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Aggregation Based on Outliers

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Abstract

Inspired by the basic fuzzy connectives min (t-norm T_M) and max (t-conorm S_M), we introduce and study outliers-based extended aggregation functions. Simply said, A is an (a,b)-outliers-based extended aggregation function if for each arity $n \geq a+b$, its output values depend on the number a minimal and b maximal input values only. We focus on associative outliers-based extended aggregation functions, including t-norms, t-conorms, uninorms, nullnorms, as well as on outliers-based extended OWA operators and related outliers-based extended aggregation functions.

Keywords: Aggregation function, Choquet integral, Extended aggregation function, Nullnorm, Outlier-based-aggregation, OWA operator, Triangular conorm, Triangular norm, Uninorm.

1 Preliminaries

Aggregation of real inputs without apriori given cardinality, such as, e.g., the arithmetic mean, geometric mean, t-norms or t-conorms, is usually realized by means of extended aggregation functions [2, 7]. In this paper we restrict our consideration to aggregations on the scale $[0,1]$, and then an extended aggregation function A is a mapping $A\colon \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ such that for each $n \in \mathbb{N}$, $A_n = A|_{[0,1]^n}$ is an n-ary aggregation function, i.e.,

- $A_n(\mathbf{0}) = A_n(0,\dots,0) = 0$ and $A_n(\mathbf{1}) = A_n(1,\dots,1) = 1$,
- A_n is increasing, i.e., $A_n(\mathbf{x}) \leq A_n(\mathbf{y})$ whenever $\mathbf{x} = (x_1,\dots,x_n), \mathbf{y} = (y_1,\dots,y_n) \in [0,1]^n$ and $x_i \leq y_i$ for each $i = 1,\dots,n$.

Throughout the paper, the class of all extended aggregation functions on the scale $[0,1]$ will be denoted by $\mathscr{A}$ and for any fixed $n \in \mathbb{N}$, the class of all n-ary aggregation functions will be denoted by $\mathscr{A}_n$.

In some situations, only outliers of the input data sample are necessary to be considered in aggregation process for obtaining the output value. For example, the fuzzy disjunction S_M (max) proposed by Zadeh [15] requires considering a unique value, namely the maximal value. On the other hand, the t-norm T_D (drastic product) requires the knowledge of the two minimal input values only. The aim of this contribution is to provide formalization of outliers-based aggregation and then to study such aggregation functions in some particular subclasses, for example, in the class of all extended triangular norms and conorms, uninorms, nullnorms, OWA operators, etc.

The contribution is organized as follows. In Section 2, (a,b)-outliers-based extended aggregation functions are introduced and some general results are given. In Section 3, (a,b)-outliers-based extended t-norms, t-conorms, uninorms and nullnorms are sudied. Section 4 is devoted to extended OWA opertors and to some related classes of aggregation functions. Finally, some concluding remarks are added.

2 (a,b)-outliers-based extended aggregation functions

The following definition was introduced by the authors in [9].

Definition 2.1. *Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and let $a,b \in \mathbb{N}_0$, $a+b \geq 1$. An extended aggregation function $A\colon \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ is called an (a,b)-outliers-based extended aggregation function whenever for each $\mathbf{x} \in [0,1]^n$ with $n \geq a+b$, we have*

$$A(\mathbf{x}) = B_A(x_{(1)},\dots,x_{(a)},x_{(n-b+1)},\dots,x_{(n)}) \qquad (1)$$

where $B_A\colon [0,1]^{a+b} \to [0,1]$ is a fixed $(a+b)$-ary ag-

gregation function and $x_{(i)}$ is the ith smallest input from the sample $\mathbf{x} = (x_1,\dots,x_n)$.

The class of all (a,b)-outliers-based extended aggregation functions will be denoted by $\mathscr{A}_{(a,b)}$.

Obviously, for each input vector $\mathbf{x} = (x_1,\dots,x_n)$ there is a permutation $(\cdot)\colon \{1,\dots,n\} \to \{1,\dots,n\}$ such that $x_{(1)} \le \cdots \le x_{(n)}$. Due to (1), each (a,b)-outliers-based extended aggregation function A is necessarily symmetric for each arity $n \ge a+b$. Note that for $n < a+b$, Definition 2.1 does not bring any constraint for A.

