


Research Article

On Lie Symmetry Analysis of Certain Coupled Fractional Ordinary Differential Equations

 K. Sethukumarasamy^{1,*}, P. Vijayaraju¹, P. Prakash^{2,} 
¹Department of Mathematics, Anna University, Chennai – 600 025, India

²Department of Mathematics, Amrita School of Engineering, Amrita Vishwa Vidyapeetham, Coimbatore – 641 112, India

ARTICLE INFO

Article History

Received 11 November 2020

Accepted 13 March 2021

Keywords

 System of fractional ODEs
 Lie group formalism
 Laplace transform technique
 Mittag-Leffler function
 Exact solution

2010 AMS Subject Classification

 34A08
 76M60
 26A33
 33E12

ABSTRACT

In this article, we explain how to extend the Lie symmetry analysis method for n -coupled system of fractional ordinary differential equations in the sense of Riemann-Liouville fractional derivative. Also, we systematically investigated how to derive Lie point symmetries of scalar and coupled fractional ordinary differential equations namely (i) fractional Thomas-Fermi equation, (ii) Bagley-Torvik equation, (iii) two-coupled system of fractional quartic oscillator, (iv) fractional type coupled equation of motion and (v) fractional Lotka-Volterra ABC system. The dimensions of the symmetry algebras for the Bagley-Torvik equation and its various cases are greater than 2 and for this reason we construct optimal system of one-dimensional subalgebras. In addition, the exact solutions of the above mentioned fractional ordinary differential equations are explicitly derived wherever possible using the obtained symmetries.

© 2021 The Authors. Published by Atlantis Press B.V.

 This is an open access article distributed under the CC BY-NC 4.0 license (<http://creativecommons.org/licenses/by-nc/4.0/>).

1. INTRODUCTION

Fractional calculus has been effectively used in recent years to study many complex nonlinear phenomena. A natural phenomenon may depend not only on the instantaneous time but also on the past history of time. Such a phenomenon can be successfully modeled by the theory of differential equations involving derivatives and integrals of arbitrary order. With the virtue of nonlocal property of fractional derivatives, the fractional differential equations (FDEs) have been used to describe such scenarios precisely in the area of science and engineering such as viscoelasticity [3], hydrodynamics [64], quantum mechanics [36] and so on [16,24–26,32,42,63].

The derivation of the exact solution of the differential equation is important because the geometrical and physical meaning of the problem can be easily obtained. In literature, no well-defined analytic methods exist for deriving the exact solutions of fractional ordinary differential equations (FODEs). However, the derivation of the exact solution of FDEs is not an easy task since the properties of a fractional derivative are harder than the classical derivative. Recently, several research groups have been developed to derive the exact and numerical solutions of FDEs such as invariant subspace method [11,21,39,51–53,57,58,71], variational iteration method [44], homotopy perturbation method [45], operational matrix method [55], collocation method [40] and so on [14,23,68,69].

Among those methods, the Lie symmetry analysis method is an algorithmic approach that provides an efficient tool to construct an exact solution of FDEs in a systematic way. Initially, this method was proposed by Norwegian mathematician Sophus Lie during the 19th century and was further developed by Ovsianikov [49] and others [5,10,27,31,34,38,46–48,56,62]. The Lie symmetry analysis method is to find continuous transformations of one or more parameters leaving the differential equation invariant in the new coordinate system wherein the resulting differential equation is easier to solve. Gazizov et al. [18] extended the Lie symmetry analysis of differential equations to FDEs. Recently, many mathematicians and physicists have investigated and developed the theory of Lie symmetry analysis for FODEs and fractional partial differential equations (FPDEs) [4,6,7,9,15,18–20,29,30,37,54,59–61,70].

We would like to mention that only a very few applications of Lie point symmetries of scalar and coupled FODEs have been investigated. To the best of our knowledge, no one has extended the Lie symmetry analysis to n -coupled FODEs. The main aim of this article is to demonstrate

 *Corresponding author. Email: mathsethu@gmail.com

how the Lie symmetry approach provides an efficient tool to derive exact solutions of scalar and coupled nonlinear FODEs. More precisely, Lie symmetries of fractional Thomas-Fermi equation, Bagley-Torvik equation, two-coupled system of fractional quartic oscillator, fractional type coupled equation of motion and fractional Lotka-Volterra ABC system are derived. Using the obtained symmetries, we explicitly derived the exact solution of the above-mentioned FODEs wherever possible. In addition, the dimensions of the symmetry algebras for the Bagley-Torvik equation and its various cases are greater than 2 and for this reason, we construct the optimal system of one-dimensional subalgebras.

