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Symmetry Functions with Respect to Any Point in R^n and Their Properties

Firdaus Ubaidillah*

Department of Mathematics Faculty of Mathematics and Natural Sciences
University of Jember – Jember 68121 Indonesia
**Corresponding author. Email: firdaus_u@yahoo.com*

ABSTRACT

A function $f : R \to R$ is said to be an odd function if $f(-x) = -f(x)$ for every x in R. The graph of an odd function is symmetric with respect to the origin, that is the point (0,0). The aims of this paper are to generalize odd functions on R^n and introduce symmetry functions with respect to any point in R^n. Further, this paper discusses some properties of odd functions on R^n and symmetry functions with respect to any point in R^n.

Keywords: *Odd function, Symmetry function, Symmetric graph.*

1. INTRODUCTION

A real-valued function $f: R \to R$ is said to be *odd* (respectively, *even*) on R if $f(-x) = -f(x)$ (respectively, $f(-x) = f(x)$) for all x in R. The graph of a odd (respectively, even) function is symmetric with respect to the origin (respectively, y-axis). In 2009, Bo Lin and Men gave several basic properties of odd functions [1]. In 2011, Balaich and Ondrus generalized odd and even complex-valued functions [2]. Ubaidillah generalized even real-valued functions on R^n [3].

In this paper, I will present definitions of an odd function on R^n and a symmetry function with respect to any point in R^n, some properties and examples of odd functions on R^n and symmetry functions with respect to any point in R^n.

2. PRELIMINARIES

The notations and terminologies used in this section from [4]. R and N denote the set of all real numbers and positive integer numbers, respectively. For every $n \in N$, let R^n denote the n-fold Cartesian product of R with itself; i.e.,

$$R^n = \{(x_1, x_2, \dots, x_n): x_i \in R \text{ for } i = 1,2, \dots, n\}.$$

The positive integer n is called the *dimension* of R^n, an element $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ of R^n is called *point* or *vector*, and the numbers x_i are called *components*, of $\boldsymbol{x}$. Two vectors $\boldsymbol{x} = (x_1, x_2, \dots, x_n) \in R^n$ and $\boldsymbol{y} = (y_1, y_2, \dots, y_n) \in R^n$ are said to be *equal* if and only if $x_i = y_i$ for every $i = 1,2, \dots, n$. The *zero vector* is the vector $\mathbf{0} := (0,0, \dots, 0)$. When $n = 2$, we usually denote the components of $\boldsymbol{x}$ by x and y.

We have already encountered the sets R^n for small n. $R^1 = R$ is the real line; we shall call its elements *scalars*. R^2 is the *xy* plane used to graph functions of the form $y = f(x)$. And R^3 is the *xyz* space used to graph functions of the form $z = f(x, y)$.

In this section, we begin to study functions of several variables by examining the algebraic structure of R^n. That structure is described in the following definition.

Definition 1. Let $\boldsymbol{x} = (x_1, x_2, \dots, x_n) \in R^n$ and $\boldsymbol{y} = (y_1, y_2, \dots, y_n) \in R^n$ be vectors and $\alpha \in R$ be a scalar.

(i) The *addition* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is the vector

$$\boldsymbol{x} + \boldsymbol{y} = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n).$$

(ii) The *subtraction* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is the vector

$$\boldsymbol{x} - \boldsymbol{y} = (x_1 - y_1, x_2 - y_2, \dots, x_n - y_n).$$

(iii) The *product* of a scalar α and a vector $\boldsymbol{x}$ is the vector

$$\alpha\boldsymbol{x} = (\alpha x_1, \alpha x_2, \dots, \alpha x_n).$$

(iv) The *scalar product* or *multiplication* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is the scalar

$$\boldsymbol{x} \cdot \boldsymbol{y} = (x_1 y_1, x_2 y_2, \dots, x_n y_n).$$

The properties in following theorem are direct consequences of Definition 1 and corresponding properties of real numbers.

Theorem 2. On the R^n, two operations addition $(+)$ and multiplication $(\cdot)$ and a product of scalar satisfy the following properties:

