Continuous OWA Operators

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Abstract

Discrete OWA operators introduced by Yager (1988) have been widely used and their theoretical as well as application aspects have been studied since their introduction. Some generalizations to continuous case have already been proposed. In this contribution we extend the approach by Jin et al. (2020).

Keywords: OWA operator, Canonical form of OWA operator, Continuous OWA operator, Decreasing form of input function.

1 Introduction

Discrete (finite, to be more precise) OWA operators are a well-known family of aggregation functions (see, e,g., the monograph [7] for an overview on aggregation functions) that attracts attention of many researchers. See, e.g., [5] and the citations therein, [3, 13]. They were proposed by Yager [14]. Later on, Grabisch [6] proved that OWA operators are expressible by Choquet integrals (see [2] for Choquet integral). Particularly, the family of OWA operators is equivalent with the family of Choquet integrals with respect to symmetric capacities.

The discrete form of an OWA operator of dimension $n \in \{2, 3, \dots\}$ is a piece-wise linear aggregation function $\mathrm{OWA}_w : \mathbb{R}^n \to \mathbb{R}$, fulfilling the following:
for a given input function $f : \{1, 2, \dots, n\} \to \mathbb{R}$ and a given weighing function $w : \{1, 2, \dots, n\} \to [0, 1]$ with $\sum_{i=1}^{n} w(i) = 1$, choose a permutation $\tau : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ such that $f(\tau(i)) \geq f(\tau(j))$ whenever $i \leq j$, and then

$$\mathrm{OWA}_w(f) = \sum_{i=1}^{n} f(\tau(i)). \tag{1}$$

Narukawa et al. [9] proposed continuous weighted OWA operators (continuous WOWA operators) which are a kind of continuous version of OWA operators. Another approach to continuous OWA operators was proposed by Jin et al. [8], called the canonical form of OWA operators. There is a partial coincidence between the continuous WOWA operators and the canonical form of OWA operators.

This contribution is a continuation of Jin et al. [8]. In [8], the authors introduced introduced a decreasing form of measurable input functions for uncountably many inputs that are related to OWA-operators. Further, by means of Choquet integral, they defined a canonical form of the decreasing forms in case these are defined on a bounded interval and with a bounded range. We are going to extend he canonical form of OWA operators for the case when the input functions are defined on an unbounded interval and/or have unbounded range.

The paper is organized as follows – in Section 2 we provide a brief introduction to the theory of capacities and of Choquet integrals and we review the most important parts of [8]. Section 3 is the main body of the paper extending the canonical form of OWA operators from [8]. Finally, Section 4 concludes the paper with a final discussion pointing out a relationship between the continuous WOWA operators by Narukawa et al. [9] and the extended version of the canonical form of OWA operators.

2 Preliminaries

In the first part we briefly review the theory of capacities and of Choquet integral. The second part is devoted to a review of the most important results on the canonical forms of OWA operators by Jin et al. [8].

2.1 Capacities and Choquet integral

We assume that reader is familiar with basic notions of fuzzy measure, capacity and integral theory. For deeper knowledge we recommend monographs [10, 12]. Here we just very briefly review some notions needed in the paper.

Definition 1. *Let $(X, \mathscr{S})$ be a measurable space and $\mu : \mathscr{S} \to [0,1]$ be a monotone set-function such that $\mu(\emptyset) = 0$ and $\mu(X) = 1$. Then μ i called a* capacity.

The triplet $(X, \mathscr{S}, \mu)$ will be called a *capacity space*. Recall that for every capacity μ, μ^d is the dual capacity if $\mu^d(A^c) = 1 - \mu(A)$ for $A \in \mathscr{S}$.

