r_1^2) + n_1^3(4m_2(n_2-1)^2 + m_2r_2^2) + 8n_1n_2(m_1(n_1-1)W(H) + m_2(n_2-1)W(G)) + n_1^2m_1 \sum_{v_r \in V(H)} \xi_H^2(v_r) + n_2^2m_2 \sum_{u_r \in V(G)} \xi_G^2(u_r) - 2(2n_1^3(n_2-1) + 2n_1n_2 W(G))m_2r_2 - 2(2n_2^3(n_1-1) + 2n_1n_2W(H))m_1r_1\Big].$$

RESULTS ON JOIN GRAPHS

The join of two graphs G and H, denoted as $G+H$, results from the union of $G \cup H$ and the inclusion of all edges connecting the vertices in $V(G)$ with those in $V(H)$. In the structure of $G+H$, the distance between any two vertices u and v within the combined graph $G+H$ is given by:

$$d_{G+H}(u,v) = \begin{cases} o, & if \;\; u = v, \\ 1, & if \;\; uv \in E(G) \;\; or \;\; uv \in E(H) \;\; or \;\; (u \in V(G) \;\; and \;\; v \in V(H)), \\ 2, & otherwise. \end{cases}$$

Moreover, the degree of a vertex v in $V(G+H)$ is

$$d_{G+H}(v) = \begin{cases} d_G(v) + |V(H)|, & if \;\; v \in V(G), \\ d_H(v) + |V(G)|, & if \;\; v \in V(H). \end{cases}$$

Theorem 13. *Let G and H be two connected graphs n_1, n_2 vertices and m_1, m_2 edges, respectivily. Then*

$$VLS(G+H) = \frac{1}{2}[(2m_1 + n_1n_2)(2n_1 + n_2 - 2) - M_1(G) - 2n_1n_2m_1 + (2m_2 + n_1n_2)(2n_2 + n_1 - 2) - M_1(H) - 2n_1n_2m_2 + M_2(G) + M_2(H) - (2n_1 + n_2 - 2)M_1(G) - (2n_2 + n_1 - 2)M_1(H) + (2n_1 + n_2 - 2)[(2n_1 + n_2 - 2) m_1 - 2n_1m_2] - (2n_2 + n_1 - 2)[(2n_2 + n_1 - 2)m_2 - 2n_2m_1 + n_1n_2(2 n_1 + n_2 - 2)] + 4m_1m_2]$$

Proof. Let u be a vertex in $V(G)$. Then from the structure of $G+H$, we obtain

$$\xi_{G+H}(u) = 2n_1 + n_2 - 2 - d_G(u).$$

Similarly, if v is a vertex of H, then

$$\xi_{G+H}(v) = 2n_2 + n_1 - 2 - d_G(v).$$

From the definition of VLS status index, we have

$$\begin{aligned}
VLS(G+H) &= \frac{1}{2}\left[\sum_{uv\in E_{G+H}} \xi_{G+H}(u) + \xi_{G+H}(v) + \xi_{G+H}(u)\cdot\xi_{G+H}(v)\right] \\
&= \frac{1}{2}\left[\sum_{uv\in E_{G+H}} \xi_{G+H}(u) + \xi_{G+H}(v) + \sum_{uv\in E_{G+H}} \xi_{G+H}(u)\cdot\xi_{G+H}(v)\right] \\
&= \frac{1}{2}[A_1 + A_2] \qquad (2)
\end{aligned}$$