(i) $\boldsymbol{x}+\boldsymbol{y}=\boldsymbol{y}+\boldsymbol{x}$ for every $\boldsymbol{x},\boldsymbol{y}\in R^n$;
(ii) $(\boldsymbol{x}+\boldsymbol{y})+\boldsymbol{z}=\boldsymbol{x}+(\boldsymbol{y}+\boldsymbol{z})$ for every $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\in R^n$;
(iii) $\boldsymbol{0}+\boldsymbol{x}=\boldsymbol{x}+\boldsymbol{0}=\boldsymbol{x}$ for every $x\in R^n$;
(iv) $\boldsymbol{x}+(-\boldsymbol{x})=(-\boldsymbol{x})+\boldsymbol{x}=\boldsymbol{0};$
(v) $\boldsymbol{x}\cdot\boldsymbol{y}=\boldsymbol{y}\cdot\boldsymbol{x}$ for every $\boldsymbol{x},\boldsymbol{y}\in R^n$;
(vi) $\alpha(\beta\boldsymbol{x})=\beta(\alpha\boldsymbol{x})=(\alpha\beta)\boldsymbol{x}$ for every $\boldsymbol{x}\in R^n$ and $\alpha,\beta\in R$;
(vii) $\alpha(\boldsymbol{x}+\boldsymbol{y})=\alpha\boldsymbol{x}+\alpha\boldsymbol{y}$ for every $\boldsymbol{x},\boldsymbol{y}\in R^n$ and $\alpha\in R$;
(viii) $\alpha(\boldsymbol{x}\cdot\boldsymbol{y})=(\alpha\boldsymbol{x})\cdot\boldsymbol{y}$ for every $\boldsymbol{x},\boldsymbol{y}\in R^n$ and $\alpha\in R$;
(ix) $1\boldsymbol{x}=\boldsymbol{x}1=\boldsymbol{x}$ for every $\boldsymbol{x}\in R^n$;
(x) for each $\boldsymbol{x}\neq\boldsymbol{0}$ in R^n there exitsts an element $\boldsymbol{x}^{-1}\in R^n$ such that $\boldsymbol{x}\cdot\boldsymbol{x}^{-1}=\boldsymbol{x}^{-1}\cdot\boldsymbol{x}=1;$
(xi) $\boldsymbol{x}\cdot(\boldsymbol{y}+\boldsymbol{z})=\boldsymbol{x}\cdot\boldsymbol{y}+\boldsymbol{x}\cdot\boldsymbol{z}$ for every $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\in R^n$;

The following definition is a norm in R^n and distance between two points in R^n.

Definition 3. Let $\boldsymbol{x},\boldsymbol{y}\in R^n$.

(i) The *norm* of $\boldsymbol{x}$ is the scalar

$$\|\boldsymbol{x}\|=\sqrt{x_1^2+x_2^2+\cdots+x_n^2}\,;$$

(ii) The *distance* between two points $\boldsymbol{x}$ and $\boldsymbol{y}$ is the scalar

$$d(\boldsymbol{x},\boldsymbol{y})=\|\boldsymbol{x}-\boldsymbol{y}\|\,.$$

The analogy between the absolute value and the norm is further reinforced by the following result.

Theorem 4. Let $\boldsymbol{x},\boldsymbol{y}\in R^n$. Then

(i) $\|\boldsymbol{x}\|\geq 0$ and $\|\boldsymbol{x}\|=0$ if and only if $\boldsymbol{x}=\boldsymbol{0};$
(ii) $\|\alpha\boldsymbol{x}\|=|\alpha|\|\boldsymbol{x}\|$ for every scalars $\alpha\in R$.
(iii) $\|\boldsymbol{x}+\boldsymbol{y}\|\leq\|\boldsymbol{x}\|+\|\boldsymbol{y}\|$.

3. RESULTS AND DISCUSSION

We begin to introduce a terminology of an odd function on R^n.

Definition 5. A function $f\colon R^n\to R$ is said to be *odd* on R^n if $f(-\boldsymbol{x})=-f(\boldsymbol{x})$ for every $\boldsymbol{x}$ in R^n.

For instance, a function $f\colon R^2\to R$ defined by $f(\boldsymbol{x})=f(x,y)=2xy^2-x^2y$ is an odd function on R^2, because

$$\begin{aligned} f(-\boldsymbol{x}) &= f(-x,-y)\\ &= 2(-x)(-y)^2-(-x)^2(-y)\\ &= -2xy^2+x^2y\\ &= -[2xy^2-x^2y]\\ &= -f(\boldsymbol{x}).\end{aligned}$$

While the function $g\colon R^3\to R$ defined by $g(\boldsymbol{x})=g(x,y,z)=3xy^2z-x^2\sin(y+z)$ is not an odd function on R^3, because

$$\begin{aligned} g(-\boldsymbol{x}) &= g(-x,-y,-z)\\ &= 3(-x)(-y)^2(-z)-(-x)^2\sin(-y-z)\\ &= 3xy^2z+x^2\sin(y+z)\\ &\neq -g(\boldsymbol{x}).\end{aligned}$$

The graph of an odd function on R^n is symmetric with respect to the point $\boldsymbol{x}=\boldsymbol{0}\in R^n$. The graph of the odd function $f(\boldsymbol{x})=f(x,y)=2xy^2-x^2y$ is illustrated in Figure 1.

