A Class of Hamiltonian Cubic Planar Graphs: a Brief Exploration of Their Properties

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Abstract. We study properties of a class of 2-connected bipartite planar cubic graphs G_{db} obtained by operating on connected plane graphs with minimum degree two. We show that G_{db} is a hamiltonian graph with $2^{|V(G)|} + 2^{|E(G)|} - 1$ different perfect matchings, and demonstrate how G_{db} can be decomposed into unions of K_2's and 2-factors. Additional results include how any hamiltonian cycle in G_{db} induces a spanning non-crossing closed trail T in a graph obtained in an intermediate step of constructing G_{db}. The different kinds of subgraphs induced by the non-crossing trail in G are also discussed. We also explore the connection between hamiltonian cycles in G_{db} and hamiltonian cycles in a set of graphs G_v^* called vertex envelopes. Specifically, we show that certain hamiltonian cycles in G_{db} can be easily transformed into hamiltonian cycle in G_v^*. We end by discussing how additional classes of hamiltonian graphs can be obtained by operating on G_{db}.
Keywords: bipartite planar cubic graphs, hamiltonian cycles.

INTRODUCTION

The graph notation and terminology used can be found in West [1]. We explore properties of cubic planar graphs obtained by operating on connected plane graphs of minimum degree two. These derived graphs, which we denote by G_{db}, are 2-connected, cubic, bipartite, planar, and hamiltonian. We study decompositions of G_{db} and show that G_{db} can be decomposed into a union of cycles and K_2's in at least three different ways. We also show that each graph G_{db} contains $2^{|V(G)|} + 2^{|E(G)|} - 1$ different perfect matchings. The hamiltonicity of the graphs G_{db} is easily demonstrated, and we study the subgraphs induced by hamiltonian cycles of G_{db} in G and another related graph. A connection is made between the graph G_{db} are another similarly constructed graph, G_v^* called the vertex envelope. Of particular interest will be how some hamiltonian cycles in G_{db} can be transformed into hamiltonian cycle of G_v^*.

Studying of decompositions and hamiltonian cycles is influenced in part by the Berge-Fulkerson conjecture, Gallai's conjecture, and Bannet's conjecture, all of which are stated below. Among other goals, this study aims to build on work done by other researchers, see for example [2], who studied path and acyclic path decomposition numbers, and [3] who studied hamiltonicity in vertex envelopes.

Conjecture 1. *The Berge-Fulkerson conjecture [4]: If G is a bridgeless cubic graph, then there exist 6 perfect matchings $M_1, \ldots, M_6$ of G with the property that every edge of G is contained in exactly two of $M_1, \ldots, M_6$.*

Conjecture 2. *Gallai's conjecture [5]: If G is a connected graph on n vertices, then G can be decomposed into $\lceil n/2 \rceil$ paths.*

Conjecture 3. *Barnette's conjecture [6]: Every planar, cubic, bipartite, 3-connected graph is hamiltonian.*

In deciding the hamiltonicity of graphs derived from a graph G, researchers often investigate characteristics of graph G that are sufficient for the derived graph to be hamiltonian. Examples of this approach abound, for example, showing that the vertex envelope of a cubic plane graph G is hamiltonian if G contains an edge dominating subgraph with certain properties [3], and finding a dominating cycle in G to show the line graph of G is hamiltonian [7].

DEFINITIONS AND PRELIMINARY RESULTS

A graph G is planar if it can be drawn without crossings, and a plane graph is a planar embedding of G. Faces of a plane graph are the maximal regions of the plane that do not contain any point used in the embedding. The boundary

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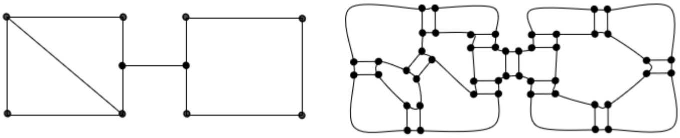

FIGURE 1. A graph G and its derived graph G_{db}

of a face F in a plane graph is a closed walk around the edges of the face, and the length $l(F)$ of the face is the number of edges in the boundary.

