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On a New Class of Trivariate Copulas

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Abstract

Starting with three bivariate copulas and some auxiliary univariate functions, we determine a construction method of trivariate copulas, which generalizes a class of copulas previously introduced by M. Úbeda-Flores (2005). Specifically, we provide a characterization of this class of 3–copulas and we discuss some of its properties. Various related examples of parametric and semiparametric families can be obtained, including generalizations of EFGM copulas.

Keywords: Copula, Extremal Dependence, Fréchet class, Eyraud-Farlie-Gumbel-Morgenstern distribution.

1 Introduction

The interest about copulas is largely due to the fact that they may provide valuable tools to construct flexible stochastic models for multivariate random vectors. However, while an extended list of possible bivariate models is nowadays available, constructions of copulas in higher dimensions are still less popular, despite some powerful methods like factor, hierarchical and vine copulas (see, e.g., [1, 4, 8]). In particular, the compatibility problem of copulas, i.e. the determination of all copulas with some fixed lower-dimensional margins, is still considered under some simplifying assumptions and special cases (see, e.g., [6]).

In the case of trivariate copulas, for instance, an interesting construction has been considered in [12]. Given some bivariate copulas C_{12}, C_{13} and C_{23}, 3–copulas of type

$$C(x,y,z) = zC_{12}(x,y) + yC_{13}(x,z) + xC_{23}(y,z) - 2xyz \quad (1)$$

are studied. Notably, C_{12}, C_{13} and C_{23} are also the bivariate margins of C, which represents hence an element of the class of Fréchet class determined by these margins (see, e.g., [5]).

Here, we would like to extend the trivariate copulas of type (1). This extension requires three additional univariate functions that, together with three 2–copulas, allow to more flexibility in the whole class. To this end, first, we define a trivariate mapping in the most general case and give a sufficient and necessary condition for it to be a 3–copula. Then we restrict our attention to a special subclass that is easier to handle, in the sense that the 3-increasing property is easier to verify. We give a few examples showing how some parametric and semiparametric families of 3-copulas can be obtained. Notably, some of these families are related to Eyraud-Farlie-Gumbel-Morgenstern (EFGM) distributions (see, e.g. [11]). Various dependence properties are hence presented, with particular focus to the (multivariate) extremal dependence.

2 The general construction

We refer to [2, 9] for the definition of n–copulas ($n \geq 2$). In particular, given a copula C, we denote by $V_C(R)$ the C–volume of any rectangle $R \subseteq [0,1]^n$.

Since we are going to focus on 3–copulas, we recall that a function $C : [0,1]^3 \to [0,1]$ is a 3–copula if and only if C satisfies the *boundary conditions*, i.e., for every $s,t \in [0,1]$,

$$C(s,t,0) = C(s,0,t) = C(0,s,t) = 0,$$
$$C(1,1,t) = C(1,t,1) = C(t,1,1) = t,$$

and the C–volume of any rectangle $R = [x_1,x_2] \times [y_1,y_2] \times [z_1,z_2] \subseteq [0,1]^3$ is positive (*increasing prop-*

erty), i.e.,

$$\begin{aligned}V_C(R) = &- C(x_1,y_1,z_1) + C(x_1,y_1,z_2) + C(x_1,y_2,z_1)\\ &- C(x_1,y_2,z_2) + C(x_2,y_1,z_1) - C(x_2,y_1,z_2)\\ &- C(x_2,y_2,z_1) + C(x_2,y_2,z_2) \geq 0.\end{aligned}$$

We denote by Φ the class of all continuous and strictly increasing functions f from $[0,1]$ into $[0,1]$ such that $f(0)=0$ and $f(1)=1$. Let f, g and h be in Φ and let A, B and C be 2–copulas. Define the following mapping $F : [0,1]^3 \to \mathbb{R}$

$$\begin{aligned}F(x,y,z) :=& f(z)A(x,y) + g(y)B(x,z) + h(x)C(y,z)\\ &- xyz - f(z)g(y)h(x),\end{aligned} \tag{2}$$

for every $(x,y,z) \in [0,1]^3$.