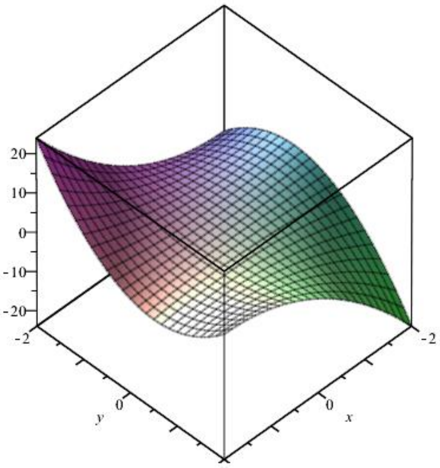

Figure 1 The graph of the function $f(x,y)=2xy^2-x^2y$.

Theorem 6. If f is an odd function on R^n, then $f(\boldsymbol{0})=0$.

Proof. Since $\boldsymbol{0}=-\boldsymbol{0}$ and f is an odd function on R^n, we have

$$f(\boldsymbol{0})=f(-\boldsymbol{0})=-f(\boldsymbol{0}).$$

Therefore, we have $2f(\boldsymbol{0})=0$. Thus we conclude $f(\boldsymbol{0})=0$. ■

Theorem 7. If f and g are odd functions on R^n, then

(i) αf, for every $\alpha\in R$; and

(ii) $f+g$,

are odd functions on R^n.

Proof. (i) Suppose that f is an odd function on R^n and $\alpha\in R$. Then for every $\boldsymbol{x}\in R^n$, we obtain

$$\begin{aligned}(\alpha f)(-\boldsymbol{x}) &= \alpha[f(-\boldsymbol{x})]\\ &= \alpha[-f(\boldsymbol{x})]\\ &= -\alpha[f(\boldsymbol{x})]\end{aligned}$$

$$= -(\alpha f)(\boldsymbol{x})$$

Thus, αf is an odd function on R^n.

(ii). Suppose that f and g are odd functions on R^n. Then, for every $\boldsymbol{x} \in R^n$, we obtain

$$(f+g)(-\boldsymbol{x}) = f(-\boldsymbol{x}) + g(-\boldsymbol{x})$$
$$= -f(\boldsymbol{x}) - g(\boldsymbol{x})$$
$$= -(f(\boldsymbol{x}) + g(\boldsymbol{x}))$$
$$= -(f+g)(\boldsymbol{x})$$

Then, $f+g$ is an odd function on R^n. ■

Theorem 8. Let $D = \{\boldsymbol{x} \in R^n : \|\boldsymbol{x}\| \leq r\}$ be a closed ball centered at $\mathbf{0}$ of radius r and let $f: D \to R$ be a function that integrable on D. If f is an odd function on R^n, then

$$\int_D f(\boldsymbol{x})\ d\boldsymbol{x} = 0.$$

As illustrations of the Theorem 8, given two examples integral of odd function that defined on closed ball centered at $\mathbf{0}$ of radius $r > 0$.

Example 9. Let $D = \{\boldsymbol{x} \in R^2 : \|\boldsymbol{x}\| \leq 1\}$ and let $f: D \to R$ be a function that defined by $f(x,y) = \sin(x+y)$.

It is clear that f is an odd function on R^2 and is integrable on D. By changing the order of integration and a property of odd function, that is $\int_{-r}^{r} f(x)\ dx = 0$, we obtain

$$\int_D f(\boldsymbol{x})\ d\boldsymbol{x} = \int_D f(\boldsymbol{x})\ dA$$
$$= \int_{x=-1}^{1} \left(\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sin(x+y)\, dy \right) dx$$
$$= \int_{x=-1}^{1} \left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} [\sin x \cos y + \cos x \sin y]\, dy \right) dx$$
$$= \int_{x=-1}^{1} \left(\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sin x \cos y\, dy \right) dx$$
$$= \int_{y=-1}^{1} \left(\int_{x=-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \sin x \cos y\, dx \right) dy$$
$$= \int_{y=-1}^{1} 0\ dy = 0.$$

Example 10. Let $D = \{\boldsymbol{x} \in R^3 : \|\boldsymbol{x}\| \leq a\}$ and let $g: D \to R$ be a function that defined by $g(x,y,z) = \mathrm{x} + \mathrm{yz}^2$.

