On b-Coloring of Some Graphs

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Abstract. Let $G=(V,E)$ be a graph with vertex set V and edge set E. A proper $k-vertex$ coloring of a graph $G=(V,E)$ is a partition $P=\{V_1,V_2,...V_k\}$ of V into independent set. Graph G is b-coloring if there is a vertex in each color class, which is adjacent to at least one vertex in every other color class. The b-chromatic number of graph G, denoted by $\varphi(G)$, is the largest integer k such that there is a b-coloring with k colors. In this paper, we study b-coloring of king's tour graph $K_{n,m}$, tribun graph T_n, diamond ladder graph Dl_n, three cycle ladder graph TCl_n, and chain graph K_4P_n.
Keywords: b-coloring, b-chromatic number, vertex coloring.

INTRODUCTION

One of the oldest and best-known problems of graph theory is graph coloring. In 1852, de Morgan wrote letter to his friend Hamilton informing him that one of his students had observed that when coloring the counties on an administrative map of England only four color were necessary to ensure that adjacent counties were given different colors the origins of graph coloring [1]. A graph coloring is an assignment of colors to the elements of the graphs concerned. Some types of graph coloring are Vertex Coloring, Edge Coloring, Total Coloring, Map Coloring, Interval Coloring, Clique Coloring, and Bipartite Graph Coloring. Vertex coloring is an assignment of colors in a way that two adjacent vertices in G must have different color.

Definition 1 *[2] The chromatic number of graph G is denoted by $\chi(G)$, is the minimum number of colors needed to produce a proper coloring of a graph G.*

Many variants of colorings, besides the classical vertex colorings, Harrary, et.al [2] gave an information about a-colorings.

Definition 2 *[2] An a-coloring of a graph G is a proper vertex coloring of G such that, for any pair of colors, there is at least one edge of G whose end vertices are colored with this pair of colors.*

Definition 3 *[3] A b-coloring of a graph G by k colors is a proper vertex coloring such that there is a vertex in each color class, which is adjacent to at least one vertex in every other color class.*

Definition 4 *[3] The b-chromatic number of graph G is such that there is a b-coloring of G denoted by $\varphi(G)$. It is the largest integer k.*

In this paper, we will use this lemma as general bound of the b-chromatic number of b-coloring:

Lemma 1 *[4] For any graph G, $\chi(G)\le\varphi(G)\le\Delta(G)+1$, where the $\Delta(G)$ is the maximum degree of vertices in G.*

Some researchers have determined b-chromatic number of some graph. Kalpana and Vijayalakshmi [5] determined the b-coloring of central graph of some graphs, Shaebani and Saeed [6] determined a note on b-coloring of Kneser graphs, Yang, et.al [7] determined b-coloring of infinite graph, Vinitha, et.al [8] determined b-chromatic number of theta graph families, Nagarathinam and Parvathi [9] determined b-coloring on graph operators and b-coloring line, middle and total graph of tadpole graph. Some b-coloring graphs has researched are central graph of triangular snake

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graph, sunlet graph, helm graph, double triangular snake graph, gear graph, and closed helm graph [5]; fan graph [7]; theta graph [8]; middle and total graph of tadpole[[9]; cartesian product of *Km* cartesian G, *Km* cartesian *Cn*, *Km* cartesian *Pn*, *Kn* cartesian *Kn* [10]; etc.

In this paper, we will analyze the new result of the *b*-chromatic number of *b*-coloring of king's tour graph $K_{n,m}$, tribun graph T_n, diamond ladder graph Dl_n, three cycle ladder graph TCl_n, and chain graph K_4P_n.

Definition 5 *[11] King's tour graph $K_{n,m}$ for $n,m \geq 2$. is the graph with n,m vertices in which each vertex represents a square in an mxn chessboard, and each edge corresponds to a legal move by king.*

An example of king's tour graph $K_{n,m}$ is provided in Figure 1.