The article is organized as follows: For the sake of completeness, in section 2, we recall certain basic definitions pertaining to the fractional calculus. The Lie symmetry analysis of the coupled system of FODEs is presented. The applicability and effectiveness of this method have been illustrated through the above mentioned FODEs in section 3. In section 4, we summarize our results as concluding remarks of this paper.

2. PRELIMINARIES

In this section, we present some basic concepts and results of the fractional calculus. We also present brief details of symmetry analysis for scalar and coupled system of FODEs.

Definition 2.1. [50] Let $x(t) \in L^1[a, b]$ and $\alpha \in \mathbb{R}^+$. Then the Riemann-Liouville (R-L) fractional differential operator of order $\alpha > 0$ is defined by

$${}_a D_t^\alpha x(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\xi)^{n-\alpha-1} x(\xi) d\xi, & \text{if } \alpha \in (n-1, n), n \in \mathbb{N} \\ x^{(n)}(t), & \text{if } \alpha = n \in \mathbb{N} \end{cases} \tag{2.1}$$

for $t > a$.

Note that $L^1[a, b]$ denotes the space of absolutely integrable functions on $[a, b]$.

Note 1. [32,50] The R-L fractional derivative of order α for $x(t) = t^\beta$ is as follows:

$${}_0 D_t^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, \beta > -1, \alpha \geq 0.$$

Note 2. The Leibniz formula for the R-L fractional derivative is of the following form:

$${}_a D_t^\alpha (x_1(t)x_2(t)) = \sum_{k=0}^\infty \binom{\alpha}{k} {}_a D_t^{\alpha-k} x_1(t) D_t^k x_2(t), \alpha > 0, \tag{2.2}$$

where $\binom{\alpha}{k} = \frac{\Gamma(1+\alpha)}{\Gamma(\alpha-k+1)\Gamma(k+1)}$.

Note 3. [50] Let $\alpha \in (n-1, n], n \in \mathbb{N}$. Then the Laplace transformation of R-L derivative (2.1) is given by

$$L \{ {}_0 D_t^\alpha x(t) \} = s^\alpha x(s) - \sum_{k=0}^{n-1} s^k D^{\alpha-k-1} x(t)|_{t=0}, \Re(s) > 0, \tag{2.3}$$

where $x(s) = L \{ x(t) \} = \int_0^\infty e^{-st} x(t) dt$.

Definition 2.2. [43] The generalized Mittag-Leffler function with three parameters is defined as

$$\mathbf{E}_{\rho, \mu}^\gamma(z) = \sum_{k=0}^\infty \frac{(\gamma)_k z^k}{\Gamma(\rho k + \mu) k!}, \rho, \mu, \gamma \in \mathbb{C}, \Re(\rho) > 0, \Re(\mu) > 0, \tag{2.4}$$

where $(\gamma)_k = \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)}$ and $(\gamma)_0 = 1$ for $\Re(\gamma) > 0$.

Note 4. The Laplace transformation of the Mittag-Leffler functions are given by [6]

$$(i) L \{ t^{\mu-1} \mathbf{E}_{\rho, \mu}^\gamma(\pm at^\rho) \} = \begin{cases} \frac{s^{\rho\gamma-\mu}}{(s^\rho \mp a)^\gamma}, & \text{if } \mu \neq \rho\gamma, \Re(s) > |a|^{\frac{1}{\rho}}, \\ \frac{1}{(s^\rho \mp a)^\gamma}, & \text{if } \mu = \rho\gamma, \Re(s) > |a|^{\frac{1}{\rho}}. \end{cases}$$

$$(ii) L \left\{ t^{\mu-1} E_{\rho,\mu}(\mp at^\rho) \right\} = \left[\frac{s^{\rho-\mu}}{s^\rho \pm a}; t \right].$$

$$(iii) L \left\{ t^{\rho-\delta} \sum_{r=0}^{\infty} (-a)^r t^{(\rho-\mu)r} E_{\rho,\rho+(\rho-\mu)r-\delta+1}^{r+1}(-bt^\rho) \right\} = \left[\frac{s^{\delta-1}}{s^\rho + as^\mu + b}; t \right].$$