Definition 2. *Let $(X, \mathscr{S}, \mu)$ be a capacity space. If for arbitrary system of measurable sets $A_1 \subset A_2 \subset \cdots \subset A_i \subset \dots$ the following holds*

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i \to \infty} \mu(A_i), \tag{2}$$

the capacity μ is called semi-continuous from below. *If for arbitrary system of measurable sets $A_1 \supset A_2 \supset \cdots \supset A_i \supset \dots$ the following holds*

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{i \to \infty} \mu(A_i), \tag{3}$$

the capacity μ is called semi-continuous from above. *If for all $A, B \in \mathscr{S}$*

$$\mu(B) = 0 \implies \mu(A \cup B) = \mu(A),$$

then μ is a null-additive capacity.

Notation 1. For arbitrary function $f : X \to \mathbb{R}$ we denote

$$f^+(x) = f(x) \vee 0 \quad \text{and} \quad f^-(x) = (-f)^+(x).$$

In the whole paper, Borel measurable functions will be called measurable (skipping the term Borel).

Definition 3 ([12])**.** *Let $f : X \to \mathbb{R}$ be a function and $\mathrm{rng}(f)$ be its range (called also co-domain). If $\mathrm{rng}(f)$ is a finite set, then f is said to be a* simple function. *If f is simple and moreover $\{x \in X; f(x) = a\} \in \mathscr{S}$ for all $a \in \mathrm{rng}(f)$, then we say that f is a* simple measurable function. *The set of all simple measurable functions on $(X, \mathscr{S})$ will be denoted by $\mathbb{SM}_{(X,\mathscr{S})}$.*

Definition 4 ([12])**.** *A function $f : X \to \mathbb{R}$ is said to be* measurable *if there exists a function $F : \mathbb{N} \to \mathbb{SM}_{(X,\mathscr{S})}$ such that, for all $x \in X$,*

$$f(x) = \lim_{i \to \infty} F_i(x),$$

where $\mathbb{N}$ denotes the set of all natural numbers and F_i denotes the function $F(i)$.

In the whole paper we will write F_i instead of $F(i)$, meaning the *i*-th element of the sequence F.

Definition 5 (see, e.g., [2, 10])**.** *Let $f : X \to [0, \infty[$ be a measurable function and μ be a capacity. The functional*

$$(C) \int f \, \mathrm{d}\mu = \int_0^\infty \mu(\{x \in X; f(x) \geq t\}) \, \mathrm{d}t,$$

where the right-hand-side integral is the Riemann one, is called the Choquet integral.

Remark 1. *Since the function f in Definition 5 is measurable, function $\tilde{f} : [0, \infty[\to [0,1]$ defined by $\tilde{f}(t) = \mu(\{x; f(x) \geq t\})$ is well defined. Moreover, it is decreasing and thus Riemann integrable. This justifies the definition of Choquet integral. The Choquet integral can be expressed also by*

$$(C) \int f \, \mathrm{d}\mu = \int_0^\infty \mu(\{x \in X; f(x) > t\}) \, \mathrm{d}t,$$

see [4, 10].

There are two possibilities how to extend the Choquet integral to functions achieving also negative values. The symmetric and the asymmetric way. In this paper we will deal with the asymmetric extension.

Definition 6 ([4])**.** *Let $f : X \to \mathbb{R}$ be a measurable function and μ be a capacity. The functional*

$$(AC) \int f \, \mathrm{d}\mu = (C) \int f^+ \, \mathrm{d}\mu - (C) \int f^- \, \mathrm{d}\mu^d,$$

where at least one of the values $(C) \int f^+ \, \mathrm{d}\mu$ and $(C) \int f^- \, \mathrm{d}\mu^d$, is finite, is called the Asymmetric Choquet Integral. *The measurable function f such that $(AC) \int f \, \mathrm{d}\mu$ does exist, is called* integrable.

Since there will be no confusion, we will denote also the asymmetric Choquet integral of f by $(C) \int f \, \mathrm{d}\mu$.

An important property of the asymmetric Choquet integral is the following.

Lemma 1. *Let μ be a capacity, $c \in \mathbb{R}$ a constant and f be a measurable function such that $(C) \int f \, \mathrm{d}\mu$ does exist. Then*

$$(C) \int f \, \mathrm{d}\mu + c = (C) \int (f + c \cdot \chi_X) \, \mathrm{d}\mu, \tag{4}$$

where χ_X is the characteristic function of X.