Let G be a connected plane graph with minimum degree two. We form a new graph G_b by duplicating each edge e of G followed by replacing each vertex v in the graph double edges with a face of length $2d_G(v)$, maintaining the adjacencies induced by adjacencies in G. We denote the new graph by G_{db} and state without proof the following result which summarizes some properties of $G_{bd}$and follow easily from the construction.

Proposition 4. *Let G be a connected plane graph with minimum degree two. Then*

1. *For each $e \in E(G)$ there is a face F_e of G_{db} with $l(F_e) = 4$, and the two edges e' and e'' corresponding to e are on the boundary of F_e.*

2. *For each vertex $v \in V(G)$, there is a face F_v of G_{db} satisfying $l(F_v) = 2d_G(v)$.*

3. *For each face F of G, there is a face F' of G_{db} with $l(F') = 2l(F)$.*

4. *The cycles induced by faces of type F_e form a 2-factor of G_{db}. Similarly for the faces of type F_v and type F'.*

5. *G_{db} is cubic and bipartite for all connected graphs G with minimum degree two.*

6. *G_{db} is simple if G is loopless.*

Proposition 5. *Let G be a connected plane graph with minimum degree two. Then G_{db} is a cubic bipartite planar graph of order $|V(G_{db})| = 4|E(G))$ and size $|E(G_{db})| = 6|E(G))$.*

Proof. By construction, each vertex of G_{db} lies on the boundary of a face of length four, and there are $|E(G)|$ such faces. Therefore, the order of G_{db} is $4|E(G))$. Since for each edge e of G there is a face F_e of length four, and for each vertex v of G there is an additional $d_G(v)$ edges not on the F_e faces, the number of edges of G_{db} is $4|E(G))| + \sum_{v \in V(G)} d_G(v) = 4|E(G))| + 2|E(G)) = 6|E(G))$.

RESULT

Decompositions of G_{db}

A collection $\mathscr{H}$ of edge-disjoint subgraphs $H_1, H_2, ..., H_n$ of a graph G is a decomposition of G if every edge of G belongs to exactly one H_i [2].

Proposition 6. *Let G be a connected plane graph with minimum degree two. Then G_{db} can be decomposed into a union of cycles and K_2's.*

Proof. Let $\mathscr{Q}$ be the collection of all cycles of G_{db} induced by edges on the boundaries of faces of type F_e, F_v, or type F'. Then each component of $G_{db} - \mathscr{Q} = \mathscr{K}$ is a K_2. Hence $\mathscr{Q} \cup \mathscr{K}$ is the required decomposition of G_{db}.

Theorem 7. *G_{db} has $2^n + 2^m - 1$ different perfect matchings, where $n = |V(G)|$ and $m = |E(G)|$.*

Proof. We begin with a perfect matching M_1 that uses only edges on the boundaries of faces of type F_e, for each edge e of G. We denote the edges on the boundary of some face F_e with e_1, e_2, e_3 and e_4, and assume e_1 and e_3 are in M_1. We can form another perfect matching of G_{db} by replacing e_1 and e_3 with e_2 and e_4. Doing this with each F_e gives an additional $\binom{m}{1}$ perfect matchings, where $m = |E(G)|$. Switching out of the edges e_1 and e_3 can be done in two, three, or more faces F_e including switching out the edges in all m faces of type F_e. Hence including M_1 there are 1 + $\binom{m}{1}$ + $\binom{m}{2}$ + $\binom{m}{3}$ + ... + $\binom{m}{m}$ = $1 + \sum_{k=1}^{m} \binom{m}{k} = 1 + (2^m - 1) = 2^m$ Next, we start with the perfect matching M_1 again and note that is it also composed only of edges on faces of type F_v, but not on faces of type F'. We form additional perfect matchings by switching out edges in one or more of the faces F_v. Since there are $|V(G)| = n$ faces of type F_v, the number of additional perfect matchings is $\binom{n}{1}$ + $\binom{n}{2}$ + $\binom{n}{3}$ + ... + $\binom{n}{n}$ = $\sum_{k=1}^{n} \binom{n}{n} = 2^n - 1$. Therefore, the number of perfect matchings in G_{db} has $2^n + 2^m - 1$.