2.1. Symmetry Analysis for FODE

Consider the following generalized FODE of the form

$${}_a D_t^{\alpha+k} x = G \left(t, x, {}_a D_t^\alpha x, {}_a D_t^{\alpha+1} x, \dots, {}_a D_t^{\alpha+k-1} x \right), \alpha \in \mathbb{R}^+, t > 0, k \in \mathbb{N} \cup \{0\}, \tag{2.5}$$

where t and x are independent and dependent variables respectively. Assume that equation (2.5) is invariant under the following one parameter (ϵ) Lie group of continuous point transformations [6,18,19,29,54]

$$\begin{aligned} \bar{t} &\mapsto t + \epsilon \tau(t, x) + O(\epsilon^2), \\ \bar{x} &\mapsto x + \epsilon \zeta(t, x) + O(\epsilon^2), \\ \bar{x}^{(k)} &\mapsto x^{(k)} + \epsilon \zeta^{(k)} + O(\epsilon^2), \quad k \in \mathbb{N}, \\ {}_a D_t^\alpha \bar{x} &\mapsto {}_a D_t^\alpha x + \epsilon \zeta^{(\alpha)} + O(\epsilon^2), \\ &\vdots \\ {}_a D_t^{\alpha+k} \bar{x} &\mapsto {}_a D_t^{\alpha+k} x + \epsilon \zeta^{(\alpha+k)} + O(\epsilon^2), \quad k \in \mathbb{N}. \end{aligned} \tag{2.6}$$

We apply the transformations (2.6) in (2.5) and omit the higher powers of ϵ . Then by comparing the coefficients of ϵ on both the sides of the resulting equation we obtain

$$\left[\zeta^{(\alpha+k)} - \tau \frac{\partial G}{\partial t} - \zeta \frac{\partial G}{\partial x} - \sum_{r=0}^k \zeta^{(\alpha+r-1)} \frac{\partial G}{\partial {}_a D_t^{\alpha+r-1} x} - \sum_{r=0}^k \zeta^{(r)} \frac{\partial G}{\partial x^{(r)}} \right] \Big|_{H=0} = 0,$$

where $H = {}_a D_t^{\alpha+k} x - G, k \in \mathbb{N} \cup \{0\}$, which is the invariant equation for (2.5). Therefore, the infinitesimal generator becomes

$$X = \tau(t, x) \frac{\partial}{\partial t} + \zeta(t, x) \frac{\partial}{\partial x},$$

where

$$\begin{aligned} \tau(t, x) &= \left. \frac{d\bar{t}}{d\epsilon} \right|_{\epsilon=0}, \\ \zeta(t, x) &= \left. \frac{d\bar{x}}{d\epsilon} \right|_{\epsilon=0}. \end{aligned}$$

Therefore, the prolongation formula for (2.5) can be written as

$$Pr^{(\alpha+k)} X(H) = \left[\zeta^{(\alpha+k)} - XG - \sum_{r=0}^k \zeta^{(\alpha+r-1)} \frac{\partial G}{\partial {}_a D_t^{\alpha+r-1} x} - \sum_{r=0}^k \zeta^{(r)} \frac{\partial G}{\partial x^{(r)}} \right] \Big|_{H=0} = 0,$$

where $H = {}_a D_t^{\alpha+k} x - G, k \in \mathbb{N} \cup \{0\}$ and the infinitesimals are calculated as follows [6,18,19,29,54]

$$\zeta^{(k)} = D(\zeta^{(k-1)}) - x^{(k)} D(\tau) = D^k (\zeta - \tau x^{(1)}) + \tau x^{(k+1)}, \quad k \in \mathbb{N}, \tag{2.7}$$

$$\begin{aligned} \zeta^{(\alpha+k)} &= {}_a D_t^{\alpha+k} (\zeta - \tau x^{(1)}) + \tau {}_a D_t^{\alpha+k-1} x \\ &= {}_a D_t^{\alpha+k} (\zeta) + \sum_{n=0}^{\infty} \binom{\alpha+k}{n} \left(\frac{n - (\alpha+k)}{n+1} \right) {}_a D_t^{\alpha+k-n} x D_t^{n+1} (\tau), \quad \alpha > 0. \end{aligned} \tag{2.8}$$

Note 5. The lower limit of the R-L fractional derivative (2.1) is fixed. Therefore, it should be invariant at $t = a$, under the given transformation (2.6). Thus, we obtain the invariant condition as

$$\tau(a, x) = a. \tag{2.9}$$

Definition 2.3. A function $F(x, t)$ is an invariant of FODE (2.5) if $F(x, t)$ is an invariant surface, that is $XF = 0$ which implies the following

$$\left(\tau \frac{\partial F}{\partial t} + \zeta \frac{\partial F}{\partial x} \right) = 0.$$