Proposition 1. *The function F given by* (2) *is a copula if, and only if,*

$$\begin{aligned}&\frac{V_A([x_1,x_2]\times[y_1,y_2])}{(g(y_2)-g(y_1))\cdot(h(x_2)-h(x_1))}\\ &+\frac{V_B([x_1,x_2]\times[z_1,z_2])}{(f(z_2)-f(z_1))\cdot(h(x_2)-h(x_1))}\\ &+\frac{V_C([y_1,y_2]\times[z_1,z_2])}{(f(z_2)-f(z_1))\cdot(g(y_2)-g(y_1))} \geq\\ &\frac{V_{\Pi_3}([x_1,x_2]\times[y_1,y_2]\times[z_1,z_2])}{(f(z_2)-f(z_1))\cdot(g(y_2)-g(y_1))\cdot(h(x_2)-h(x_1))}+1,\end{aligned} \tag{3}$$

for every $x_1 \leq x_2$, $y_1 \leq y_2$ and $z_1 \leq z_2$.

Proof. Notice that, for every $t \in [0,1]$

$$F(1,1,t) = f(t) + t + t - t - f(t) = t,$$

and, analougsly, $F(1,t,1) = t = F(t,1,1)$. Morevorer, for every $s,t \in [0,1]$

$$F(s,t,0) = 0 = F(s,0,t) = F(0,s,t).$$

Therefore, F satisfies the boundary conditions for 3–copulas.

In order to prove that F satisfies the 3–increasing property, let

$$R := [x_1,x_2]\times[y_1,y_2]\times[z_1,z_2]$$

be a box in $[0,1]^3$. We have

$$\begin{aligned}V_F(R) = &- F(x_1,y_1,z_1) + F(x_1,y_1,z_2) + F(x_1,y_2,z_1)\\ &- F(x_1,y_2,z_2) + F(x_2,y_1,z_1) - F(x_2,y_1,z_2)\\ &- F(x_2,y_2,z_1) + F(x_2,y_2,z_2).\end{aligned}$$

Therefore, if either $x_1 = x_2$ or $y_1 = y_2$ or $z_1 = z_2$, then we have $V_F(R) = 0$. In the other cases,

$$\begin{aligned}V_F(R) = &(f(z_2)-f(z_1))\,V_A([x_1,x_2]\times[y_1,y_2])\\ &+(g(y_2)-g(y_1))\,V_B([x_1,x_2]\times[z_1,z_2])\\ &+(h(x_2)-h(x_1))\,V_C([y_1,y_2]\times[z_1,z_2])\\ &-V_{\Pi_3}([x_1,x_2]\times[y_1,y_2]\times[z_1,z_2])\\ &-(f(z_2)-f(z_1))\,(g(y_2)-g(y_1))\,(h(x_2)-h(x_1)).\end{aligned}$$

Therefore $V_F(R) \geq 0$ if, and only if, (3) holds. □

Notice that, if $f = g = h$ is the identity mapping on $[0,1]$, we obtain the family of trivariate copulas given in [12].

Now, we would like to discuss the dependence properties of the 3-copulas given by (2). For this purpose, we recall that, given two n-copulas C_1 and C_2, C_1 is said to be *more concordant* than C_2 if, for every $\mathbf{u} \in [0,1]^n$, $C_1(\mathbf{u}) \geq C_2(\mathbf{u})$ and $\overline{C_1}(\mathbf{u}) \geq \overline{C_2}(\mathbf{u})$, where $\overline{C_1}$ and $\overline{C_2}$ are the *survival functions* associated to C_1 and C_2 respectively. Recall that the survival function of a copula C is defined as the function $\overline{C}(\mathbf{u}) := \mathbb{P}(\mathbf{U} > \mathbf{u})$, where $\mathbf{u} \in [0,1]^n$ and $\mathbf{U}$ is a random vector distributed according to C. The following result holds.