Hamiltonicity of G_{db}

We begin this section by proving a result that gives necessary and sufficient conditions for hamiltonicity of a graph G. Let $A = \{A_1, A_2, ..., A_k\}, k \geq 1$, be a finite collection of sets that are not necessarily distinct. The intersection graph $I(A)$ is defined as $V(I(A)) = A$ and $E(I(A)) = \{A_n A_m | A_n, A_m \in A \text{ and } A_n \cap A_m \neq \emptyset\}$

Theorem 8. *Let G be a graph. H is a hamiltonian cycle of a graph G if and only if there is a set* $\mathscr{K} = \{C_1, ..., C_n\}$ *of cycles of G, with* $\cup_{i=1}^{n} V(C_i) = V(G)$ *and*

1. *any two distinct cycles in* $\mathscr{C}$ *have at most one edge in common.*
2. $I(\mathscr{K})$ *is a tree.*

Furthermore, such an H consists of precisely those edges that belong to exactly one of the cycles $C_1, ..., C_n$.

Proof. Suppose H is a hamiltonian cycle G. Then $C_1 = H$ satisfies stated properties.

Conversely, if $\mathscr{K} = \{C_1\}$, then C_1 is a hamiltonian cycle of the graph G. We therefore assume $|\mathscr{K}| > 1$. Then $I(S)$ has an end vertex c_1. We number the cycles so that c_1 corresponds to C_1, and let $\mathscr{K}_1 = \mathscr{K} - \{C_1\}$. We break the cycle generated by $\mathscr{K}_1$ at the edge it has in common with C_1, attaching C_1, and removing the common edge. The cycle constructed s hamiltonian cycle of G.

Theorem 9. *Let G be a plane simple connected graph with minimum degree two. Then* G_{db} *is hamiltonian.*

Proof. Let T be a spanning tree of G, and denote the set of cycles of G_{db} corresponding to the vertices and edges of T by $\mathscr{C}$. Then $I(\mathscr{C})$ is a tree. Hence G_{db} is hamiltonian by Theorem 8.

Let H be a hamiltonian cycle constructed from a spanning tree T. The following is true.

1. For each $e \in E(T)$, both e' and e'' are in H.
2. If $e \in E(G) - E(T)$, then e' and e'' are not in H.
3. H separates F_v if and only if $d_T(v) = d_G(v)$.
4. If $d_T(v) = 1$, then all but one edge of F_v are in H.

Theorem 10. *Let* T_1 *and* T_2 *be two distinct spanning trees of G. Then the respective hamiltonian cycles* H_1 *and* H_2 *of* G_{db} *are distinct.*

Proof. Let T_1 and T_2 be two distinct spanning tree of G. Then there is at least one edge $e \in E(G)$ satisfying $e \in E(T_1) - E(T_2)$. By construction of $H_1$and H_2, the edges e' and e'' of G_{db} corresponding to e are in H_1 and they are not in H_2. There H_1 and H_2 are distinct.

Theorem 11. *Let H be a hamiltonian cycle of* G_{db}. *Then H induces a spanning non-crossing closed trail T in* G_b.

Proof. Let H be a hamiltonian cycle of G_{db}. We carry out a marking procedure on the edges of H by going along H and placing an arrow indicating the direction in which each edge is traversed. We then form the graph G_d by shrinking each type F_v face of G_{db} into a vertex. This transforms H into a non-crossing closed trail T with $V(T) = V(G)$ as required.

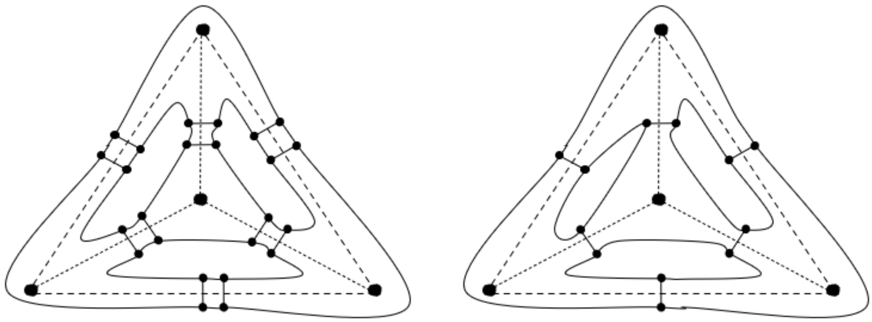

FIGURE 2. G_{db} and G_v^* of a graph G

Let e be a bridge of a graph G. Then e' and e'' form an edge-cut of G_{db}. Therefore, any hamiltonian cycle of G_{db} contains e' and e''. We also note that if $v \in V(G)$ is a cut-vertex, then there are two edges, say e_1 and e_2, on the boundary of F_v that form an edge-cut of G_{db}. Hence for each cut vertex v of G there are two edges contained in every hamiltonian cycle of G_{db}.