Notation 2. *For any set A, χ_A will denote its characteristic function.*

Definition 7. *Let μ_1, μ_2 be two capacities defined on a measurable space $(X, \mathscr{S})$. We say that μ_2 is absolutely continuous with respect to μ_1, notation $\mu_2 \ll \mu_1$, if for all $A \in \mathscr{S}$ we have that*

$$\mu_1(A) = 0 \quad \Rightarrow \quad \mu_2(A) = 0.$$

2.2 Canonical form of OWA operators

In this part we briefly review the most important notions and results by Jin et al. [8].

Notation 3. We denote $(X_1, \mathscr{S}_1)$ a measurable space fulfilling

- $\emptyset \neq X_1 \subset \mathbb{R}$ is a bounded set achieving its minimum and maximum denoted by

$$x_m = \min X_1, \quad x_M = \max X_1, \quad x_m < x_M,$$

- $\mathscr{S}_1$ is a σ-algebra such that $\mathscr{B}_{X_1} \subset \mathscr{S}_1 \subset 2^{X_1}$ where $\mathscr{B}_{X_1}$ is the σ-algebra of Borel subsets of X_1.

Moreover, we will assume that the capacity space $(X_1, \mathscr{S}_1, \mu)$ is equipped with a null-additive capacity semi-continuous from below and such that for arbitrary disjoint $A, B \in \mathscr{S}$

$$\mu(A \cup B) = \mu(A) \iff \mu(B) = 0. \tag{5}$$

Definition 8. *Let $(X_1, \mathscr{S}_1)$ be a measurable space. Any bounded measurable function $f : X_1 \to \mathbb{R}$ is said to be* input function. *The set of all input functions on $(X_1, \mathscr{S}_1)$ will be denoted by $\mathfrak{F}_{(X_1, \mathscr{S}_1)}$.*

Definition 9 ([8]). *Let $\mathscr{M} = (X_1, \mathscr{S}_1, \mu)$ be a capacity space and $f \in \mathfrak{F}_{(X_1, \mathscr{S}_1)}$. We say that $g_{(f,\mu)} \in \mathfrak{F}_{(X_1, \mathscr{S}_1)}$ is a* decreasing form of f in $\mathscr{M}$ *if the following are satisfied:*

(i) $g_{(f,\mu)}(x_1) \geq g_{(f,\mu)}(x_2)$ for all $x_1, x_2 \in X_1$, $x_1 \leq x_2$,

(ii) $\mu(\{x \in X_1; f(x) \geq t\}) = \mu(\{x \in X_1; g_{(f,\mu)}(x) \geq t\})$ holds for all $t \in \mathbb{R}$.

Proposition 1 ([8]). *Let $f \in \mathfrak{F}_{(X_1, \mathscr{S}_1)}$. Assume, in a capacity space $\mathscr{M} = (X_1, \mathscr{S}_1, \mu)$, f is integrable and there exists a decreasing form $g_{(f,\mu)}$ of f. Then*

$$(C) \int f \, d\mu = (C) \int g_{(f,\mu)} \, d\mu. \tag{6}$$

Corollary 1 ([8]). *Let $f \in \mathfrak{F}_{(X_1, \mathscr{S}_1)}$ be an input function. Assume in a capacity space $\mathscr{M}$ with a capacity μ satisfying equivalence (5) there exist two decreasing forms of f in the capacity space $\mathscr{M}$, denoted by $\hat{g}_{(f,\mu)}$ and $\tilde{g}_{(f,\mu)}$. Then*

$$\mu(\{x \in X_1; \hat{g}_{(f,\mu)}(x) \neq \tilde{g}_{(f,\mu)}(x)\}) = 0. \tag{7}$$