Proposition 2. *Let $f,g,h \in \Phi$ and F_1 and F_2 be two 3-copulas defined, for every $(x,y,z) \in [0,1]^3$ as*

$$\begin{aligned}F_1(x,y,z) :=& f(z)A_1(x,y) + g(y)B_1(x,z) + h(x)C_1(y,z)\\ &- xyz - f(z)g(y)h(x);\end{aligned}$$

$$\begin{aligned}F_2(x,y,z) :=& f(z)A_2(x,y) + g(y)B_2(x,z) + h(x)C_2(y,z)\\ &- xyz - f(z)g(y)h(x).\end{aligned}$$

for suitable 2-copulas A_1,A_2,B_1,B_2,C_1,C_2 such that $A_1 \geq A_2$, $B_1 \geq B_2$ and $C_1 \geq C_2$. Then F_1 is more concordant than F_2.

Proof. For every $(x,y,z) \in [0,1]^3$, we have $F_1(x,y,z) \geq F_2(x,y,z)$ if and only if

$$\begin{aligned}f(z)(A_1(x,y)-A_2(x,y)) + g(y)(B_1(x,z)-B_2(x,z))\\ +h(x)(C_1(y,z)-C_2(y,z)) \geq 0,\end{aligned}$$

and $\overline{F_1}(x,y,z) \geq \overline{F_2}(x,y,z)$ if and only if

$$\begin{aligned}&(1-f(z))A_1(x,y) + (1-g(y))B_1(x,z)\\ &+(1-h(x))C_1(y,z) \geq (1-f(z))A_2(x,y)\\ &+(1-g(y))B_2(x,z) + (1-h(x))C_2(y,z).\end{aligned}$$

These two conditions are both satisfied under our assumptions, hence F_1 is more concordant than F_2. □

Let now F be defined as in (2). In order to determine whether F is a copula, condition (3) has to be satisfied. Let us assume that A, B and C are three *absolutely continuous* 2-copulas with densities given by $a(x,y), b(x,z)$ and $c(y,z)$, respectively, where $x,y,z \in [0,1]$. Let us also assume that f,g and h are differentiable functions on $[0,1]$, with continuous derivatives. We shall find another condition, which is easily seen to be equivalent to (3), but involves densities.

To this end, consider that, under previous assumptions, if F is a copula, then it is an absolutely continuous 3-copula. In fact, it is possible to compute the mixed third partial derivative of F. Note that, since $a(x,y) = \frac{\partial^2 A(x,y)}{\partial x \partial y}$, $b(x,z) = \frac{\partial^2 B(x,z)}{\partial x \partial z}$ and $c(y,z) = \frac{\partial^2 C(y,z)}{\partial y \partial z}$ regardless of the order of the derivatives (and almost everywhere on the respective domains), the mixed third partial derivative of F is in its turn independent of the order of the derivatives, and it is given by

$$\frac{\partial^3 F(x,y,z)}{\partial x \partial y \partial z} = f'(z)a(x,y) + g'(y)b(x,z) + h'(x)c(y,z) - 1 - f'(z)g'(y)h'(x). \tag{4}$$

Since it is straightforward to verify that, for every $(x,y,z) \in [0,1]^3$,

$$F(x,y,z) = \int_0^x du \int_0^y dv \int_0^z \frac{\partial^3 F(u,v,w)}{\partial u \partial v \partial w} dw,$$

the following result holds.

Proposition 3. *Let $f,g,h \in \Phi$ be differentiable functions on $[0,1]$, with continuous derivatives. Let A,B and C be three absolutely continuous 2-copulas with densities given by $a(x,y), b(x,z)$ and $c(y,z)$, respectively, where $x,y,z \in [0,1]$. Then the function F defined as in (2) is a copula if and only if, for every $(x,y,z) \in [0,1]^3$,*

$$f'(z)a(x,y) + g'(y)b(x,z) + h'(x)c(y,z) - 1 - f'(z)g'(y)h'(x) \geq 0. \tag{5}$$

Moreover, in this case, F is an absolutely continuous copula with density given by (4).

Example 1. If $A = B = C = \Pi$, the independence copula, then the function F of type (2) has the simple form

$$F(x,y,z) := xyf(z) + xzg(y) + h(x)yz - xyz - f(z)g(y)h(x).$$

However, such an F is not always a copula. If we consider for instance, $f(x) = g(x) = h(x) = x^2$, then the density given by (4) is negative for $x = y = 0.2$ and $z = 0.1$.