A comparison of G_{db} and the Vertex Envelope of a graph G

Figure 2 shows the graphs G_{db} and the vertex envelope G_v^* both superimposed with a graph G shown with dashed lines. In this section we compare the properties of the graphs G_{db} and vertex envelopes G_v^*. We begin by noting that since every edge of G_{db} belongs to some cycle, G_{db} is 2-connected. On the other hand, G_v^* is 3-connected according to [3]. Below is a comparison of other properties.

1. If $v \in V(G)$, there is a face F_v of G_{db} and G_v^* such that $l(F_v) = 2d_G(v)$.
2. If F is a face of G, then there is a corresponding face F' whose length is $l(F') = 2l(F)$ in G_{db} and $l(F') = l(F)$ G_v^*.
3. The faces of type F_v form a 2-factor in either graph.
4. Faces of type F' also form a 2-factor in either graph.

Figure 3 illustrates how the graph G_v^* can be obtained from G_{db} by collapsing all faces of size four corresponding to each edge e of G by merging the edges e_1 and e_2 into one edge f, as suggested in the figure.

Theorem 12. *Suppose G_{db} contains a hamiltonian cycle H such that for any $e \in E(G)$, $e', e'' \in E(H)$. Then G_v^* is hamiltonian.*

Proof. On the left side of Figure 4 are the two non-isomorphic ways that a hamiltonian cycle H that uses both edges e' and e'' can run through the vertices. Since $G)db$ can be transformed into G_v^* by merging edges e_1 and e_2, the hamiltonian cycle H can therefore be transformed into a hamiltonian cycle of G_v^* as illustrated in the figure.

Theorem 13. *Suppose G contains an independent set of vertices whose deletion leaves a tree. Then the vertex envelope G_v^* is hamiltonian.*

Proof. Consider the tree T obtained by deleting the independent set of vertices I from G. Then the set of cycles $\mathscr{C}$ consisting of cycles C_v and C_e induced by faces F_v and F_e of G_{db} corresponding to the vertices and edges of T, and cycles C_f induced be faces F_f of the edges f incident with vertices of I form a vertex cycle cover of G_{db}. Each cycle

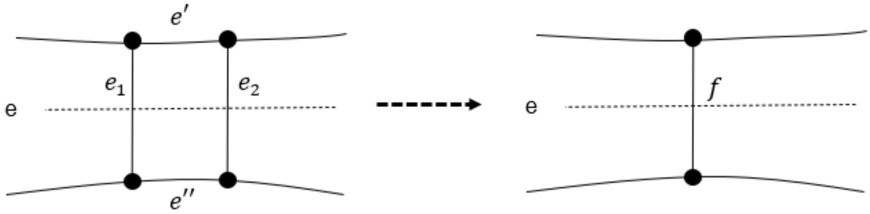

FIGURE 3. G_{db} and G_v^* of a graph G

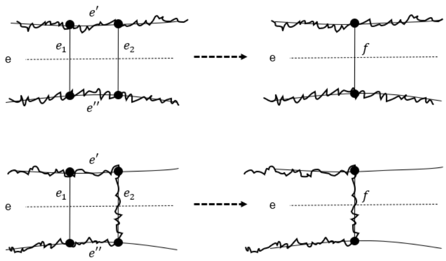

FIGURE 4. Transforming hamiltonian cycle of G_{db} to a hamiltonian cycle of G_v^*

C_f is adjacent to exactly one cycle C_v in $\mathscr{C}$, where $v \in V(T)$ is a neighbor of a vertex in I. Each cycle C_e is adjacent to exactly two cycles C_x and C_y, where $e = xy \in E(G)$. Each cycle C_v is adjacent to $d_G(v)$ cycles corresponding to the edges of G incident with v. By Theorem 8, a hamiltonian cycle H of G_{db} can be constructed from $\mathscr{C}$. By construction, the hamiltonian cycle H uses the edges e' and e'' for each edge e of G, and therefore by Theorem 12, G_v^* is hamiltonian.