Let us stress that in this contribution we work with extended aggregation functions on $[0,1]$, but for simplicity, when no confusion can arise, the word "extended" will sometimes be dropped.

The following result can be seen as a construction method for (a,b)-outliers-based extended aggregation functions.

Theorem 2.1. *Let $a,b \in \mathbb{N}$ and let $C,D\colon \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ be $(a,0)$- and $(0,b)$-outliers-based extended aggregation functions, respectively. If $E \in \mathscr{A}_2$ is a binary aggregation function then the extended function $A\colon \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ given for each $\mathbf{x} \in [0,1]^n$, $n \ge a+b$, by*

$$A(\mathbf{x}) = E\left(C\left(x_{(1)},\dots,x_{(a)}\right), D\left(x_{(n-b+1)},\dots,x_{(n)}\right)\right) \tag{2}$$

and such that $A|_{[0,1]^n} \in \mathscr{A}_n$ for any $n < a+b$, is an (a,b)-outliers-based extended aggregation function.

Example 2.1. *Let $A\colon \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ be given by*

$$A(\mathbf{x}) = \frac{1}{2}(x_{(1)}x_{(n)} + x_{(2)}x_{(n-1)}),$$

if $n \ge 2$, and if $n = 1$, $A(x) = x$, for each $x \in [0,1]$. Then A is a $(2,2)$-outliers-based extended aggregation function which cannot be expressed in the form (2).

The following general results can be proved.

Proposition 2.1.

(i) *Let a,b,c,d be as in Definition 2.1 and such that $(a,b) \le (c,d)$. Then $\mathscr{A}_{(a,b)} \subseteq \mathscr{A}_{(c,d)}$.*

(ii) *Let A be an extended aggregation function, $A \in \mathscr{A}_{(a,b)}$. Then its dual $A^d\colon \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$, defined by $A^d(x_1,\dots,x_n) = 1 - A(1-x_1,\dots,1-x_n)$, belongs to $\mathscr{A}_{(b,a)}$.*

Proposition 2.2. *Let A be a symmetric extended aggregation function. Then*

(i) *$A \in \mathscr{A}_{(1,0)}$ if and only if $A(\mathbf{x},y) = A(\mathbf{x})$ for all $\mathbf{x} \in [0,1]^n$, $n \in \mathbb{N}$, and each $y \in [0,1]$ such that $y \ge \min\{x_1,\dots,x_n\}$.*

(ii) *$A \in \mathscr{A}_{(0,1)}$ if and only if $A(\mathbf{x},y) = A(\mathbf{x})$ for all $\mathbf{x} \in [0,1]^n$, $n \in \mathbb{N}$, and each $y \in [0,1]$ such that $y \le \max\{x_1,\dots,x_n\}$.*

(iii) *$A \in \mathscr{A}_{(1,1)}$ if and only if $A(\mathbf{x},y) = A(\mathbf{x})$ for all $\mathbf{x} \in [0,1]^n$, $n \ge 2$, and each $y \in [0,1]$ such that $\min\{x_1,\dots,x_n\} \le y \le \max\{x_1,\dots,x_n\}$ (considering $n \ge 2$) .*

Due to Proposition 2.2, it is not difficult to see that

- $\mathscr{A}_{(1,0)} = \{f \circ T_M \mid f \in \mathscr{A}_1\}$, where $T_M = \min$ is the greatest t-norm [8],
- $\mathscr{A}_{(0,1)} = \{f \circ S_M \mid f \in \mathscr{A}_1\}$, where $S_M = \max$ is the weakest t-conorm [8],
- $\mathscr{A}_{(1,1)} = \{B(T_M,S_M) \mid B \in \mathscr{A}_2\}$ (considering $n \ge 2$).