Note 6. [28] The Lie bracket $[X_1, X_2]$ of operators

$$X_1 = \tau_1 \frac{\partial}{\partial t} + \zeta_1 \frac{\partial}{\partial x} \quad \text{and} \quad X_2 = \tau_2 \frac{\partial}{\partial t} + \zeta_2 \frac{\partial}{\partial x}$$

is given by

$$[X_1, X_2] = X_1(X_2) - X_2(X_1) = (X_1(\tau_2) - X_2(\tau_1)) \frac{\partial}{\partial t} + (X_1(\zeta_2) - X_2(\zeta_1)) \frac{\partial}{\partial x}. \tag{2.10}$$

Definition 2.4. [28] A Lie algebra is a real linear space \mathcal{L} together with a binary operation $[\cdot, \cdot] : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ satisfying the following properties:

- (i) $[X, Y] = -[Y, X]$,
- (ii) $[aX + bY, Z] = a[X, Z] + b[Y, Z]$,
- (iii) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$,

for $X, Y, Z \in \mathcal{L}$ and $a, b \in \mathbb{R}$. The binary operation $[\cdot, \cdot]$ is known as Lie bracket defined in equation (2.10).

2.2. Symmetry Analysis for Coupled System of FODEs

Let us first present the Lie symmetry analysis for the two-coupled system of FODEs in the sense of R-L fractional derivative. Then we shall extend this method for the n -coupled system of FODEs.

2.2.1. Symmetry analysis for two-coupled system of FODEs

Consider the following two-coupled system of FODE in the sense of R-L fractional derivative

$$\begin{aligned} {}_a D_t^\alpha x_1(t) &= f_1(t, x_1, x_2), \\ {}_a D_t^\alpha x_2(t) &= f_2(t, x_1, x_2), \quad \alpha > 0, \end{aligned} \tag{2.11}$$

where ${}_a D_t^\alpha x_i(t) = \frac{d^\alpha x_i}{dt^\alpha}$ for $i = 1, 2$ and f_1, f_2 are arbitrary functions. Here we assume that system (2.11) is invariant under one-parameter (ϵ) Lie group of continuous point transformations

$$\begin{aligned} \bar{t} &\mapsto t + \epsilon \tau(t, x_1, x_2) + O(\epsilon^2), \\ \bar{x}_1 &\mapsto x_1 + \epsilon \zeta_{x_1}(t, x_1, x_2) + O(\epsilon^2), \\ \bar{x}_2 &\mapsto x_2 + \epsilon \zeta_{x_2}(t, x_1, x_2) + O(\epsilon^2), \\ {}_a D_t^\alpha \bar{x}_1 &\mapsto {}_a D_t^\alpha x_1 + \epsilon \zeta_{x_1}^{(\alpha)}(t, x_1, x_2) + O(\epsilon^2), \\ {}_a D_t^\alpha \bar{x}_2 &\mapsto {}_a D_t^\alpha x_2 + \epsilon \zeta_{x_2}^{(\alpha)}(t, x_1, x_2) + O(\epsilon^2), \end{aligned} \tag{2.12}$$

where $\zeta_{x_1}^{(\alpha)}$ and $\zeta_{x_2}^{(\alpha)}$ are α th extended infinitesimals that are derived in the following theorem.

Theorem 2.1. The α th ($\alpha \in \mathbb{R}^+$) extended infinitesimals for the two-coupled system of FODEs in the sense of R-L fractional derivative are given by

$$\begin{aligned} \zeta_{x_1}^{(\alpha)} &= {}_a D_t^\alpha (\zeta - \tau x_1^{(1)}) + \tau {}_a D_t^{\alpha+1}(x_1), \\ \zeta_{x_2}^{(\alpha)} &= {}_a D_t^\alpha (\zeta - \tau x_2^{(1)}) + \tau {}_a D_t^{\alpha+1}(x_2). \end{aligned} \tag{2.13}$$