Proposition 2 ([8]). *Assume $\mathscr{M}$ is a capacity space where μ is a semi-continuous from above capacity that fulfils equivalence (5). Then for every $f \in \mathfrak{F}_{(X_1, \mathscr{S}_1)}$ there exists a decreasing form $g_{(f,\mu)}$ in $\mathscr{M}$ if and only if for every measurable set $E \neq \emptyset$ there exist elements $a_1, a_2 \in X_1$ such that*

$$\mu(E) = \mu([x_m, a_1] \cap X) = \mu^d([a_2, x_M] \cap X). \tag{8}$$

Remark 2. As it was proven in [8], there are two possibilities for the capacity space $(X_1, \mathscr{S}_1, \mu)$ in order every input function to have a decreasing form:

1. There exists a finite set $\bar{X} \in \mathscr{S}_1$ such that $\mu(\bar{X}) = \mu(X_1)$ and moreover, for all $A, B \in \mathscr{S}_1$ we have

$$\mu(A \cap \bar{X}) = \mu(B \cap \bar{X}) \tag{9}$$

whenever $\text{card}(A \cap \bar{X}) = \text{card}(B \cap \bar{X})$, and $\mu(\{x\}) > 0$ for all $x \in \bar{X}$, meaning that μ is a symmetric capacity on $2^{\bar{X}}$.

2. For all $x \in X_1$, $\mu(\{x\}) = 0$.

Definition 10. *Let $f \in \mathfrak{F}_{(X_1, \mathscr{S}_1)}$ and $g_{(f,\mu)}$ be a decreasing form of f in a capacity space $\mathscr{M}$ with a capacity μ fulfilling equivalence (5). Further, let $\mathbf{w} \ll \mu$ be a probability measure on the measurable space $(X_1, \mathscr{S}_1)$. Then $\mathbf{w}$ is said to be* OWA weighing function *in $\mathscr{M}$ and*

$$\text{OWA}_{\mathbf{w}}(f) = \int g_{(f,\mu)} \, d\mathbf{w}$$

is the canonical OWA form *of the input function f in $\mathscr{M}$.*

Corollary 2. *Let X_1 be an infinite countable set and $\mathscr{M}$ be a capacity space equipped with a capacity fulfilling equivalence (5) that is semi-continuous from above, and there exists no finite set $\bar{X} \subset X_1$ with $\mu(\bar{X}) = \mu(X_1)$. There exists $f \in \mathfrak{F}_{(X_1, \mathscr{S}_1)}$ with no canonical OWA form of f in $\mathscr{M}$.*

Definition 11 ([8]). *Let $\mu : \mathscr{S} \to [0,1]$ be a capacity. We say that a capacity $\nu : \mathscr{S} \to [0,1]$ is μ-symmetric if for arbitrary $A, B \in \mathscr{S}$*

$$\nu(A) = \nu(B) \quad \text{whenever} \quad \mu(A) = \mu(B). \tag{10}$$

Proposition 3 ([8]). *Let $f \in \mathfrak{F}_{(X_1, \mathscr{S}_1)}$ be an input function, μ be a capacity fulfilling the equivalence (5) and such that $\mu(\{x\}) = 0$ for all $x \in X_1$. Further, let $\mathbf{w} \ll \mu$ be a probability measure on $(X_1, \mathscr{S}_1)$. Denote $\tilde{\mathbf{w}}_{(\mu, \mathbf{w})}$ a μ-symmetric capacity such that*

$$\mathbf{w}([x_m, a] \cap X_1) = \tilde{\mathbf{w}}_{(\mu, \mathbf{w})}([x_m, a] \cap X)1) \tag{11}$$

for all $a \in [x_m, x_M]$. Then

$$\text{OWA}_{\mathbf{w}}(f) = (C) \int f^+ \, d\tilde{\mathbf{w}}_{(\mu, \mathbf{w})} - (C) \int f^- \, d\tilde{\mathbf{w}}^d_{(\mu, \mathbf{w})}. \tag{12}$$

Yet, we recall the definition of continuous WOWA operators from [9], since there is a partial coincidence between the canonical OWA form and the continuous WOWA operators which we will point out in Conclusions.