Where,

$$\begin{aligned}
A_1 &= \sum_{uv\in E_{G+H}} \xi_{G+H}(u) + \xi_{G+H}(v) \\
&= \sum_{u\in V_{G+H}} d_{G+H}(u)\xi_{G+H}(u) \\
&= \sum_{u\in V_G} (d_G(u) + n_2)(2n_1 + n_2 - 2 - d_G(u)) \\
&\quad + \sum_{u\in V_H} (d_H(u) + n_1)(2n_2 + n_1 - 2 - d_H(u)) \\
&= \sum_{u\in V_G}\left(2n_1 + n_2 - 2)d_G(u) - (d_G(u))^2 + n_2(2n_1 + n_2 - 2) - n_2 d_G(u)\right) \\
&\quad + \sum_{u\in V_H}\left(2n_2 + n_1 - 2)d_H(u) - (d_H(u))^2 + n_1(2n_2 + n_1 - 2) - n_1 d_H(u)\right) \\
&= \sum_{u\in V_G}(2n_1 + n_2 - 2)d_G(u) - \sum_{u\in V_G}(d_G(u))^2 + \sum_{u\in V_G} n_2(2n_1 + n_2 - 2) - n_2 \\
&\quad \sum_{u\in V_G} d_G(u) + \sum_{u\in V_H}(2n_2 + n_1 - 2)d_H(u) - \sum_{u\in V_H}(d_H(u))^2 + \sum_{u\in V_H} n_1(2n_2 \\
&\quad + n_1 - 2) - n_1\sum_{u\in V_H} d_H(u) \\
&= (2m_1 + n_1n_2)(2n_1 + n_2 - 2) - M_1(G) - 2n_1n_2m_1 + (2m_2 + n_1n_2)(2n_2 \\
&\quad + n_1 - 2) - M_1(H) - 2n_1n_2m_2
\end{aligned}$$

The edge set of $G+H$ can be partitioned into three subsets, namely,

$$E_1 = \{uv \in E(G+H) | uv \in E(G)\},$$

$$E_2 = \{uv \in E(G+H) | uv \in E(H)\} and$$

$$E_3 = \{uv \in E(G+H) | u \in v(G), v \in V(H)\}.$$

The contribution of the edges in E_1,

$$\begin{aligned}
A_2 &= \sum_{uv\in E_1} \xi_{G+H}(u)\cdot\xi_{G+H}(v) + \sum_{uv\in E_2} \xi_{G+H}(u)\cdot\xi_{G+H}(v) + \\
&\quad \sum_{uv\in E_3} \xi_{G+H}(u)\cdot\xi_{G+H}(v) \\
&= B_1 + B_2 + B_3
\end{aligned}$$

The contribution of the edges in E_1 is given by

$$\begin{aligned} B_1 &= \sum_{uv \in E_1} \xi_{G+H}(u) \cdot \xi_{G+H}(v) \\ &= \sum_{uv \in E_G} (2n_1 + n_2 - 2 - d_G(u))(2n_1 + n_2 - 2 - d_G(v)) \\ &= (2n_1 + n_2 - 2)^2 m_1 - (2n_1 + n_2 - 2) M_1(G) + M_2(G) \end{aligned}$$

The contribution of the edges in E_2 is given by

$$\begin{aligned} B_2 &= \sum_{uv \in E_2} \xi_{G+H}(u) \cdot \xi_{G+H}(v) \\ &= \sum_{uv \in E_H} (2n_2 + n_1 - 2 - d_G(v))(2n_2 + n_1 - 2 - d_G(v)) \\ &= (2n_2 + n_1 - 2)^2 m_2 - (2n_2 + n_1 - 2) M_1(H) + M_2(H) \end{aligned}$$

Similarly, the contribution of the edges in E_3 is given by

$$\begin{aligned} B_3 &= \sum_{uv \in E_3} \xi_{G+H}(u) \cdot \xi_{G+H}(v) \\ &= \sum_{u \in V_G} \sum_{v \in V_H} (2n_1 + n_2 - 2 - d_G(u))(2n_2 + n_1 - 2 - d_H(v)) \\ &= (2n_1 + n_2 - 2)(2n_2 + n_1 - 2) n_1 n_2 - 2n_1 m_2 (2n_1 + n_2 - 2) - 2n_2 m_1 (2n_2 \\ &\quad + n_1 - 2) + 4 m_1 m_2 \end{aligned}$$

The total contribution of the edges in $G+H$ and its A_2 is given by

$$\begin{aligned} A_2 = M_2(G) &+ M_2(H) - (2n_1 + n_2 - 2) M_1(G) - (2n_2 + n_1 - 2) M_1(H) + (2n_1 \\ &+ n_2 - 2) \quad [(2n_1 + n_2 - 2) m_1 - 2n_1 m_2] - (2n_2 + n_1 - 2)[(2n_2 + n_1 - 2) \\ &m_2 - 2n_2 m_1 + n_1 n_2 \quad (2n_1 + n_2 - 2)] + 4 m_1 m_2 \end{aligned}$$