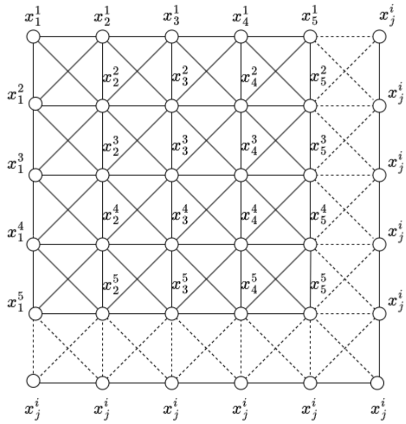

FIGURE 1. The illustration of $K_{n,m}$

Definition 6 *[12] Tribun graph T_n with $n \geq 2$. is a connected graph with vertex set. Tribun graph have vertex, $V(T_n)$=$\{x_j, 1 \leq j \leq n+1\} \cup \{y_j, 1 \leq j \leq n\} \cup \{z_j, 1 \leq j \leq 2n+1\}$ and edge set $E(T_n) = \{z_j z_{j+1}, 1 \leq i \leq 2n\} \cup \{x_j z_j, 1 \leq i \leq 4n+1\} \cup \{y_j z_j, 1 \leq i \leq 2n\}$.*

For an example of tribun graph T_n provided in Figure 2.

Definition 7 *[13] A diamond ladder graph Dl_n is a family of ladder graph. Diamond ladder graph has vertex set $V(Dl_n)$=$\{x_j, 1 \leq j \leq n\} \cup \{y_j, 1 \leq j \leq 2n\} \cup \{z_j, 1 \leq j \leq n\}$ and edge set $E(Dl_n) = \{x_j x_{j+1}, 1 \leq j \leq n-1\} \cup \{z_j z_{j+1}, 1 \leq j \leq n-1\} \cup \{x_j z_j, 1 \leq j \leq n\} \cup \{x_j y_j, 1 \leq j \leq 2n\} \cup \{y_j z_j, 1 \leq j \leq 2n\} \cup \{y_{2j} y_{2j+1}, 1 \leq j \leq n-1\}$.*

For an example of diamond ladder graph Dl_n provided in Figure 3.

Definition 8 *[14] Three cycle ladder graph TCl_n is a family of ladder graph. Three cycle ladder graph has vertex set $V(TCl_n)$=$\{z_i, 1 \leq i \leq n+1\} \cup \{x_i, 1 \leq i \leq n\} \cup \{y_j, 1 \leq i \leq n+1\}$ and edge set $E(TCl_n) = \{y_i x_{i+1}, 1 \leq i \leq n\} \cup \{y_i z_i, 1 \leq i \leq n+1\} \cup \{z_i x_i, 1 \leq i \leq n\} \cup \{x_i y_i, 1 \leq i \leq n\} \cup \{y_i y_i, 1 \leq i \leq n\} \cup \{z_i x_i, 1 \leq i \leq n\}$.*

For an example of three cycle ladder graph TCl_n provided in Figure 4.

Definition 9 *[15] Chain graph is a point shackle symbolized by shack$(K4, v, n)$, so that shack$(K4, v, n)$ has the same meaning as K_4P_n. Let k be the round positive. The shackle graph is denoted by shack $(G_1, G_2, ..., G_k)$.*

For an example of chain graph K_4P_n provided in Figure 5.