3 Some associative (a,b)-outliers-based extended aggregation functions

In this section we will discuss extended t-norms and their dual t-conorms [8], uninorms [14], and also nullnorms [1]. Binary forms of all these aggregation functions are associative, thus they univocally generate the related extended aggregation functions. Note that unary forms of all these aggregation functions coincide with the identity function on $[0,1]$. Due to Proposition 2.2, it is evident that T_M is the only t-norm contained in $\mathscr{A}_{(1,0)}$, and S_M is the only t-conorm in $\mathscr{A}_{(0,1)}$. Note that proper uninorms or nullnorms belong neither to $\mathscr{A}_{(0,1)}$ nor to $\mathscr{A}_{(1,0)}$.

If we denote by $\mathscr{T}, \mathscr{S}, \mathscr{U}$ and $\mathscr{V}$ the class of all extended t-norms, t-conorms, uninorms and nullnorms, respectively, then:

- $\mathscr{T} \cap \mathscr{A}_{(a,b)} = \mathscr{T} \cap \mathscr{A}_{(a,0)}$ whenever $a \in \mathbb{N}$;
- $\mathscr{S} \cap \mathscr{A}_{(a,b)} = \mathscr{S} \cap \mathscr{A}_{(0,b)}$ whenever $b \in \mathbb{N}$;
- if $U \in \mathscr{U} \cap \mathscr{A}_{(a,b)}$ then $a,b \in \mathbb{N}$;
- if $V \in \mathscr{V} \cap \mathscr{A}_{(a,b)}$ then $a,b \in \mathbb{N}$.

Theorem 3.1. *Let $T \in \mathscr{T}$ and let $\delta \in \mathscr{A}_1$ be its diagonal section, i.e., $\delta(x) = T(x,x)$ for each $x \in [0,1]$.*

(i) *If $T \in \mathscr{A}_{(2,0)}$ then $\delta \circ \delta = \delta$.*

(ii) *If $\delta \circ \delta = \delta$, and if for all $x,y \in [0,1]$*

$$T(x,y) = \begin{cases} \min\{x,y\} & if\ \max\{x,y\} = 1, \\ \delta(\min\{x,y\}) & otherwise, \end{cases}$$

then for each $\mathbf{x} \in [0,1]^n$, $n \geq 2$, we have

$$T(\mathbf{x}) = \begin{cases} T_M(\mathbf{x}) & if\ x_{(2)} = 1, \\ \delta(T_M(\mathbf{x})) & otherwise, \end{cases}$$

and $T \in \mathscr{A}_{(2,0)}$.

There are also some other constructions of extended t-norms belonging to $\mathscr{A}_{(2,0)}$.

Example 3.1. *Let T^{nM} be the nilpotent minimum [3, 11],*

$$T^{nM}(\mathbf{x}) = \begin{cases} 0 & if\ x_{(1)} + x_{(2)} \leq 1, \\ x_{(1)} & otherwise. \end{cases}$$

Then $T^{nM} \in \mathscr{A}_{(2,0)}$.

The following example shows a method for constructing extended t-norms from $\mathscr{A}_{(a,0)}$.

Example 3.2. *Let $t\colon [0,1] \to [0,\infty[$ be a strictly decreasing function continuous on $[0,1[$, such that $t(1) = 0$ and*

$$t(1^-) = \lim_{x \to 1^-} t(x) \geq \frac{t(0)}{a},\ a \geq 2.$$

Then $T \in \mathscr{A}_{(a,0)}$, where T is a t-norm generated by t, i.e.,

$$T(\mathbf{x}) = t^{-1}\left(\min\left\{t(0), \sum_{i=1}^{n} t(x_i)\right\}\right).$$

Note that if $a = 2$, then $T = T_D$ is the drastic product, and its diagonal is given by

$$\delta(x) = \begin{cases} 1 & if\ x = 1, \\ 0 & otherwise, \end{cases}$$

compare with Theorem 3.1.