The 2–margins of the mapping F are given by

$$F_{1,2}(x,y) = F(x,y,1) = A(x,y) + g(y)x + h(x)y - xy - g(y)h(x);$$

$$F_{1,3}(x,z) = F(x,1,z) = f(z)x + B(x,z) + h(x)z - xz - f(z)h(x);$$

$$F_{2,3}(y,z) = F(1,y,z) = f(z)y + g(y)x + C(y,z) - yz - f(z)g(y).$$

It is known that, if F is a 3–copula, then $F_{1,2}, F_{1,3}$ and $F_{2,3}$ are 2–copulas. In particular, these marginals can be expressed in the form

$$C_{f,g}(x,y) := C(x,y) + (f(x) - x)(y - g(y)), \tag{6}$$

for a 2–copula C and for suitable f,g in Φ. Bivariate copulas of this form belong to the large class of copulas introduced in [7], which, on the other hand, generalizes the results given in [10].

3 A special subclass

In general, given three 2-copulas A,B and C and given three functions $f,g,h \in \Phi$, it looks very challenging to determine whether or not the function F, defined as in (2), is a 3-copula. For this reason, in this section we restrict our attention to a special subclass of functions of the kind (2) for which it is easier to check whether we obtain a 3-copula. Specifically, when we set $g = h = id_{[0,1]}$ and $B = C = \Pi$, eq. (2) becomes

$$F(x,y,z) := f(z)(A(x,y) - xy) + xyz. \tag{7}$$

If we denote by $\mathscr{C}_n$ the space of all the n-copulas, (7) defines a functional from $\Phi \times \mathscr{C}_2$ to the space of all continuous functions on $[0,1]^3$. Moreover, for a fixed f, such a function is continuous with respect to L^∞ norm (in their respective space). We are going to show a few cases in which this functional takes values in $\mathscr{C}_3$, namely, for suitable $(f,A) \in \Phi \times \mathscr{C}_2$, the function F defined as in (7) is a 3-copula. The following proposition, which provides a sufficient and necessary condition for F to be a copula, follows quite easily from Proposition 1.

Proposition 4. *Let $f \in \Phi$ and $A \in \mathscr{C}_2$. Then the function F defined, for every $(x,y,z) \in [0,1]^3$, as*

$$F(x,y,z) := f(z)(A(x,y) - xy) + xyz$$

is a 3-copula if and only if, for every rectangle $R := [x_1,x_2] \times [y_1,y_2] \times [z_1,z_2]$ of $[0,1]^3$,

$$\frac{f(z_2) - f(z_1)}{z_2 - z_1} \left(\frac{V_A(R')}{V_\Pi(R')} - 1 \right) \geq -1. \tag{8}$$

where $R' := [x_1,x_2] \times [y_1,y_2]$.

If $F(x,y,z) := f(z)(A(x,y) - xy) + xyz$, for suitable f and A, turns out to be a copula, then its bivariate margins are

$$\begin{aligned} F_{12}(x,y) &= F(x,y,1) = A(x,y); \\ F_{13}(x,z) &= F(x,1,z) = xz = \Pi(x,z); \\ F_{23}(y,z) &= F(1,y,z) = yz = \Pi(y,z). \end{aligned}$$

This means that F defines a copula model for a random vector (U,V,W) of uniformly distributed random variables on $[0,1]$ such that (U,V) is distributed according to A (in symbols, $(U,V) \sim A$), whereas $(U,W) \sim \Pi$ and $(V,W) \sim \Pi$, namely W is independent of both U and V.

Note that, since (by Proposition 1) $F(x,y,z) = f(z)(A(x,y) - xy) + xyz$ always satisfies the boundary conditions, Equation (8) is equivalent to the 3-increasing property of F. We shall point out that, for a fixed $A \in \mathscr{C}_2$, not every $f \in \Phi$ allows us to obtain a copula in $\mathscr{C}_3$; hence the function f needs to be chosen according to the properties of A. From now on, unless otherwise stated, F will always denote the mapping defined as in (7).