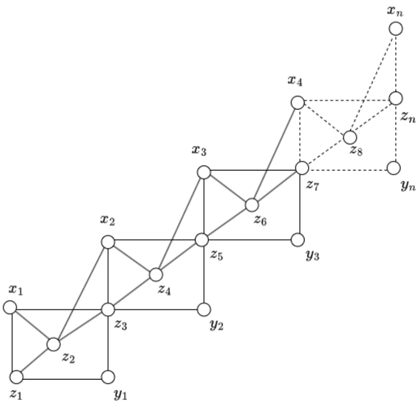

FIGURE 2. The illustration of T_n

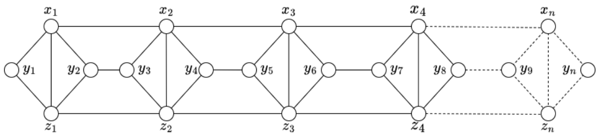

FIGURE 3. The illustration of Dl_n

METHOD

In this study, the method used was a pattern recognition method, Pattern recognition method is a method that used to find the rexact value of some b-chromatic number of king's tour graph, tribun graph, diamond ladder graph, three cycle ladder graph, and chain graph. So that it meets the b-chromatic number on some graphs define and axiomatic deductive method. The axiomatic deductive method is a research method used by applying the principles deductive proof that apply in mathematical logic using axioms, lemmas, and theorems that already exists and then applied in solving a problem related to b-coloring on some graphs that has been determined [16]. The steps used are as follows:
1. Formulate the research problem to be discussed.
2. Study and understand literature sources related to b-coloring and b-chromatic number of some graphs.
3. Determine the pattern of the graphs.
4. Proving the b-chromatic number theorem in the specified graphs.
5. Formulate the conclusions that had been obtained.

Furthermore, several definitions and theorems that were used in this study were presented.

RESEARCH FINDING

In this paper, we discuss some new results of the b-chromatic number of king's tour $K_{n,m}$, tribun T_n, diamond ladder Dl_n, three cycle ladder TCL_n, and chain graph K_4P_n.

Theorem 1 *Let $K_{n,m}$ be king's tour graph. For every positive integer $n,m \geq 4$, $\varphi(K_{n,m}) = 9$.*

Proof. King's tour graph $K_{n,m}$ is a connected graph with vertex set $V(K_{n,m})$=$\{x^i_j, 1 \leq i \leq m, 1 \leq j \leq n\}$ and edge set $E(K_{n,m})$=$\{x^i_j x^i_{j+1}, 1 \leq j \leq n, 1 \leq i \leq m-1\} \cup \{x^i_j x^{i+1}_j, 1 \leq j \leq n-1, 1 \leq i \leq m\} \cup \{x^i_j x^{i+1}_{j+1}, 1 \leq j \leq n-1, 1 \leq$

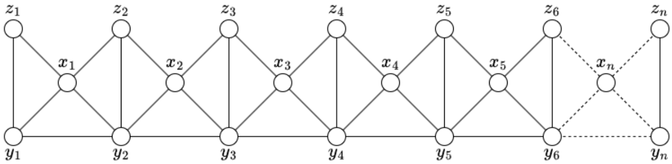

FIGURE 4. The illustration of TCl_n

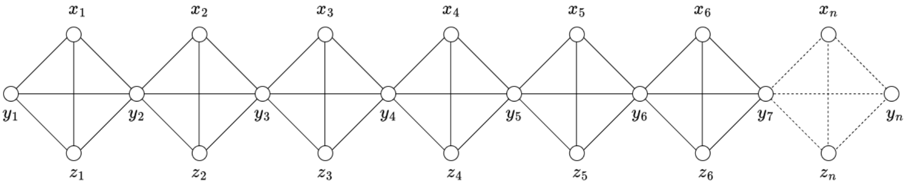

FIGURE 5. The illustration of K_4P_n

$i \leq m-1\} \cup \{x^i_j x^{i+1}_{j-1}, 1 \leq j \leq n-1, 1 \leq i \leq m-1\}$. The cardinality of the vertex set $K_{n,m}$ is $|V(K_{n,m})| = nm$ and the cardinality of the edges set $K_{n,m}$ is $|E(K_{n,m})| = 4mn - 3m - 3n + 2$. The chromatic number and maximum degree of $K_{n,m}$ are $\chi(K_{n,m})$=9 and Δ $K_{n,m}$=9.

Furthermore we will prove the b-chromatic number of $K_{n,m}$, to show the $\varphi(K_{n,m})$ we have Lemma 1. Based on the Lemma 1 we have $\chi(K_{n,m}) \leq \varphi(K_{n,m})) \leq \Delta(K_{n,m})) + 1$, the first step we identify based on the chromatic number and maximum degree of $K_{n,m}$, we know $\chi(K_{n,m}) = 9$ and $\Delta(K_{n,m}) = 8$. Furthermore, we have the bounds of b-chromatic number of $K_{n,m}$ are $\varphi(K_{n,m}) \geq \chi(K_{n,m}) = 9$ and $\varphi(K_{n,m}) \leq \Delta(K_{n,m}) + 1 = 8 + 1 = 9$.