Remark 3.1. *Due to the duality of t-norms and t-conorms, similar results are valid for t-conorms. For example,*

- *if $S \in \mathscr{S} \cap \mathscr{A}_{(0,2)}$ then $\delta = \delta \circ \delta$, where $\delta(x) = S(x,x)$ for each $x \in [0,1]$.*
- *Similarly, if $s\colon [0,1] \to [0,\infty[$ is a strictly increasing function and such that it is continuous on $]0,1]$, $s(0) = 0$ and $s(0^+) \leq \frac{s(1)}{b}$, then S generated by s belongs to $\mathscr{A}_{(0,b)}$.*

Note that then

$$S(\mathbf{x}) = s^{-1}\left(\min\left\{s(1), \sum_{i=1}^{n} s(x_i)\right\}\right).$$

Now, let U be an extended uninorm with neutral element $e \in]0,1[$. Then, due to [5], U induces an extended t-norm T and an extended t-conorm S given, for each $\mathbf{x} \in [0,1]^n$, $n \in \mathbb{N}$, by

$$T(\mathbf{x}) = \frac{U(e \cdot \mathbf{x})}{e},$$

and

$$S(\mathbf{x}) = \frac{U(e + (1-e)x_1, \ldots, e + (1-e)x_n) - e}{1-e},$$

respectively.

Based on Theorem 2.1, we get the following interesting result for uninorms.

Theorem 3.2. *Let $U \in \mathscr{U}$ be an extended uninorm with neutral element $e \in]0,1[$ and let $a,b \in \mathbb{N}$. Then $U \in \mathscr{A}_{(a,b)}$ if and only if the extended t-norm T induced by U belongs to $\mathscr{T} \cap \mathscr{A}_{(a,0)}$, and the extended t-conorm S induced by U belongs to $\mathscr{S} \cap \mathscr{A}_{(0,b)}$.*

Clearly, if $U \in \mathscr{U} \cap \mathscr{A}_{(1,1)}$, then necessarily $T = T_M$ and $S = S_M$, i.e., U is an idempotent uninorm. Note that idempotent uninorms were completely characterized in [10]. We have the following result.

Corollary 3.1. *Extended idempotent uninorms can be characterized as follows:*

$$\begin{aligned} & \{U \in \mathscr{U} \mid \forall\, c \in [0,1], n \in \mathbb{N} : U_n(c,\ldots,c) = c\} \\ = \ & \mathscr{U} \cap \mathscr{A}_{(1,1)}. \end{aligned}$$

Remark 3.2. *Similarly to the case of extended t-norms and t-conorms, we have a necessary condition for extended uninorms $U \in \mathscr{U} \cap \mathscr{A}_{(2,2)}$, namely, the involutivity of their diagonal section δ, i.e., $\delta \circ \delta = \delta$, where $\delta(x) = U(x,x)$, $x \in [0,1]$.*

For extended nullnorms we have the following result.

Theorem 3.3. *Let $V \in \mathscr{V}$ be an extended nullnorm with annihilator $\alpha \in]0,1[$, and $a,b \in \mathbb{N}$. Then $V \in \mathscr{A}_{(a,b)}$ if and only if the extended t-norm T given by*

$$T(\mathbf{x}) = \frac{V(\alpha + (1-\alpha)x_1, \ldots, \alpha + (1-\alpha)x_n) - \alpha}{1-\alpha},$$

$\mathbf{x} \in [0,1]^n$, $n \in \mathbb{N}$, belongs to $\mathscr{T} \cap \mathscr{A}_{(a,0)}$,
and the extended t-conorm S given by

$$S(\mathbf{x}) = \frac{V(\alpha \cdot \mathbf{x})}{\alpha},\ \mathbf{x} \in [0,1]^n, n \in \mathbb{N},$$

belongs to $\mathscr{S} \cap \mathscr{A}_{(0,b)}$.

Observe that the only proper nullnorms from $\mathscr{A}_{(1,1)}$ are just α-medians [4] given by

$$V_\alpha(\mathbf{x}) = \mathrm{med}(x_1, \alpha, x_2, \alpha, \ldots, x_{n-1}, \alpha, x_n).$$

4 OWA operators based on outliers

Recall that OWA operators in their n-ary form were introduced by Yager in [13]. An n-ary OWA operator $\mathrm{OWA}_{\mathbf{w}}$ can be written as

$$\mathrm{OWA}_{\mathbf{w}}(x_1,\ldots,x_n)=\sum_{i=1}^{n} w_i x_{(i)},$$

where $\mathbf{w}=(w_1,\ldots,w_n)\in[0,1]^n$ with $\sum_{i=1}^{n} w_i=1$, is its weighting vector.