Denote $g = g_1 + g_2$ with $g_1(x,y,z) = x$ and $g_2(x,y,z) = yz^2$. It is clear that g_1 and g_2 are odd functions on R^2. By changing the order of integration, we obtain

$$\int_D g(x,y,z)\ d\boldsymbol{x} = \int_D g(x,y,z)\ dV$$
$$= \int_D [g_1(x,y,z) + g_2(x,y,z)]\ dV$$
$$= \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2+y^2}}^{\sqrt{a^2-x^2-y^2}} [x + yz^2]\, dzdydx$$
$$= \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-(x^2+y^2)}}^{\sqrt{a^2-(x^2+y^2)}} x\ dz\, dy\, dx$$
$$+ \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-(x^2+y^2)}}^{\sqrt{a^2-(x^2+y^2)}} yz^2\, dz\, dy\, dx$$
$$= \int_{z=-a}^{a} \int_{-\sqrt{a^2-z^2}}^{\sqrt{a^2-z^2}} \left(\int_{-\sqrt{a^2-(y^2+z^2)}}^{\sqrt{a^2-(y^2+z^2)}} x\, dx \right) dy\, dz$$
$$+ \int_{x=-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left(\int_{-\sqrt{a^2-(x^2+z^2)}}^{\sqrt{a^2-(x^2+z^2)}} yz^2\, dy \right) dzdx$$
$$= \int_{z=-a}^{a} \int_{-\sqrt{a^2-z^2}}^{\sqrt{a^2-z^2}} 0\ dy\, dz$$
$$+ \int_{x=-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 0\ dz\, dx$$
$$= 0 + 0 = 0.$$

We now introduce a terminology of a symmetry function with respect to any point in R^n and its properties. We begin with a definition.

Definition 11. Let $\boldsymbol{a} \in R^n$ and let $f: R^n \to R$ be a function. Function f is said to be *symmetry with respect to the point* $\boldsymbol{a}$ in R^n if there is a function h on R^n that defined by

$$h(\boldsymbol{x}) = f(\boldsymbol{x} + \boldsymbol{a}), \quad \text{for all } \boldsymbol{x} \text{ in } R^n$$

such that h is an odd function on R^n.

For instance, a function $f: R^3 \to R$ that defined by $f(\boldsymbol{x}) = f(x, y, z) = xy^2 - y^2 - z + 2$ is a symmetry function with respect to the point $\boldsymbol{a} = (1,0,2) \in R^3$, because

$$
\begin{aligned}
h(\boldsymbol{x}) &= f(\boldsymbol{x} + \boldsymbol{a}) \\
&= f(x + 1, y, z + 2) \\
&= (x + 1)y^2 - y^2 - (z + 2) + 2 \\
&= xy^2 - z
\end{aligned}
$$

is an odd function on R^3.

The graph of a symmetry function with respect to any point $\boldsymbol{a} \in R^n$ is symmetric with respect to the point $\boldsymbol{a} \in R^n$.

Theorem 12. If f is a symmetry function with respect to the point $\boldsymbol{a} \in R^n$, then $f(\boldsymbol{a}) = 0$.

Proof. Since f is a symmetry function with respect to $\boldsymbol{a} \in R^n$, based on Definition 11, there is h an odd function on R^n such that

$$h(x) = f(x + a).$$

Based on the Theorem 6, we have $h(\boldsymbol{0}) = 0$. Then

$$0 = h(\boldsymbol{0}) = f(\boldsymbol{0} + \boldsymbol{a}) = f(\boldsymbol{a}).$$

Thus, it satisfies $f(\boldsymbol{a}) = 0$. ■

Theorem 13. Let $\boldsymbol{a} \in R^n$, let $D = \{\boldsymbol{x} \in R^n : \|\boldsymbol{x} - \boldsymbol{a}\| \leq r\}$, and let $f: D \to R$ be a function that integrable on D. If f is a symmetry function with respect to the point $\boldsymbol{a} \in R^n$, then

$$\int_D f(\boldsymbol{x})\ d\boldsymbol{x} = 0.$$

Proof. Suppose that $\boldsymbol{x} - \boldsymbol{a} = \boldsymbol{u}$. Then, we have $d\boldsymbol{x} = d\boldsymbol{u}$, $D' = \{\boldsymbol{u} \in R^n : \|\boldsymbol{u}\| \leq r\}$, and $f(\boldsymbol{x}) = f(\boldsymbol{u} + \boldsymbol{a})$. Since f is a symmetry function with respect to the point $\boldsymbol{a} \in R^n$, based on the Definition 11, there is an odd function h on R^n such that $h(\boldsymbol{u}) = f(\boldsymbol{u} + \boldsymbol{a})$. Therefore, we have

$$\int_D f(\boldsymbol{x})\ d\boldsymbol{x} = \int_{D'} f(\boldsymbol{u} + \boldsymbol{a})\ d\boldsymbol{u} = \int_{D'} h(\boldsymbol{u})\ d\boldsymbol{u}.$$

Based on the Theorem 8, we obtain

$$\int_{D'} h(\boldsymbol{u})\ d\boldsymbol{u} = 0.$$

Thus, we conclude $\int_D f(\boldsymbol{x})\ d\boldsymbol{x} = 0$. ■

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