□

Put A_1 and A_2 in (1)

$$\begin{aligned} VLS(G+H) = \frac{1}{2}[&(2m_1 + n_1 n_2)(2n_1 + n_2 - 2) - M_1(G) - 2n_1 n_2 m_1 + (2m_2 + n_1 \\ &n_2)(2n_2 + n_1 - 2) - M_1(H) - 2n_1 n_2 m_2 + M_2(G) + M_2(H) - (2 \\ &n_1 + n_2 - 2) M_1(G) - (2n_2 + n_1 - 2) M_1(H) + (2n_1 + n_2 - 2)[(2 \\ &n_1 + n_2 - 2) m_1 - 2n_1 m_2] - (2n_2 + n_1 - 2)[(2n_2 + n_1 - 2) m_2 - \\ &2n_2 m_1 + n_1 n_2 (2n_1 + n_2 - 2)] + 4 m_1 m_2] \end{aligned}$$

COMPOSITION OF GRAPHS

The composition of two graphs, denoted as $G[H]$, is formed as follows: The vertex set of $G[H]$ is the Cartesian product of the vertex sets of G and H, denoted as $V(G) \times V(H)$, and two vertices, (u_i, v_r) and (u_k, v_s), in $G[H]$ are adjacent if and only if either $u_i u_k \in E(G)$ or $u_i = u_k$ and $v_r v_s \in E(H)$ [20].

Theorem 14. *Let G and H be two connected graphs n_1, n_2 vertices and m_1, m_2 edges, respectivily. Then*

$$\begin{aligned} VLS(G[H]) = \frac{1}{2}\Bigg[& n_2^3 S_1(G) + 4n_2m_1W(G) - n_1M_1(H) + 4n_2^2m_1(n_2-1) + 4n_1m_2 \\ & (n_2-1) - 4m_1m_2n_2 + n_2^4S_2(G) + 2n_2^2(n_2(n_2-1)-m_2)S_1(G) + 8 \\ & n_2m_2(n_2-1)W(G) - 2n_2W(G)M_1(H) - 2n_1(n_2-1)M_1(H) + n_1 \\ & M_2(H) + n_2^2m_2 \sum_{u_i \in V(G)} (\xi_G(u_i))^2 + 4(n_2-1)^2(n_1m_2 + m_1n_2^2) + 4m_1 \\ & m_2(m_2 - 2n_2(n_2-1))\Bigg]. \end{aligned}$$

Proof. For the composition of two graphs, the degree of a vertex (u,v) of $G[H]$ is given by $d_{G[H]}((u,v)) - n_2d_G(u) + d_H(v)$. Moreover, the distance between two vertices (u_i,v_r) and (u_k,v_s) of $G[H]$ is

$$d_{G[H]}((u_i,v_r),(u_k,v_s)) = \begin{cases} d_G(u_i,u_k) & u_i \neq u_k. \\ 2 \;\; u_i = u_k, & v_rv_s \notin E(H). \\ 1 \;\; u_i = u_k, & v_rv_s \in E(H). \end{cases}$$

Let (u_i,v_r) be a vertex of $G[H]$. Then

$$\begin{aligned} \xi_{G[H]}((u_i,v_r)) &= \sum_{(u_k,v_s)\in V(G[H])} d_{G[H]}((u_i,v_r),(u_k,v_s)) \\ &= \sum_{(u_k,v_s)\in V(G[H]),u_i\neq u_k} d_G(u_i,v_k) \\ &\quad + \sum_{(u_i,v_s)\in V(G[H])} d_{G[H]}((u_i,v_r),(u_i,v_s)) \\ &= n_2\xi_G(u_i) + d_H(v_r) + 2(n_2-1-d_H(v_r)) \\ &= n_2\xi_G(u_i) + 2(n_2-1) - d_H(v_r) \end{aligned} \quad (3)$$