Conjecture 14. *Suppose G_{db} contains a hamiltonian cycle H such that for any $e \in E(G)$, $e', e'' \in E(H)$. Then G contains a set S of independent vertices such that $G - S$ is a tree.*

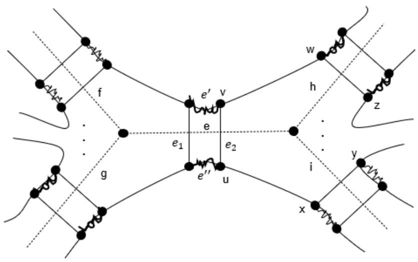

FIGURE 5. G_{db} and G_v^* of a graph G

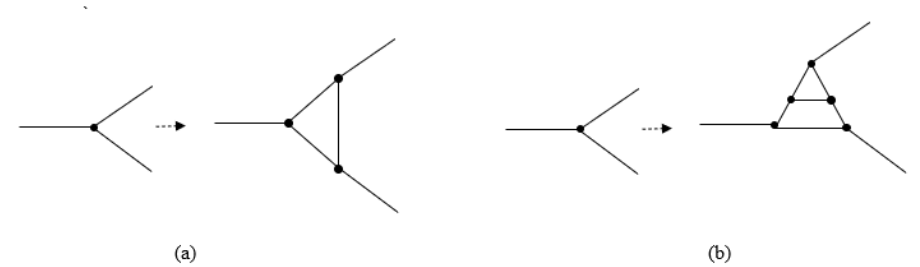

FIGURE 6. Operating on vertices of G_{db}

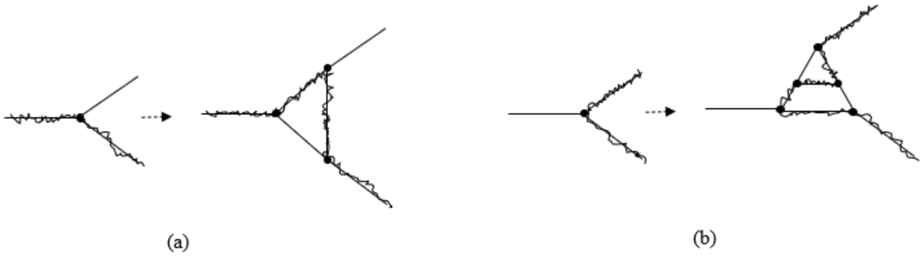

FIGURE 7. Extending the hamiltonian cycle of G_{db}

Infinite classes of hamiltonian graphs obtained from G_{db}

The following result help to illustrate that additionally hamiltonian graphs that are not necessarily bipartite can be obtained by operating on the graphs G_{db}.

Proposition 15. *The graph obtained by operating one or more vertices of G_{db} as shown in Figure 6 is hamiltonian.*

Proof. Figure 7 shows the non-isomorphic ways of how a hamiltonian cycle can be extended to a hamiltonian cycle in the graphs obtained by applying either of the operations illustrated in Figure 6.

CONCLUSION

As seen earlier the graphs G_{db} contain multiple perfect matchings, specifically, $2^{|V(G)|} + 2^{|E(G)|} - 1$ different perfect matchings, and can be decomposed into a union of cycles and K_2's. We also showed that every graph G_{db} is hamiltonian, and the easiness of finding hamiltonian cycles in G_{db} can be exploited to generate other infinite classes of hamiltonian planar cubic graphs that are not necessarily bipartite, as seen in Proposition 15. In Theorem 12 it was shown that if G_{db} contains a hamiltonian cycle H such that for any $e \in E(G)$, $e', e'' \in E(H)$, then the vertex envelope G_v^* is hamiltonian. This result is encouraging in that the hamiltonicity of other classes of graphs can be studied by examining hamiltonicity in G_{db}.

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