OWA operators as extended aggregation functions are determined by the related weighting triangles

$$\Delta=\{w_{i,n}\mid n\in\mathbb{N},\, i\in\{1,\ldots,n\}\},$$

where, for each $n\in\mathbb{N}$, $(w_{1,n},\ldots,w_{n,n})\in[0,1]^n$ is an n-ary weighting vector, $\sum_{i=1}^{n} w_{i,n}=1$. An extended OWA operator $\mathrm{OWA}_\Delta\colon \bigcup_{n\in\mathbb{N}}[0,1]^n\to[0,1]$ can be written as

$$\mathrm{OWA}_\Delta(x_1,\ldots,x_n)=\sum_{i=1}^{n} w_{i,n}x_{(i)}.$$

More details can be found, e.g., in [2].

The following result is immediate.

Proposition 4.1. *Let $(a,b)\in\mathbb{N}_0\times\mathbb{N}_0$, $a+b\geq 1$, and let OWA_Δ be an extended OWA operator. Then $\mathrm{OWA}_\Delta\in\mathscr{A}_{(a,b)}$ if and only if for each $n\geq a+b$ we have*

- $w_{i,n}=w_{i,a+b}$ *for all $i\in\{1,\ldots,a\}$,*
- $w_{i,n}=0$ *for each $i\in\{a+1,\ldots,n-b\}$,*
- $w_{i,n}=w_{i+a+b-n,a+b}$ *for all $i\in\{n-b+1,\ldots,n\}$.*

Based on the above proposition, one obtains that all outliers-based extended OWA operators can be represented as is given in Theorem 2.1. Indeed, $\mathrm{OWA}_\Delta\in\mathscr{A}_{(a,b)}$ if and only if there are $\mathrm{OWA}_{\Delta^{(1)}}\in\mathscr{A}_{(a,0)}$, $\mathrm{OWA}_{\Delta^{(2)}}\in\mathscr{A}_{(0,b)}$ and $\lambda\in[0,1]$ such that for each $n\geq a+b$,

$$w_{i,n}=\begin{cases} w^{(1)}_{i,a}\cdot\lambda & \text{for } i\in\{1,\ldots,a\},\\ w^{(2)}_{i-n+b,b}\cdot(1-\lambda) & \text{for } i\in\{n-b+1,\ldots,n\},\\ 0 & \text{otherwise.}\end{cases}$$

Recall that OWA operators are related to the Choquet integrals [6], more precisely, the Choquet integrals with respect to symmetric capacities (i.e., symmetric Choquet integrals) coincide with OWA operators. Thus, an extended Choquet integral Ch_M, $M=(m_n)$ being a system of capacities m_n acting on $\{1,\ldots,n\}$, belongs to $\mathscr{A}_{(a,b)}$ only if the capacities m_n are symmetric whenever $n\geq a+b$, and the related weighting triangle $\Delta_M=(w_{i,n})$ given, for all $n\geq a+b$, by

$$w_{i,n}=m_n(\{i,\ldots,n\})-m_n(\{i+1,\ldots,n\})$$

(with convention $\{n+1,\ldots,n\}=\emptyset$) satisfies the constraints of Proposition 4.1.

Summarizing the above results for OWA operators OWA_Δ (Choquet integrals), we can see that $\mathrm{OWA}_\Delta=Ch_M\in\mathscr{A}_{(a,b)}$ if and only if there is an $(a+b)$-ary weighting vector $(v_1,\ldots,v_{a+b})$ such that for all $n\geq a+b$ and all $\mathbf{x}\in[0,1]^n$, we have

$$\mathrm{OWA}_\Delta(\mathbf{x})=Ch_M(\mathbf{x})=\sum_{i=1}^{a} v_i x_{(i)}+\sum_{i=a+1}^{a+b} v_i x_{(n+i-a-b)}. \tag{3}$$