If we consider, in the expression of F, any bivariate copula $A \in \mathscr{C}_2$ that is *not fully supported* on $[0,1]^2$, we can state the following result.

Proposition 5. *Let $A \in \mathscr{C}_2$ be any copula whose support is not the whole unit square $[0,1]^2$. Then the only copula model of the kind* (7) *involving A is obtained with $f = id_{[0,1]}$, i.e.*

$$F(x,y,z) = zA(x,y), \quad (x,y,z) \in [0,1]^3.$$

Proof. Consider a copula $A \in \mathscr{C}_2$ that is not fully supported on $[0,1]^2$ and choose any rectangle $R' = [x_1,x_2] \times [y_1,y_2]$ such that $V_A(R') = 0$. Then, for every $z_1, z_2 \in [0,1]$ with $z_1 < z_2$, considering the rectangle $R := [x_1,x_2] \times [y_1,y_2] \times [z_1,z_2]$ of $[0,1]^3$, we have

$$\begin{aligned} V_F(R) \geq 0 &\iff \frac{f(z_2) - f(z_1)}{z_2 - z_1}\left(\frac{V_A(R')}{V_\Pi(R')} - 1\right) \geq -1 \\ &\iff \frac{f(z_2) - f(z_1)}{z_2 - z_1} \leq 1. \end{aligned}$$

Then, in order for F to be a copula, we must have

$$\sup_{z_1 < z_2} \frac{f(z_2) - f(z_1)}{z_2 - z_1} \leq 1.$$

Actually, this last condition, together with the conditions $f(0) = 0$ and $f(1) = 1$, forces every incremental ratio to be equal to 1, resulting in $f = id_{[0,1]}$. In order to see why, let us assume, by contradiction, that there exist $z_1, z_2 \in [0,1], z_1 < z_2$, such that the related incremental ratio is strictly less than 1. Hence we have

$$\frac{f(z_2) - f(z_1)}{z_2 - z_1} < 1 \Longrightarrow f(z_2) < f(z_1) + z_2 - z_1.$$

Now, since $f(0) = 0$ and $f(1) = 1$, either $z_1 > 0$ or $z_2 < 1$; without loss of generality, we can assume $z_2 < 1$ (the other case can be treated analogously). Note that

$$\frac{f(z_1) - f(0)}{z_1 - 0} = \frac{f(z_1)}{z_1} \leq 1 \Longrightarrow f(z_1) \leq z_1,$$

so that, since we previously found $f(z_2) < f(z_1) + z_2 - z_1$, this yields $f(z_2) < z_2$. But then

$$\frac{f(1) - f(z_2)}{1 - z_2} = \frac{1 - f(z_2)}{1 - z_2} > 1,$$

which is absurd since every incremental ratio must be less than or equal to 1. This proves that, for every $z \in [0,1]$, $f(z) = z$, so this is the only copula model of the kind (7) involving A. □

Let us now focus our attention on the semiparametric family of 2-copulas given by

$$A_{\theta,\phi}(x,y) := xy + \theta\phi(x)\phi(y), \quad (x,y) \in [0,1]^2,$$

where $\theta \in [0,1]$ and $\phi : [0,1] \to [-1,1]$ is assumed to be a 1-Lipschitz function such that $\phi(0) = \phi(1) = 0$. We would like to build a copula model of the kind (7) involving $A_{\theta,\phi}$. As already pointed out, we should be careful with the choice of the function $f \in \Phi$. Moreover, we expect the choice of the function f to be somehow related or influenced by the parameter θ. Having set $f(z) := z^\alpha, z \in [0,1], \alpha > 0$, we can state the following result.