To verify $\varphi(K_{n,m}) \leq 9$, we show the function of vertex colors as follow:

$$f(x^i_j)= \begin{cases} 1, & \text{if } i \equiv 1 \bmod 3,\ j \equiv 2 \bmod 3 \\ 2, & \text{if } i \equiv 1 \bmod 3,\ j \equiv 0 \bmod 3 \\ 3, & \text{if } i \equiv 1 \bmod 3,\ j \equiv 1 \bmod 3 \\ 4, & \text{if } i \equiv 0 \bmod 3,\ j \equiv 1 \bmod 3 \\ 5, & \text{if } i \equiv 0 \bmod 3,\ j \equiv 0 \bmod 3 \\ 6, & \text{if } i \equiv 0 \bmod 3,\ j \equiv 2 \bmod 3 \\ 7, & \text{if } i \equiv 2 \bmod 3,\ j \equiv 1 \bmod 3 \\ 8, & \text{if } i \equiv 2 \bmod 3,\ j \equiv 2 \bmod 3 \\ 9, & \text{if } i \equiv 2 \bmod 3,\ j \equiv 0 \bmod 3 \end{cases}$$

Based on the function determined, we have $|f(x_i) \cup f(y_i) \cup f(z_i)| = 9$. Then verified that $\varphi(K_{m,n}) \geq 5$. Based on the lower and upper bound, we have $5 \leq \varphi(K_{m,n}) \leq 9$. It can be concluded that $\varphi(K_{m,n}) = 9$ for $n \geq 5$. It completes the proof.

An illustration of b-coloring of $K_{n,m}$ can be seen in Figure 6 and Figure 7.

Theorem 2 *Let T_n be tribun graph. For every positive integer $n \geq 5$, $\varphi(T_n) = 7$.*

Proof. Tribun graph T_n is a connected graph with vertex set $V(T_n)$=$\{x_j, 1 \leq j \leq n+1\} \cup \{y_j, 1 \leq j \leq n\} \cup \{z_j, 1 \leq j \leq 2n+1\}$ and edge set $E(T_n) = \{z_j z_{j+1}, 1 \leq i \leq 2n\} \cup \{x_j z_j, 1 \leq i \leq 4n+1\} \cup \{y_j z_j, 1 \leq i \leq 2n\}$. The cardinality of the vertex set T_n is $|V(T_n)| = (n+1) + n + (2n+1)$ and the cardinality of the edges set T_n is $|E(T_n)| = 8n+1$. The chromatic number and maximum degree of T_n are $\chi(T_n) = 5$ and $\Delta(T_n) = 6$. The chromatic number and maximum degree of T_n are $\chi(T_n) = 5$ and $\Delta(K_{n,m}) = 6$.

The b-chromatic number of T_n for $n \geq 5$ is $\varphi(T_n) = 7$, then find the lower and upper bound of T_n. Based on Lemma 1, we have

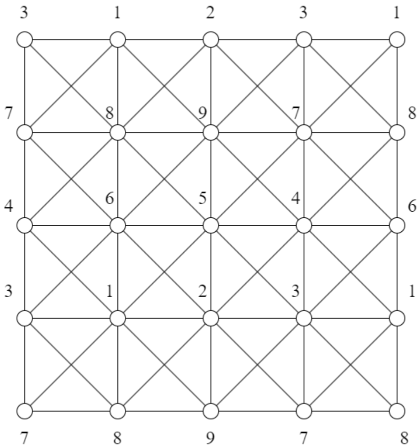

FIGURE 6. *b*-chromatic number of *b*-coloring $\varphi(K_{5,5}) = 9$.

$$\chi(T_n) \leq \varphi(T_n) \leq \Delta(T_n) + 1$$

We have to verify that $n \geq 5$ and $\varphi(T_n) \leq 7$. Based on the Definition 2 and Lemma 1, we have

$$\begin{aligned} \varphi(T_n) &\leq \Delta(T_n) + 1 \\ \varphi(T_n) &\leq 6 + 1 \\ \varphi(T_n) &\leq 7 \end{aligned}$$