There are two immediate generalizations of (3). Instead of OWA operators one can consider QOWA (quasi-OWA) operators based on continuous strictly monotone functions $f\colon[0,1]\to[-\infty,\infty]$, where

$$\mathrm{QOWA}_{f,\Delta}=f^{-1}\left(\mathrm{OWA}_\Delta(f(x_1),\ldots,f(x_n))\right).$$

For example, considering $f(x)=\log x$, the related $\mathrm{QOWA}_{f,\Delta}$ is the weighted ordered geometric mean. Transforming (3), we see that $\mathrm{QOWA}_{\log,\Delta}$ belongs to $\mathscr{A}_{(a,b)}$ if and only if there is an $(a+b)$-ary weighting vector $(v_1,\ldots,v_{a+b})$ such that for each $\mathbf{x}\in[0,1]^n$, $n\geq a+b$, we have

$$\mathrm{QOWA}_{\log,\Delta}(\mathbf{x})=\left(\prod_{i=1}^{a} x_{(i)}^{v_i}\right)\cdot\left(\prod_{i=a+1}^{a+b} x_{(n+i-a-b)}^{v_i}\right).$$

Another approach to generalizing outliers-based OWA operators is based on replacing the Choquet integral by some other fuzzy integral. For example, one can consider the Sugeno integral [12], and then $Su_M\in\mathscr{A}_{(a,b)}$ if and only if there is an $(a+b)$-ary weighting vector $(v_1,\ldots,v_{a+b})$ such that for each $\mathbf{x}\in[0,1]^n$, $n\geq a+b$, we have

$$Su_M(\mathbf{x})=\left(\bigvee_{i=1}^{a}\left(u_i\wedge x_{(i)}\right)\right)\vee\left(\bigvee_{i=a+1}^{a+b}\left(u_i\wedge x_{(n+i-a-b)}\right)\right),$$

where $u_i=\sum_{j=i}^{a+b} v_j$.

Example 4.1. *Consider $a=b=1$ and $(v_1,v_2)=(0.5,0.5)$. Then all the following extended aggregation functions belong to $\mathscr{A}_{(1,1)}$:*

- $\mathrm{OWA}_\Delta(\mathbf{x})=Ch_M(\mathbf{x})=\frac{x_{(1)}+x_{(n)}}{2}=\frac{\min(\mathbf{x})+\max(\mathbf{x})}{2}$

- $\text{QOWA}_{\log,\Delta}(\mathbf{x}) = x_{(1)}^{0.5} \cdot x_{(n)}^{0.5} = \sqrt{\min(\mathbf{x}) \cdot \max(\mathbf{x})}$
- $Su_M(\mathbf{x}) = \left(x_{(1)} \wedge 1\right) \vee \left(x_{(n)} \wedge 0.5\right)$

$$\begin{aligned} &= (\min(\mathbf{x})) \vee (\max(\mathbf{x}) \wedge 0.5) \\ &= \text{med}(\min(\mathbf{x}), 0.5, \max(\mathbf{x})) \\ &= \text{med}_{0.5}(\min(\mathbf{x}), \max(\mathbf{x})) \end{aligned}$$

5 Concluding remarks

We have introduced and discussed outliers-based extended aggregation functions related to some particular subclasses of extended aggregation functions, including extended t-norms, t-conorms, uninorms, nullnorms, OWA operators, etc. Note that for any $A \in \mathscr{A}_{(a,b)}$, also the transforms $A_\varphi \in \mathscr{A}_{(a,b)}$, $\varphi\colon [0,1] \to [0,1]$ being an automorphism and $A_\varphi(\mathbf{x}) = \varphi^{-1}(A(\varphi(x_1), \ldots, \varphi(x_n)))$. Moreover, if $\varphi\colon [0,1] \to [0,1]$ is a decreasing bijection then $A_\varphi \in \mathscr{A}_{(b,a)}$. In special cases, also some other construction methods could be applied. For example, in the case of ordinal sums of t-norms (t-conorms) [8], if all summands belong to $\mathscr{A}_{(a,b)}$, then the related ordinal sums are also outliers-based aggregation functions from $\mathscr{A}_{(a,b)}$. Possible applications of our research we expect in big data processing when instead of the huge amount of output data it is enough to consider $a+b$ outlying values only.

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