From the definition of VLS status index, we have

$$\begin{aligned} VLS(G[H]) &= \frac{1}{2}\left[\sum_{uv\in E_{G[H]}} \xi_{G[H]}(u) + \xi_{G[H]}(v) + \xi_{G[H]}(u)\cdot\xi_{G[H]}(v)\right] \\ &= \frac{1}{2}\left[\sum_{uv\in E_{G[H]}} \xi_{G[H]}(u) + \xi_{G[H]}(v) + \sum_{uv\in E_{G[H]}} \xi_{G[H]}(u)\cdot\xi_{G[H]}(v)\right] \\ &= \frac{1}{2}[A_1 + A_2] \end{aligned} \quad (4)$$

Where,

$$
\begin{aligned}
A_1 &= \sum_{uv \in E_{G[H]}} \xi_{G[H]}(u) + \xi_{G[H]}(v) \\
&= \sum_{(u_i,v_r) \in V(G \otimes H)} d_{G[H]}((u_i,v_r))\xi_{G[H]}((u_i,v_r)) \\
&= \sum_{u_i \in V(G)} \sum_{v_r \in V(H)} (n_2 d_G(u_i) + d_H(v_r))(n_2 \xi_G(u_i) + 2(n_2-1) \\
&\quad - d_H(v_r)) \qquad by\ (1) \\
&= \sum_{u_i \in V(G)} \sum_{v_r \in V(H)} [n_2^2 d_G(u_i)\xi_G(u_i) + 2(n_2-1)n_2 d_G(u_i) - n_2 d_G(u_i) \\
&\quad d_H(v_r) + n_2 d_H(v_r)\xi_G(u_i) + 2(n_2-1)d_H(v_r) - (d_H(v_r))^2] \\
&= \sum_{u_i \in V(G)} \sum_{v_r \in V(H)} n_2^2 d_G(u_i)\xi_G(u_i) + 2(n_2-1)n_2 \sum_{u_i \in V(G)} \sum_{v_r \in V(H)} d_G(u_i) \\
&\quad - n_2 \sum_{u_i \in V(G)} \sum_{v_r \in V(H)} d_G(u_i) d_H(v_r) + n_2 \sum_{u_i \in V(G)} \sum_{v_r \in V(H)} d_H(v_r)\xi_G(u_i) \\
&\quad + 2(n_2-1) \sum_{u_i \in V(G)} \sum_{v_r \in V(H)} d_H(v_r) - \sum_{u_i \in V(G)} \sum_{v_r \in V(H)} (d_H(v_r))^2
\end{aligned}
$$

By the definitions of first status connectivity, Wiener index and first Zagreb index, we obtain

$$
\begin{aligned}
A_1 &= n_2^3 S_1(G) + 4n_2 m_1 W(G) - n_1 M_1(H) + 4n_2^2 m_1 (n_2-1) + 4n_1 m_2 (n_2-1) \\
&\quad - 4m_1 m_2 n_2.
\end{aligned}
$$

and

$$
\begin{aligned}
A_2 &= \sum_{uv \in E_{G[H]}} \xi_{G[H]}(u) \cdot \xi_{G[H]}(v) \\
&= \sum_{u_i \in V(G)} \sum_{v_r v_s \in E(H)} \xi_{G[H]}((ui,v_r))\xi_{G[H]}((u_i,v_s)) \\
&\quad + \sum_{u_i u_k \in E(H)} \sum_{v_r \in V(H)} \sum_{v_s \in V(H)} \xi_{G[H]}((ui,v_r))\xi_{G[H]}((u_i,v_s)) \\
&= B_1 + B_2
\end{aligned}
$$