Then verified $\varphi(T_n) \leq 7$, then we will find $\varphi(T_n) \geq 5$ by determining the function, we have:

$$f(x_i) = \begin{cases} 1, & \text{if } i \equiv 2 \bmod 4 \\ 2, & \text{if } i \equiv 3 \bmod 4 \\ 3, & \text{if } i \equiv 0 \bmod 4 \\ 6, & \text{if } i \equiv 1 \bmod 4 \end{cases} \quad f(y_i) = \begin{cases} 1, & \text{if } i \equiv 1 \bmod 7 \\ 2, & \text{if } i \equiv 2 \bmod 7 \\ 3, & \text{if } i \equiv 3 \bmod 7 \\ 4, & \text{if } i \equiv 4 \bmod 7 \\ 5, & \text{if } i \equiv 5 \bmod 7 \\ 6, & \text{if } i \equiv 6 \bmod 7 \\ 7, & \text{if } i \equiv 0 \bmod 7 \end{cases} \quad f(z_i) = \begin{cases} 3, & \text{if } i \equiv 0 \bmod 3 \\ 5, & \text{if } i \equiv 1 \bmod 3 \\ 7, & \text{if } i \equiv 2 \bmod 3 \end{cases}$$

Based on the function determined, we have $|f(x_i) \cup f(y_i) \cup f(z_i)| = 7$. Then verified that $\varphi(T_n) \geq 5$. Based on the lower and upper bound, we have $5 \leq \varphi(T_n) \leq 7$. It can be concluded that $\varphi(T_n) = 7$ for $n \geq 5$. It completes the proof.

For an illustration of *b*-coloring of T_n can be seen in Figure 8.

Theorem 3 *Let Dl_n be diamond ladder graph. For every positive integer $n \geq 4$, $\varphi(Dl_n) = 6$.*

Proof. Diamond ladder graph Dl_n is a connected graph with vertex set $V(Dl_n)$=$\{x_j, 1 \leq j \leq n\} \cup \{y_j, 1 \leq j \leq 2n\} \cup \{z_j, 1 \leq j \leq n\}$ and edge set $E(Dl_n) = \{x_j x_{j+1}, 1 \leq j \leq n-1\} \cup \{z_j z_{j+1}, 1 \leq j \leq n-1\} \cup \{x_j z_j, 1 \leq j \leq n\} \cup \{x_j y_j, 1 \leq j \leq 2n\} \cup \{y_j z_j, 1 \leq j \leq 2n\} \cup \{y_{2j} y_{2j+1}, 1 \leq j \leq n-1\}$. The cardinality of vertex set Dl_n is $|V(Dl_n)| = 4n$ and the cardinality of the edges set $|V(Dl_n)| = 4n$ is $|E(Dl_n)| = 8n-3$. The chromatic number and maximum degree of Dl_n are $\chi(Dl_n) = 4$ and $\Delta(Dl_n) = 5$.

The *b*-chromatic number of Dl_n for $n \geq 4$ is $\varphi(Dl_n) = 6$, then find the lower and upper bound of Dl_n. Based on Lemma 1, we have

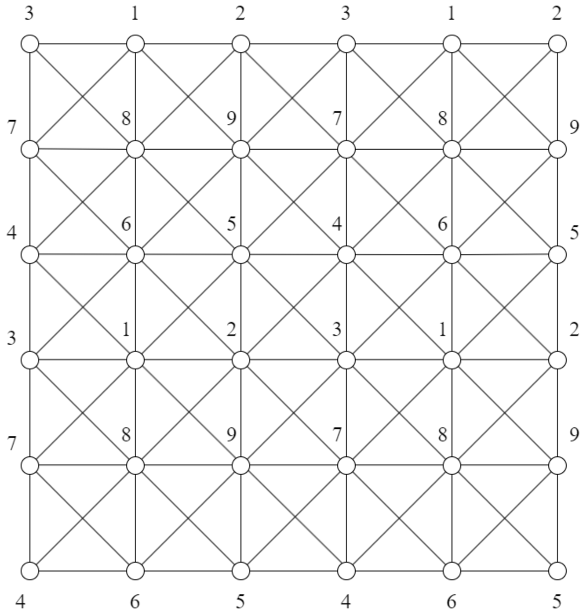

FIGURE 7. *b*-chromatic number of *b*-coloring $\varphi(K_{6,6}) = 9$.