Proposition 6. *Let $\theta \in [0,1]$ and $\phi : [0,1] \to [-1,1]$ be a 1-Lipschitz function such that $\phi(0) = \phi(1) = 0$. Consider the 2-copula*

$$A_{\theta,\phi}(x,y) := xy + \theta\phi(x)\phi(y). \quad (x,y) \in [0,1]^2;$$

then, for every $\alpha \in \left[1, \frac{1}{\theta}\right]$, there exists a trivariate copula model of the kind (7)*, having $A_{\theta,\phi}, \Pi$ and Π as its bivariate margins, given by*

$$F_{\alpha,\theta,\phi}(x,y,z) := \theta\phi(x)\phi(y)z^\alpha + xyz, \quad (x,y,z) \in [0,1]^3. \tag{9}$$

Proof. We are going to check the 3-increasing property for F, in order to see what kind of limitations we get on the choice of α. Considering an arbitrary rectangle $R := [x_1,x_2] \times [y_1,y_2] \times [z_1,z_2]$ of $[0,1]^3$ and applying (8), we have $V_F(R) \geq 0$ if and only if

$$\frac{f(z_2) - f(z_1)}{z_2 - z_1}\left(\theta \frac{\phi(x_2) - \phi(x_1)}{x_2 - x_1}\frac{\phi(y_2) - \phi(y_1)}{y_2 - y_1}\right) \geq -1. \tag{10}$$

Taking the absolute value of the last product and applying the 1-Lipschitz property of ϕ, one has

$$\frac{f(z_2)-f(z_1)}{z_2-z_1}\left(\theta\frac{|\phi(x_2)-\phi(x_1)|}{x_2-x_1}\frac{|\phi(y_2)-\phi(y_1)|}{y_2-y_1}\right)\leq \theta\sup_{z\in[0,1]}f'(z)\cdot 1\cdot 1\leq 1$$

provided that f fulfils

$$\sup_{z\in[0,1]}f'(z)\leq\frac{1}{\theta}. \tag{11}$$

At this point, condition (11) gives us all the information we need about how to choose α. Note that, if $\alpha\in(0,1)$, $f'(z)$ is a decreasing function, hence

$$\sup_{z\in[0,1]}f'(z)=\lim_{z\to 0^+}\alpha z^{\alpha-1}=+\infty,$$

which violates (11). Thus, we must choose $\alpha\geq 1$ in order for $f'(z)$ to be an increasing function such that

$$\sup_{z\in[0,1]}f'(z)=f'(1)=\alpha,$$

which, together with (11), means that we need to choose $\alpha\leq\frac{1}{\theta}$, and that completes the proof. □

Let us consider another copula model involving the *Eyraud-Farlie-Gumbel-Morgenstern* (EFGM) family of bivariate copulas. We recall that this is a parametric family of copulas, whose equation is given by

$$E_\lambda(x,y):=xy(1+\lambda(1-x)(1-y)),\quad (x,y)\in[0,1]^2,$$

where $\lambda\in[-1,1]$. We would like to build a copula model of the kind (7) involving E_λ, and as it turns out, this case has some similarities with the model built in Proposition 6.

Proposition 7. *Let $\lambda\in[-1,1]$ and consider the 2-copula*

$$E_\lambda(x,y):=xy(1+\lambda(1-x)(1-y)),\quad (x,y)\in[0,1]^2.$$

Then, for every $\alpha\in\left[1,\frac{1}{|\lambda|}\right]$, there exists a trivariate copula model of the kind (7), having E_λ,Π and Π as its bivariate margins, given for all $(x,y,z)\in[0,1]^3$ by

$$F_{\alpha,\lambda}(x,y,z):=\lambda xy(1-x)(1-y)z^\alpha+xyz. \tag{12}$$

Proof. First of all, set, for every $t\in[0,1]$, $\phi(t):=t(1-t)$, and note that the EFGM copula E_λ can be rewritten as

$$E_\lambda(x,y):=xy+\lambda\phi(x)\phi(y),\quad (x,y)\in[0,1]^2.$$

Also, note that $\phi(0)=\phi(1)=0$ and that, for every $t\in[0,1]$ $\phi'(t)=1-2t\in[-1,1]$, which implies that ϕ is a 1-Lipschitz function that satisfies all the requirements needed to build a semiparametric bivariate copula model as in Proposition 6. The only difference between A_θ and E_λ is in the parameters: in Proposition 6 we had $\theta\in[0,1]$, whereas now $\lambda\in[-1,1]$ can also take negative values. Again, we set $f(z):=z^\alpha, z\in[0,1], \alpha>0$. The corresponding trivariate function of the kind (7) is given, for every $(x,y,z)\in[0,1]^3$, by

$$\begin{aligned}F(x,y,z)&:=f(z)\left(E_\lambda(x,y)-xy\right)+xyz\\&=\lambda\phi(x)\phi(y)z^\alpha+xyz.\end{aligned}$$