Proof. The Leibniz rule for fractional derivative is given by

$${}_a D_t^\alpha (\phi(t)\psi(t)) = \sum_{n=0}^{\infty} \binom{\alpha}{n} {}_a D_t^{\alpha-n} \phi(t)\psi^{(n)}(t), \quad \alpha > 0, \tag{2.14}$$

where the binomial coefficient is given by $\binom{\alpha}{n} = \frac{\Gamma(1+\alpha)}{\Gamma(\alpha-n+1)\Gamma(n+1)}$. By substituting $\phi(t) = 1$ and $\psi(t) = \bar{x}_1(\bar{t})$ in Leibniz rule given by equation (2.14) we obtain

$${}_a D_t^\alpha (\bar{x}_1(\bar{t})) = \sum_{n=0}^{\infty} \binom{\alpha}{n} \frac{(\bar{t} - a)^{n-\alpha}}{\Gamma(n - \alpha + 1)} \bar{x}_1^{(n)}(\bar{t}). \tag{2.15}$$

Similarly, we obtain

$${}_a D_t^\alpha (\bar{x}_2(\bar{t})) = \sum_{n=0}^{\infty} \binom{\alpha}{n} \frac{(\bar{t} - a)^{n-\alpha}}{\Gamma(n - \alpha + 1)} \bar{x}_2^{(n)}(\bar{t}). \tag{2.16}$$

By the definition of extended α th infinitesimals, we have

$$\zeta_{x_1}^{(\alpha)} = \frac{d}{d\epsilon} [{}_a D_t^\alpha (\bar{x}_1(\bar{t}))]_{\epsilon=0}, \tag{2.17}$$

$$\zeta_{x_2}^{(\alpha)} = \frac{d}{d\epsilon} [{}_a D_t^\alpha (\bar{x}_2(\bar{t}))]_{\epsilon=0}. \tag{2.18}$$

Substituting equation (2.15) in (2.17), we obtain

$$\zeta_{x_1}^{(\alpha)} = \frac{d}{d\epsilon} \left[\sum_{n=0}^{\infty} \binom{\alpha}{n} \frac{(\bar{t} - a)^{n-\alpha}}{\Gamma(n - \alpha + 1)} \bar{x}_2^{(n)}(\bar{t}) \right]_{\epsilon=0}. \tag{2.19}$$

After the use of infinitesimal transformation given by equation (2.12) and simplification, the above equation (2.19) becomes

$$\zeta_{x_1}^{(\alpha)} = \sum_{n=0}^{\infty} \binom{\alpha}{n} \frac{(t - a)^{n-\alpha} \zeta^{(n)}}{\Gamma(n - \alpha + 1)} + \sum_{n=0}^{\infty} \frac{\tau(n - \alpha)(t - a)^{n-\alpha-1} x_1^{(n)}(t)}{\Gamma(n - \alpha + 1)}. \tag{2.20}$$

Substituting equation (2.7) in the above equation (2.20), we obtain

$$\begin{aligned} \zeta_{x_1}^{(\alpha)} &= \sum_{n=0}^{\infty} \binom{\alpha}{n} \frac{(t - a)^{n-\alpha} D^n (\zeta - \tau x_1^{(1)}(t))}{\Gamma(n - \alpha + 1)} + \sum_{n=0}^{\infty} \frac{\tau(t - a)^{n-\alpha} x_1^{(n+1)}(t)}{\Gamma(n - \alpha + 1)} \\ &\quad + \sum_{n=0}^{\infty} \binom{\alpha}{n} \frac{\tau(n - \alpha)(t - a)^{n-\alpha-1} x_1^{(n)}(t)}{\Gamma(n - \alpha + 1)}. \end{aligned} \tag{2.21}$$

In the first term of the above equation (2.21) we apply (2.15). Then replacing n by $n - 1$ in the second term and substituting $n = 0$ in the third term of the above equation (2.21) we obtain

$$\begin{aligned} \zeta_{x_1}^{(\alpha)} &= {}_a D_t^\alpha (\zeta - \tau x_1^{(1)}) + \sum_{n=1}^{\infty} \binom{\alpha}{n-1} \frac{\tau(t - a)^{n-\alpha-1} x_1^{(n)}(t)}{\Gamma(n - \alpha)} - \frac{\tau \alpha (t - a)^{-\alpha-1} x_1(t)}{\Gamma(1 - \alpha)} \\ &\quad + \sum_{n=0}^{\infty} \binom{\alpha}{n} \frac{\tau(n - \alpha)(t - a)^{n-\alpha-1} x_1^{(n)}(t)}{\Gamma(n - \alpha + 1)}. \end{aligned} \tag{2.22}$$