$$\chi(Dl_n) \leq \varphi(Dl_n) \leq \Delta(Dl_n) + 1$$

We have to verify that $n \geq 4$ and $\varphi(Dl_n) \leq 6$. Based on the Definition 2 and Lemma 1, we have

$$\varphi(Dl_n) \leq \Delta(Dl_n) + 1$$
$$\varphi(Dl_n) \leq 5 + 1$$
$$\varphi(Dl_n) \leq 6$$

Then verified $\varphi(Dl_n) \leq 6$. Then we will find $\varphi(Dl_n) \geq 5$ by determining the function there by:

$$f(x_i)= \begin{cases} 4, & \text{if } i \equiv 1 \bmod 3 \\ 5, & \text{if } i \equiv 2 \bmod 3 \\ 6, & \text{if } i \equiv 0 \bmod 3 \end{cases} \quad f(y_i)= \begin{cases} 4, & \text{if } i \equiv 0 \bmod 5 \\ 5, & \text{if } i \equiv 2 \bmod 5 \\ 6, & \text{if } i \equiv 1 \bmod 5 \end{cases} \quad f(z_i)= \begin{cases} 1, & \text{if } i \equiv 1 \bmod 5 \\ 2, & \text{if } i \equiv 2 \bmod 5 \\ 3, & \text{if } i \equiv 0 \bmod 5 \end{cases}$$

Based on the function determined, we have $|f(x_i) \cup f(y_i) \cup f(z_i)| = 6$. Then verified that $\varphi(Dl_n) \geq 4$. Based on the lower and upper bound, we have $5 \leq \varphi(Dl_n) \leq 7$. It can be concluded that $\varphi(Dl_n) = 7$ for $n \geq 4$. It completes the proof.

For an illustration of *b*-coloring of Dl_n can be seen in Figure 9.

Theorem 4 *Let TCl_n be three cycle ladder graph. For every positive integer $n \geq 4$, $\varphi(TCl_n) = 6$.*

Proof. Three cycle ladder graph TCl_n is connected graph with vertex set $V(TCl_n)$=$\{z_i, 1 \leq i \leq n+1\} \cup \{x_i, 1 \leq i \leq n\} \cup \{y_j, 1 \leq i \leq n+1\}$ and edge set $E(TCl_n) = \{y_i x_{i+1}, 1 \leq i \leq n\} \cup \{y_i z_i, 1 \leq i \leq n+1\} \cup \{z_i x_i, 1 \leq i \leq n\} \cup \{x_i y_i, 1 \leq i \leq n\} \cup \{y_i y_i, 1 \leq i \leq n\} \cup \{z_i x_i, 1 \leq i \leq n\}$. The cardinality of the vertex set TCl_n is $|V(TCl_n)| = 3n+2$ and the