We need to check the 3-increasing property for F, in order to see if there are some limitations on the choice of α. Considering an arbitrary rectangle $R:=[x_1,x_2]\times[y_1,y_2]\times[z_1,z_2]$ of $[0,1]^3$ and applying (8), we have $V_F(R)\geq 0$ if and only if

$$\frac{f(z_2)-f(z_1)}{z_2-z_1}\left(\lambda\frac{\phi(x_2)-\phi(x_1)}{x_2-x_1}\frac{\phi(y_2)-\phi(y_1)}{y_2-y_1}\right)\geq -1. \tag{13}$$

Taking the absolute value of the last product and applying the 1-Lipschitz property of ϕ, one has

$$\frac{f(z_2)-f(z_1)}{z_2-z_1}\left(|\lambda|\frac{|\phi(x_2)-\phi(x_1)|}{x_2-x_1}\frac{|\phi(y_2)-\phi(y_1)|}{y_2-y_1}\right)\leq|\lambda|\sup_{z\in[0,1]}f'(z)\cdot 1\cdot 1\leq 1$$

provided that f fulfills

$$\sup_{z\in[0,1]}f'(z)\leq\frac{1}{|\lambda|}. \tag{14}$$

We have just obtained the condition we need in order to properly choose α. As we have already pointed out in Proposition 6, condition (14) is violated whenever $\alpha\in(0,1)$. Hence, we must choose $\alpha\geq 1$ in order for $f'(z)$ to be an increasing function such that

$$\sup_{z\in[0,1]}f'(z)=f'(1)=\alpha,$$

which, together with (14), means that we need to choose $\alpha\leq\frac{1}{|\lambda|}$. This completes the proof. □

Note that, assuming that the hypotheses of the last result are satisfied, the copula $F_{\alpha,\lambda}$ is only exchangeable when $\alpha=1$.

Finally, we would like to focus on the study of two measures of extremal dependence: the *extremal dependence coefficients* (EDC's). First of all, we recall how the lower and upper EDC's are defined. Consider a random vector (X,Y,Z) that is distributed according to $F_{\alpha,\lambda}$, and assume that F,G,H

are the distribution functions of X, Y, Z respectively. Then set $F_{min} := \min\{F(X), G(Y), H(Z)\}$ and $F_{max} := \max\{F(X), G(Y), H(Z)\}$. Then the lower EDC is defined as

$$\varepsilon_L := \lim_{t \to 0^+} \mathbb{P}(F_{max} \le t \,|\, F_{min} \le t), \tag{15}$$

whereas the upper EDC is defined as

$$\varepsilon_U := \lim_{t \to 1^-} \mathbb{P}(F_{min} > t \,|\, F_{max} > t), \tag{16}$$

provided that the above limits exist. We refer to [3] for further details.

Proposition 8. *Let F be the 3-copula considered in Proposition 7 and given by (12). Then $\varepsilon_L(F) = \varepsilon_U(F) = 0$.*

Proof. If $t \in (0,1)$, it follows that

$$\mathbb{P}(F_{max} \le t \,|\, F_{min} \le t) = \frac{\mathbb{P}(F_{min} \le t, F_{max} \le t)}{\mathbb{P}(F_{min} \le t)} = \frac{F_{\alpha,\lambda}(t,t,t)}{1 - \hat{F}_{\alpha,\lambda}(1-t, 1-t, 1-t)}$$

for the lower case, where $\hat{F}_{\alpha,\lambda}$ is the survival copula associated to $F_{\alpha,\lambda}$. Similarly, we have

$$\mathbb{P}(F_{min} > t \,|\, F_{max} > t) = \frac{\mathbb{P}(F_{min} > t, F_{max} > t)}{\mathbb{P}(F_{max} > t)} = \frac{\hat{F}_{\alpha,\lambda}(1-t, 1-t, 1-t)}{1 - F_{\alpha,\lambda}(t,t,t)}$$

for the upper case. Now, it is straightforward to verify that, if $t \in (0,1)$:

$$\hat{F}_{\alpha,\lambda}(1-t, 1-t, 1-t) = 1 - 3t + F_{\alpha,\lambda}(t,t,1) + F_{\alpha,\lambda}(t,1,t) + F_{\alpha,\lambda}(1,t,t) - F_{\alpha,\lambda}(t,t,t).$$