Definition 12 ([9])**.** *Let P be a probability measure on* $(\mathbb{R}, \mathscr{B})$ *with a density function p, that is,*

$$P([a,b]) = \int_a^b p(x)\,\mathrm{d}\lambda,$$

where λ *is the Lebesgue measure. Assume w is a monotone increasing function on* $[0,1]$ *and f is a continuous non-negative integrable function. The* Continuous WOWA operator *is defined by*

$$\mathrm{CWOWA}_\mu(f) = (C)\int f\,\mathrm{d}\mu \qquad (13)$$

where $\mu = w \circ P$.

The capacity μ in Definition 12 is a distortion of an absolutely continuous probability measure and $\mu \ll P$. The paper [1] is a kind of continuation of [9].

3 Extension of the model

Our intention is to extend the model of the canonical OWA form of input functions for functions with unbounded domain and also with unbounded range.

Notation 4. *By* $(X_2, \mathscr{S}_2)$ *we denote a measurable space fulfilling*

- $\emptyset \neq X_2 \subset \mathbb{R}$,

- $\mathscr{S}_2$ *is a* σ*-algebra such that* $\mathscr{B}_{X_2} \subset \mathscr{S}_2 \subset 2^{X_2}$.

Remark 3. We still assume to have a capacity μ that is null-additive, semi-continuous from above fulfilling equivalence (5).

Proposition 4. *Let* $(X_2, \mathscr{S}_2, \mu)$ *is a capacity space such that* μ *fulfils equivalence* (5) *and* $\mu(\{x\}) = 0$ *for every* $x \in X_2$. *Further, assume that for every* $A \in \mathscr{S}_2$ *there exist* $a_1, a_2 \in \mathbb{R}$ *such that*

$$\mu(]-\infty, a_1]) = \mu(A) = \mu^d([a_2, \infty[). \qquad (14)$$

Then there exists a decreasing form $g_{(f,\mu)}$ *for every measurable f.*

Proof. We provide here the main idea of the proof split into 4 steps.

1. There exists a monotone and bijective transform of $X_2 \subset \mathbb{R}$ to $\bar{X} \subset]0,1[$, denoted by φ. I.e., $\varphi : X_2 \to \bar{X}$. Denote

$$\varphi[E] = \{z \in \bar{X}; (\exists x \in E)(z = \varphi(x))\}.$$

2. Let $\bar{\mathscr{S}}$ be the smallest σ-algebra fulfilling $\{0\} \in \bar{\mathscr{S}}$, $\{1\} \in \bar{\mathscr{S}}$ and $\varphi[\mathscr{S}_2] \subset \bar{\mathscr{S}}$ where

$$\varphi[\mathscr{S}_2] = \{Z \subset \bar{X}; (\exists A \in \mathscr{S}_2)(Z = \varphi[A])\}. \quad (15)$$

Further, $\nu : \bar{\mathscr{S}} \to [0,1]$ be a capacity such that

$$nu(A) = \mu(E) \quad \text{if } A \setminus \{0,1\} = \varphi[E]. \qquad (16)$$

Then the fact that μ fulfils equality (14) implies that ν fulfils equality (8).

3. For an $(X_2, \mathscr{S}_2)$-measurable function f define

$$\bar{f}(x) = \begin{cases} f(\varphi^{-1}(x)) & \text{for } x \in \bar{X}, \\ \sup\{f(x); x \in \bar{X}\} & \text{if } x = 0, \\ \inf\{f(x); x \in \bar{X}\} & \text{if } x = 1. \end{cases} \qquad (17)$$

If we admit the values $\pm\infty$ as possible values of input functions, then $\bar{f}$ can be regarded an input function in the sense of Definition 8. Since the capacity ν fulfils equality (8), we get that $\bar{f}$ has a decreasing form $g_{(\bar{f},\nu)}$.