Now,

$$
\begin{aligned}
B_1 &= \sum_{u_i \in V(G)} \sum_{v_r v_s \in E(H)} \xi_{G[H]}((ui,v_r))\xi_{G[H]}((u_i,v_s)) \\
&= \sum_{u_i \in V(G)} \sum_{v_r v_s \in E(H)} (n_2 \xi_G(u_i) + 2(n_2-1) - d_H(v_r))(n_2 \xi_G(u_i) + 2(n_2-1) \\
&\quad - d_H(v_s)) \\
&= \sum_{u_i \in V(G)} \sum_{v_r v_s \in E(H)} [n_2^2(\xi_G(u_i))^2 + 2(n_2-1)n_2\xi_G(u_i) - n_2\xi_G(u_i)d_H(v_s) + 2 \\
&\quad (n_2-1)n_2\xi_G(u_i) + 4(n_2-1)^2 - 2(n_2-1)d_H(v_s) - n_2\xi_G(u_i)d_H(v_r) - 2 \\
&\quad (n_2-1)d_H(v_r) + d_H(v_r)d_H(v_s)] \\
&= \sum_{u_i \in V(G)} \sum_{v_r v_s \in E(H)} [n_2^2(\xi_G(u_i))^2 + 4(n_2-1)\xi_G(u_i) + 4(n_2-1) - n_2\xi_G(u_i) \\
&\quad (d_H(v_r) + d_H(v_s)) - 2(n_2-1)(d_H(v_r) + d_H(v_s)) + d_H(v_r)d_H(v_s)] \\
&= n_2^2 m_2 \sum_{u_i \in V(G)} (\xi_G(u_i))^2 + 8n_2(n_2-1)m_2 W(G) + n_1 M_2(H) - 2n_2 W(G) \\
&\quad M_1(H) - 2(n_2-1)n_1 M_1(H)
\end{aligned}
$$

and

$$B_2 = \sum_{u_iu_k\in E(H)}\sum_{v_r\in V(H)}\sum_{v_s\in V(H)} \xi_{G[H]}((ui,v_r))\xi_{G[H]}((u_i,v_s))$$
$$= \sum_{u_iu_k\in E(H)}\sum_{v_r\in V(H)}\sum_{v_s\in V(H)} (n_2\xi_G(u_i)+2(n_2-1)-d_H(v_r))(n_2\xi_G(u_k)+2$$
$$(n_2-1)-d_H(v_s))$$
$$= \sum_{u_iu_k\in E(H)}\sum_{v_r\in V(H)}\sum_{v_s\in V(H)} [n_2^2\xi_G(u_i)\xi_G(u_k)+2(n_2-1)n_2(\xi_G(u_i)+\xi_G(u_k))$$
$$+4(n_2-1)^2-n_2\xi_G(u_i)d_H(v_s)-n_2d_H(v_r)\xi_G(u_k)-2(n_2-1)(d_H(v_r)+$$
$$d_H(v_r))+d_H(v_r)d_H(v_s)]$$
$$= n_2^4S_2(G)+2n_2^2(n_2(n_2-1)-m_2)S_1(G)-8n_2m_1m_2(n_2-1)+4m_1m_2^2+4$$
$$(n_2-1)^2m_1n_2^2.$$

Therefore

$$A_2 = n_2^4S_2(G)+2n_2^2(n_2(n_2-1)-m_2)S_1(G)+8n_2m_2(n_2-1)W(G)-2n_2$$
$$W(G)M_1(H)-2n_1(n_2-1)M_1(H)+n_1M_2(H)+n_2^2m_2\sum_{u_i\in V(G)}(\xi_G(u_i))^2$$
$$+4(n_2-1)^2(n_1m_2+m_1n_2^2)+4m_1m_2(m_2-2n_2(n_2-1)).$$

Hence,

$$VLS(G[H]) = \frac{1}{2}\Bigg[n_2^3S_1(G)+4n_2m_1W(G)-n_1M_1(H)+4n_2^2m_1(n_2-1)+4n_1m_2$$
$$(n_2-1)-4m_1m_2n_2+n_2^4S_2(G)+2n_2^2(n_2(n_2-1)-m_2)S_1(G)+8$$
$$n_2m_2(n_2-1)W(G)-2n_2W(G)M_1(H)-2n_1(n_2-1)M_1(H)+n_1$$
$$M_2(H)+n_2^2m_2\sum_{u_i\in V(G)}(\xi_G(u_i))^2+4(n_2-1)^2(n_1m_2+m_1n_2^2)+4m_1$$
$$m_2(m_2-2n_2(n_2-1))\Bigg].$$

□

CONCLUSION

This inquiry significantly enriches graph theory, spotlighting the versatility and utility of the *VLS* index as an insightful structural descriptor. The findings underscore its adaptability, making a valuable contribution and inspiring continued exploration across a spectrum of scientific domains.