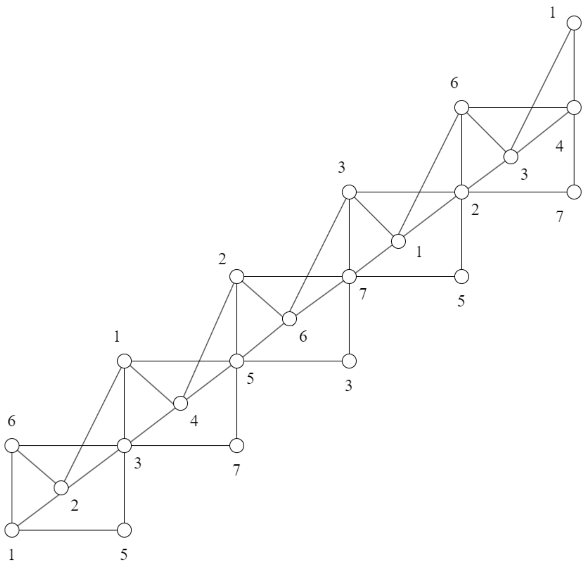

FIGURE 8. b-chromatic number of b-coloring $\varphi(T_5) = 7$

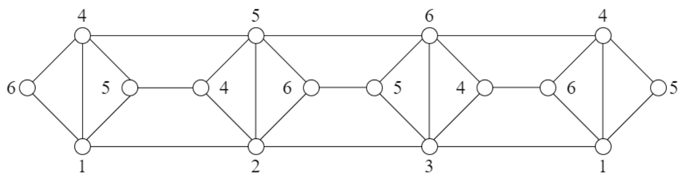

FIGURE 9. b-chromatic number of b-coloring $\varphi(Dl_4) = 6$

cardinality of the edges set TCl_n is $|E(TCl_n)| = 6n+1$. The chromatic number and maximum degree of TCl_n are $\chi(TCl_n) = 4$ and $\Delta(TCl_n) = 5$.

The b-chromatic number of TCl_n for $n \geq 4$ is $\varphi(TCl_n) = 6$, then find the lower and upper bound of TCl_n. Based on Lemma 1, we have

$$\chi(TCl_n) \leq \varphi(TCl_n) \leq \Delta(TCl_n) + 1$$

We have to verify that $n \geq 4$ and $\varphi(TCl_n) \leq 6$. Based on the Definition 2 and Lemma 1, we have

$$\varphi(TCl_n) \leq \Delta(TCl_n) + 1$$
$$\varphi(TCl_n) \leq 5 + 1$$
$$\varphi(TCl_n) \leq 6$$

Then verified $\varphi(TCl_n) \leq 6$, then we will find $\varphi(TCl_n) \geq 5$ by determining the function, we have:

$$f(x_i)=\begin{cases}4, & \text{if } i \equiv 1 \bmod 3\\ 5, & \text{if } i \equiv 0 \bmod 3\\ 6, & \text{if } i \equiv 2 \bmod 3\end{cases}\ f(y_i)=\begin{cases}1, & \text{if } i \equiv 1 \bmod 5\\ 2, & \text{if } i \equiv 2 \bmod 5\\ 3, & \text{if } i \equiv 0 \bmod 5\end{cases}\ f(z_i)=\begin{cases}4, & \text{if } i \equiv 0 \bmod 5\\ 5, & \text{if } i \equiv 2 \bmod 5\\ 6, & \text{if } i \equiv 1 \bmod 5\end{cases}$$

Based on the function determined, we have $|f(x_i) \cup f(y_i) \cup f(z_i)| = 6$. Then verified that $\varphi(TCl_n) \geq 4$. Based on the lower and upper bound, we have $4 \leq \varphi(TCl_n) \leq 6$. It can be concluded that $\varphi(TCl_n) = 6$ for $n \geq 4$. It completes the proof.

For an illustration of b-coloring of TCl_n can be seen in Figure 10.

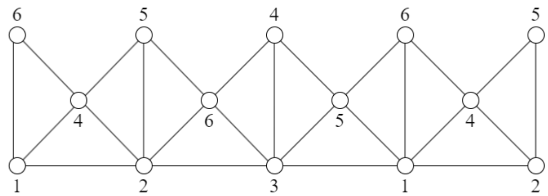

FIGURE 10. b-chromatic number of b-coloring $\varphi(TCl_4) = 6$

Theorem 5 *Let K_4P_n be chain graph. For every positive integer $n \geq 7$, $\varphi(K_4P_n) = 7$.*

Proof. Chain graph K_4P_n i connected graph with vertex set $V(K_4P_n)$=$\{x_i, 1 \leq i \leq n\} \cup \{y_i, 1 \leq i \leq n+1\} \cup \{z_j, 1 \leq i \leq n\}$ and edge set $E(K_4P_n) = \{x_iy_{i+1}, 1 \leq i \leq 2n\} \cup \{z_iy_i, 1 \leq i \leq 2n\} \cup \{y_iy_{i+1}, 1 \leq i \leq n\} \cup \{x_iz_i, 1 \leq i \leq n\}$. The cardinality of the vertex set K_4P_n is $|V(K_4P_n)| = 3n+1$ and the cardinality of the edges set K_4P_n is $|E(K_4P_n)| = 6n$. The chromatic number and maximum degree of K_4P_n are $\chi(K_4P_n) = 7$ and $\Delta(K_4P_n) = 7$.