4. For all $x \in X_2$, we define the value at x of the decreasing form $g_{(f,\mu)}$ by

$$g_{(f,\mu)}(x) = g_{(\bar{f},\nu)}(\varphi(x)).$$

$\square$

Definition 13. *Let* $(X_2, \mathscr{S}_2, \mu)$ *be a capacity space equipped with a capacity* μ *fulfilling equivalence* (5). *Further, let* $\bar{X} \subset]0,1[$ *and* $\varphi : X \to \bar{X}$ *be a monotone bijection. Denote* $\bar{\mathscr{S}}$ *the smallest* σ*-algebra such that* $\varphi[\mathscr{S}_2] \cup \{\{0\}, \{1\}\} \subset \bar{\mathscr{S}}$ *where* $\varphi[\mathscr{S}_2]$ *is defined by* (15)*, and* $\nu : \bar{\mathscr{S}} \to [0,1]$ *the capacity defined by formula* (16). *Then* $(\bar{X} \cup \{0,1\}, \bar{\mathscr{S}}, \nu)$ *will be called the* φ*-transform of* $(X_2, \mathscr{S}_2, \mu)$.

Remark 4. Let $(\bar{X} \cup \{0,1\}, \bar{\mathscr{S}})$ be the φ-transform of a measurable space $(X_2, \mathscr{S}_2)$ introduced in Notation 4. Then $(\bar{X} \cup \{0,1\}, \bar{\mathscr{S}})$ is a measurable space introduced in Notation 3.

Lemma 2. *Assume a capacity space* $(X_2, \mathscr{S}_2, \mu)$ *where* μ *fulfils equivalence* (5). *Further, let* φ *be a monotone bijection between* X_2 *and* $\bar{X} \subset]0,1[$ *and* $(X_1, \mathscr{S}_1, \nu)$ *be the* φ*-transform of* $(X_2, \mathscr{S}_2, \mu)$. *Then* μ *fulfils constraint* (14) *if and only if* ν *fulfils constraint* (8).

We skip the proof of Lemma 2 since we get the assertion directly by the construction of the φ-transform of $(X_2, \mathscr{S}_2, \mu)$.

Directly by Proposition 4 and Lemma 2 we get the following.

Corollary 3. *Assume a capacity space* $(X_2, \mathscr{S}_2, \mu)$ *where* μ *fulfils equivalence* (5). *Further, let* $\bar{X} \subset]0,1[$ *and* $\varphi : X_2 \to \bar{X}$ *be a monotone bijection. Assume* $(X_1, \mathscr{S}_1, \nu)$ *be the* φ*-transform of* $(X_2, \mathscr{S}_2, \mu)$. *Let* $f : X_2 \to \mathbb{R}$ *be* $(X_2, \mathscr{S}_2)$*-measurable. Set* $\bar{f}$ *the function defined by formula* (17). *Then* $\bar{f}$ *is* ν*-integrable if*

and only if f is μ-integrable. In that case, if constraint (14) *yields for μ, we have*

$$(C)\int f\,\mathrm{d}\mu = (C)\int g_{(f,\mu)}\,\mathrm{d}\mu$$
$$= (C)\int \bar{f}\,\mathrm{d}\nu = (C)\int g_{(\bar{f},\nu)}\,\mathrm{d}\nu,$$

where $g_{(\bar{f},\nu)}$ is a decreasing form of $\bar{f}$ and $g_{(f,\mu)}$ is a decreasing form of f.

The following example illustrates the role of the capacity μ the construction of a decreasing form. We give examples of three different capacities (particularly, probability measures) that are somehow prototypical. The capacities μ_1 and μ_2 are absolutely continuous with respect to the Lebesgue measure. While the Lebesgue measure is also absolutely continuous with respect to μ_1, it is not so with respect to μ_2. The capacity μ_3 is related to the well-known Cantor Step Function. This capacity is not absolutely continuous with respect to the Lebesgue measure.