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REFERENCES

1. M. O. Albertson, *The irregularity of a graph*, Ars Combinatoria, 46, 1997, 219-225.
2. J. Devillers and A. T. Balaban, *Topological Indices and Related Descriptors in QSAR and QSPR*, Gordon and Breach, Amsterdam, 1999.
3. I. Gutman, *Geometric approach to degree based topological indices*, MATCH Communications in Mathematical and in Computer Chemistry, 86(1), 2021, 11-16.
4. M. Randic, *Quantitative Structure-Property Relationship: boiling points and planar benzenoids*, New Journal of Chemistry, 2(10), 1996, 1001-1009.
5. F. M. Bruckler, T. Doslic, A. Graovac and I. Gutman, *On a class of distance-based molecular structure descriptors*, Chemical Physics Letters, 503, 2011, 336-338.
6. V. Lokesha, Sushmitha Jain and A.S.Maragadam, *M-polynomials for subdivision graphs of antiviral drugs using in treatment of covid-19*, International Asian congress on Contemporary Sciences-V, Azerbaijan Nakhchivan State University-İksad Publications , 2021, 1005-1018.
7. H. Winer, *Structural determination of paraffin boiling points*, Journal of american chemical society, 69, 1947, 17-20.
8. M. Eliasi and B. Taeri, *Four new sums of graphs and their Wiener indices*, Discrete Applied Mathematics, 157(1), 2009, 794-803.
9. Harary, Frank, **Graph Theory**, *Addison-Wesley Publishing Company*, 1969.
10. V. Lokesha, Suvarna, A. S. Cevik, *VL status co-index of connected graphs*, Proceedings of the Jangjeon Mathematical Society, 24(3), 2021, 285-295.
11. V. Lokesha, Suvarna, A. S. Cevik and I. N. Cangul, *VL reciprocal status index and co-index of graphs*, Journal of Mathematics, 2021, 1-10.
12. S. Suvarna, Y. Shanthakumari, V. Lokesha *VL reciprocal status index and coindex of connected graphs*, Proceeding of the Jangjeon Mathematical Society, 25(3), 2022, 319-329.
13. V. Lokesha, Suvarna, *Bounds for VL status index and coindex of graphs and validate to few specific graphs*, Applied Nonlinear Analysis and Soft Computing (Springer Nature), 2023.
14. Dafik, V. Lokesha, A. S. Maragadam, M. Manjunath, Ika Hesti Agustin, Semi-Total point Graph of Neighbourhood Edge Corona of two Graphs, European Journal of Pure and Applied Mahematics, 16(2), 2023, 1094-1109.
15. I.Gutman and N.Trinajstic, *Graph theory and molecular orbitals, total $\phi-$electron energy of alternant hydrocarbons*, chemical physics letters, 17(4), 1972, 535-538.
16. T. Deepika, *VL index and bounds for the Tensor products of $F-$sum graphs*, TWMS J. App. and Eng. Math., 11(2), 2021, 374-385.
17. Ramane, H. S. and Yyalnaik, A. S., *Status connectivity indices of graphs and its applications to the boiling point of benzenoid hydrocarbons*, Journal of applied mathematics and computing, 55(1-2), 2017, 609-627.
18. S. Klav˘zar, *On the PI index: PI-partitions and Cartesian product graphs*, MATCH Commun. Math. Comput. Chem., 57, 2007, 573-586.
19. P. Kandan, *Status connectivity indices of cartesian product of graphs*, TWMS Journal of Applied and Engineering Mathematics, 9(4), 2016, 747-754.
20. Y.N. Yeh, I. Gutman, *On the sum of all distances in composite graphs*, Discrete Math., 135, 1994, 359-365.