The b-chromatic number of K_4P_n for $n \geq 7$ is $\varphi(K_4P_n) = 7$, then find the lower and upper bound of K_4P_n. Based on Lemma 1, we have

$$\chi(K_4P_n) \leq \varphi(K_4P_n) \leq \Delta(K_4P_n) + 1$$

We have to verify that $n \geq 7$ and $\varphi(K_4P_n) \leq 7$. Based on the Definition 2 and Lemma 1, we have

$$\varphi(K_4P_n) \leq \Delta(K_4P_n) + 1$$
$$\varphi(K_4P_n) \leq 6 + 1$$
$$\varphi(K_4P_n) \leq 7$$

Then verified $\varphi(K_4P_n) \leq 7$, then we will find $\varphi(K_4P_n \geq 7$ by determining the function, we have:

$$f(x_i)=\begin{cases}1, & \text{if } i \equiv 0 \bmod 3\\ 6, & \text{if } i \equiv 1 \bmod 3\\ 7, & \text{if } i \equiv 2 \bmod 3\end{cases}\ f(y_i)=\begin{cases}1, & \text{if } i \equiv 1 \bmod 7\\ 2, & \text{if } i \equiv 2 \bmod 7\\ 3, & \text{if } i \equiv 3 \bmod 7\\ 4, & \text{if } i \equiv 4 \bmod 7\\ 5, & \text{if } i \equiv 5 \bmod 7\\ 6, & \text{if } i \equiv 6 \bmod 7\\ 7, & \text{if } i \equiv 0 \bmod 7\end{cases}\ f(z_i)=\begin{cases}4, & \text{if } i \equiv 1 \bmod 4\\ 5, & \text{if } i \equiv 2 \bmod 4\\ 6, & \text{if } i \equiv 3 \bmod 4\\ 7, & \text{if } i \equiv 0 \bmod 4\end{cases}$$

Based on the function determined, we have $|f(x_i) \cup f(y_i) \cup f(z_i)| = 7$. Then verified that $\varphi(K_4P_n) \geq 7$. Based on the lower and upper bound, we have $7 \leq \varphi(K_4P_n) \leq 7$. It can be concluded $\varphi(K_4P_n) = 7$. It completes the proof.

For an illustration of b-coloring of K_4P_n can be seen in Figure 11.

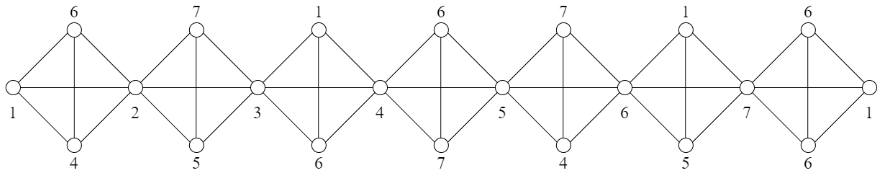

FIGURE 11. b-chromatic number of *b*-coloring $\varphi(K_4P_7) = 7$

CONCLUDING REMARK

In this paper, we have studied *b*-coloring of king's tour graph, tribun graph, diamond ladder graph, three cycle ladder graph, and chain graph. We have determined the exact value of the *b*-chromatic number of king's tour graph, tribun graph, diamond ladder graph, three cycle ladder graph, and chain graph namely $\varphi(K_{n,m})$, $\varphi(T_n)$, $\varphi(Dl_n)$, $\varphi(TCl_n)$, and $\varphi(K_4P_n)$.

OPEN PROBLEM

Determine lower and upper bound of *b*-coloring of the other graphs.

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