Example 1. Consider the measurable space $(\mathbb{R}, \mathscr{B})$ and the following measurable function

$$f(x) = \begin{cases} 1 & \text{for } x \in [-1, 0], \\ \frac{1}{2} & \text{for } x \in [1, 100], \\ 0 & \text{for } x \in]-\infty, -10[, \\ -\frac{1}{2} & \text{for } x \in [-10, -1[, \\ -1 & \text{for } x \in]0, 1[, \\ -x & \text{for } x \in]100, \infty[. \end{cases}$$

The space $(\mathbb{R}, \mathscr{B})$ will be equipped with three capacities. The capacities μ_1, μ_2, μ_3 are the probability measures given by the respective distribution functions

$$F_1(x) = \frac{e^x}{e^x + e^{-x}}, \quad x \in \mathbb{R},$$

$$F_2(x) = \begin{cases} 0 & \text{for } x \in]-\infty, 0], \\ 1 - e^{-x} & \text{for } x \in]0, \infty[, \end{cases}$$

and F_3 is a transform of the well-known Cantor step-function. We show the construction by infinitely countably many steps.
In the first step we define the value of F_3 in $]-1, 1[$ to be $\frac{1}{2}$. In the second step we define the value of F_3 in $]-100, -10[$ to be $\frac{1}{4}$ and in $[10, 100[$ to be $\frac{3}{4}$, next we define the value in $]-10000, -1000[$ to be $\frac{1}{8}$, in $]-7, -4[$ to be $\frac{3}{8}$, and symmetrically in $]4, 7[$ to be $\frac{5}{8}$ and in $]1000, 10000[$ to be $\frac{7}{8}$. I.e., we have after the 3

steps

$$F_3(x) = \begin{cases} \frac{1}{2} & \text{for } x \in]-1, 1[, \\ \frac{1}{4} & \text{for } x \in]-100, -10[, \\ \frac{3}{4} & \text{for } x \in]10, 100[, \\ \frac{1}{8} & \text{for } x \in]-10000, -1000[, \\ \frac{3}{8} & \text{for } x \in]-7, -4[, \\ \frac{5}{8} & \text{for } x \in]4, 7[, \\ \frac{7}{8} & \text{for } x \in]1000, 10000[. \end{cases}$$

In general, in the n-th step (for $n \geq 3$) we define the values of F_3 in 2^{n-1} intervals and the assigned values in these intervals are $(2i-1)2^{-n}$, respectively, for $1 \leq i \leq 2^{n-1}$. For this step we define F_3 recurrently:

$$F_3(x) = \begin{cases} \frac{1}{2^n} & \text{for } x \in]-10^{2(n-1)}, -10^{2n-3}[, \\ \frac{2^n-1}{2^n} & \text{for } x \in]10^{2n-3}, 10^{2(n-1)}[, \\ \frac{3}{2^n} & \text{for } x \in]-7 \cdot 10^{2(n-3)}, -4 \cdot 10^{2(n-3)}[, \\ \frac{2^n-3}{2^n} & \text{for } x \in]4 \cdot 10^{2(n-3)}, 7 \cdot 10^{2(n-3)}[, \\ \frac{5}{2^n} & \text{for } x \in]-9 \cdot 10^{2(n-4)}, -8 \cdot 10^{2(n-4)}[, \\ \frac{2^n-5}{2^n} & \text{for } x \in]8 \cdot 10^{2(n-4)}, 9 \cdot 10^{2(n-4)}[, \\ \frac{7}{2^n} & \text{for } x \in]-2 \cdot 10^{2(n-4)}, -10^{2(n-4)}[, \\ \frac{2^n-7}{2^n} & \text{for } x \in]10^{2(n-4)}, 2 \cdot 10^{2(n-4)}[. \end{cases}$$

Ones the values $\frac{k}{2^n}$ have been defined for odd $k \leq 2^\ell - 1$ and some $\ell \in \{0, 1, 2, \ldots, n-2\}$, we define the value for odd k, $2^\ell + 1 \leq k \leq 2^{\ell+1} - 1$. These values will be assigned in the interval $]-10^{2(n-\ell)-3}, -10^{2(n-\ell)-4}[$. This interval contains $2^{\ell-1}$ subintervals where the values of F_3 are not yet assigned. We split every of these subintervals into 3 intervals, each of the length $3^{2-2^{\ell-1}} \cdot 10^{2(n-\ell)-4}$ and we assign the value $\frac{k}{2^n}$ for $k \in \{2^\ell + 1, 2^\ell + 3, \ldots, 2^{\ell+1} - 1\}$ to the respective middle interval of every parted one.
Symmetrically, if on an interval $]-a, -b[$ a value $\frac{k}{2^n}$ is assigned, the value $1 - \frac{k}{2^n}$ is then assigned on the interval $]b, a[$.