Using (12), we can finally calculate the limits in (15) and (16).

$$\varepsilon_L = \lim_{t \to 0^+} \frac{t^3 + \lambda t^{\alpha+2}(1-t)^2}{3t - \lambda t^2(1-t)^2 - 3t^2 + t^3 + \lambda t^{\alpha+2}(1-t)^2} = \lim_{t \to 0^+} \frac{t^3(1+o(t))}{t(1+o(t))} = 0,$$

where $o(t) \to 0$ as $t \to 0^+$ (remember that $\alpha \ge 1$, hence $\alpha + 2 \ge 3$). As for the upper EDC ε_U, one has:

$$\lim_{t \to 1^-} \frac{1 - 3t + \lambda t^2(1-t)^2 + 3t^2 - t^3 - \lambda t^{\alpha+2}(1-t)^2}{1 - t^3 - \lambda t^{\alpha+2}(1-t)^2} = 1 + \lim_{t \to 1^-} \frac{3t^2 - 3t + \lambda t^2(1-t)^2}{1 - t^3 - \lambda t^{\alpha+2}(1-t)^2} = 1 - 1 = 0,$$

where we applied L'Hôpital's rule in order to solve the above limit. Thus, $\varepsilon_L = \varepsilon_U = 0$. □

Finally, we would like to show a 3D scatterplot of 200 points that we simulated from the trivariate copula given by (12), setting $\lambda = 0.5$ and $\alpha = 1.5$, as well as a scatterplot matrix in order to visualize the pairwise relationships among the variables. Both the 3D scatterplot and the scatterplot matrix have been generated by the R software. From a graphical point of view (see Figure 1), it can be noticed the absence of extremal dependence that we have just proved analytically. Also, the scatterplot matrix (see Figure 2) shows correlations on the lower panels; we actually found out the same (very weak) correlations even for different values of $\alpha \in [1,2]$.

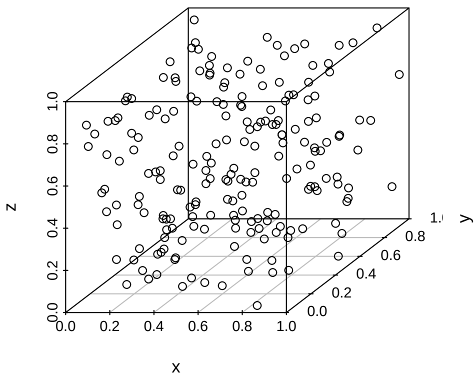

Figure 1: 3D scatterplot of 200 points simulated from the trivariate copula given by (12); here $\lambda = 0.5$ and $\alpha = 1.5$.

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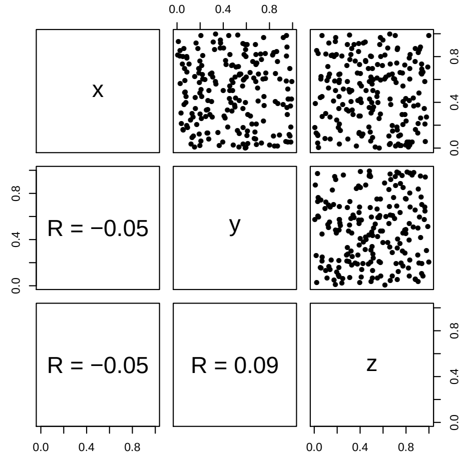

Figure 2: Scatterplot matrix for the 200 points simulated from the trivariate copula given by (12); here $\lambda = 0.5$ and $\alpha = 1.5$.

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