Now, we construct for the three capacities, μ_1, μ_2, μ_3, in fact three probability measures, always a decreasing form of f. The layout of the respective decreasing forms, with approximate values of points splitting the

real line, is the following

$$g_{(f,\mu_1)}(x) = \begin{cases} 1 & \text{for } x \in \,]-\infty, -0.243[, \\ \frac{1}{2} & \text{for } x \in [-0.243, 0[, \\ 0 & \text{for } x \in [0, 4.12 \cdot 10^{-9}[, \\ -\frac{1}{2} & \text{for } x \in [4.12 \cdot 10^{-9}, 0.243[, \\ -1 & \text{for } x \in [0.243, 100[, \\ -x & \text{for } x \in [100, \infty[, \end{cases}$$

$$g_{(f,\mu_2)}(x) = \begin{cases} \frac{1}{2} & \text{for } x \in [-\infty, 0.459[, \\ -1 & \text{for } x \in [0.459, 100[, \\ -x & \text{for } x \in [100, \infty[, \end{cases}$$

$$g_{(f,\mu_3)}(x) = \begin{cases} \frac{1}{2} & \text{for } x \in [-\infty, -10[, \\ 0 & \text{for } x \in [-10, 1[, \\ -\frac{1}{2} & \text{for } x \in [1, 100[, \\ -x & \text{for } x \in [100, \infty[. \end{cases}$$

We can define the canonical OWA form of an input function $f \in \mathfrak{F}_{(X2,\mathscr{S})}$ in the same way as in Definition 10. We are not going to give the exact formulation of the new definition. The next proposition generalizes Proposition 3.

Proposition 5. *Let $f \in \mathfrak{F}_{(X_2,\mathscr{S}_2)}$ be an input function, μ be a capacity fulfilling the equivalence* (5) *and such that $\mu(\{x\}) = 0$ for all $x \in X_2$. Further, let $\mathbf{w} \ll \mu$ be a probability measure on $(X_2, \mathscr{S}_2)$. Denote $\tilde{\mathbf{w}}_{(\mu,\mathbf{w})}$ a μ-symmetric capacity such that*

$$\mathbf{w}(]-\infty, a] \cap X_2) = \tilde{\mathbf{w}}_{(\mu,\mathbf{w})}(]-\infty, a] \cap X_2) \qquad (18)$$

for all $a \in \mathbb{R}$. Then $g_{(f,\mu)}$ is $\mathbf{w}$-integrable if and only if f is $\tilde{\mathbf{w}}_{(\mu,\mathbf{w})}$-integrable, and that case we have

$$\mathrm{OWA}_{\mathbf{w}}(f) = (C)\int g_{(f,\mu)} \, d\mathbf{w} = (C)\int f \, d\tilde{\mathbf{w}}_{(\mu,\mathbf{w})}. \quad (19)$$

4 Conclusions

In this contribution, we have extended the model of the canonical form of OWA operators to the case when the domain of input functions f is not necessarily bounded and also the range is not necessarily bounded (see Proposition 5). However, if the range of f is unbounded both from above as well as from below, the integrability in the sense of Choquet is not guaranteed.

A similar result was achieved also by Narukawa et al. [9] when proposing continuous WOWA operators (see also Definition 12). The canonical OWA form is a generalization of continuous WOWA operators in the sense that Narukawa et al. restrict their attention only to non-negative inputs, while in Proposition 5 there is no restriction concerning the range of input function. Further, Narukawa et al. consider only probability distributions (as weighing functions) that are absolutely

continuous with respect to Lebesgue measure while in approach proposed in this paper, extending results by Jin et al. [8], arbitrary continuous probability distributions as weighing functions are admissible.

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