The above equation (2.22) can be further simplified by using the relation $\binom{\alpha}{n-1} + \binom{\alpha}{n} = \binom{\alpha+1}{n}$ to get

$$\zeta_{x_1}^{(\alpha)} = {}_a D_t^\alpha (\zeta - \tau x_1^{(1)}) + \sum_{n=0}^{\infty} \binom{\alpha+1}{n} \frac{(t - a)^{n-\alpha-1} \tau x_1^{(n)}}{\Gamma(n - \alpha)}. \tag{2.23}$$

Applying (2.15) in (2.23) we obtain,

$$\zeta_{x_1}^{(\alpha)} = {}_a D_t^\alpha (\zeta - \tau x_1^{(1)}) + \tau {}_a D_t^{\alpha+1} (x_1). \tag{2.24}$$

In a similar manner, we obtain

$$\zeta_{x_2}^{(\alpha)} = {}_a D_t^\alpha (\zeta - \tau x_2^{(1)}) + \tau {}_a D_t^{\alpha+1} (x_2). \tag{2.25}$$

□

Remark 1. Applying the generalized Leibniz rule (2.2) in the extended infinitesimals given by equations (2.24) and (2.25) we obtain

$$\begin{aligned} \zeta_{x_1}^{(\alpha)} &= {}_a D_t^\alpha (\zeta_{x_1}) - \alpha D_t(\tau) {}_a D_t^\alpha (x_1) + \sum_{n=1}^{\infty} \binom{\alpha}{n} \left(\frac{n - \alpha}{n + 1} \right) {}_a D_t^{\alpha-n} (x_1) D_t^{n+1} (\tau), \\ \zeta_{x_2}^{(\alpha)} &= {}_a D_t^\alpha (\zeta_{x_2}) - \alpha D_t(\tau) {}_a D_t^\alpha (x_2) + \sum_{n=1}^{\infty} \binom{\alpha}{n} \left(\frac{n - \alpha}{n + 1} \right) {}_a D_t^{\alpha-n} (x_2) D_t^{n+1} (\tau). \end{aligned}$$

2.2.2. Symmetry analysis for n -coupled system of FODEs

In this subsection, we consider the n -coupled system of FODEs in the sense of R-L fractional derivative, having the form

$$\begin{aligned} {}_aD_t^\alpha x_1 &= f_1(t, x_1, x_2, \dots, x_n), \\ {}_aD_t^\alpha x_2 &= f_2(t, x_1, x_2, \dots, x_n), \\ &\vdots \\ {}_aD_t^\alpha x_n &= f_n(t, x_1, x_2, \dots, x_n), \end{aligned} \tag{2.26}$$

where ${}_aD_t^\alpha x_i = \frac{d^\alpha x_i}{dt^\alpha}$, for $i = 1, 2, \dots, n$, $\alpha > 0$ and f_1, f_2, \dots, f_n are arbitrary functions. Assuming that system (2.26) is invariant under one-parameter (ϵ) Lie group of continuous point transformations, we obtain

$$\begin{aligned} \bar{t} &\mapsto t + \epsilon\tau(t, x_1, x_2, \dots, x_n) + O(\epsilon^2), \\ \bar{x}_1 &\mapsto x_1 + \epsilon\zeta_{x_1}(t, x_1, x_2, \dots, x_n) + O(\epsilon^2), \\ \bar{x}_2 &\mapsto x_2 + \epsilon\zeta_{x_2}(t, x_1, x_2, \dots, x_n) + O(\epsilon^2), \\ &\vdots \\ \bar{x}_n &\mapsto x_n + \epsilon\zeta_{x_n}(t, x_1, x_2, \dots, x_n) + O(\epsilon^2), \\ {}_aD_t^\alpha \bar{x}_1 &\mapsto {}_aD_t^\alpha x_1 + \epsilon\zeta_{x_1}^{(\alpha)}(t, x_1, x_2, \dots, x_n) + O(\epsilon^2), \\ {}_aD_t^\alpha \bar{x}_2 &\mapsto {}_aD_t^\alpha x_2 + \epsilon\zeta_{x_2}^{(\alpha)}(t, x_1, x_2, \dots, x_n) + O(\epsilon^2), \\ &\vdots \\ {}_aD_t^\alpha \bar{x}_n &\mapsto {}_aD_t^\alpha x_n + \epsilon\zeta_{x_n}^{(\alpha)}(t, x_1, x_2, \dots, x_n) + O(\epsilon^2), \end{aligned} \tag{2.27}$$

where $\zeta_{x_i}^{(\alpha)}$, $i = 1, 2, \dots, n$ are infinitesimals that can be obtained in the following form

$$\zeta_{x_i}^{(\alpha)} = {}_aD_t^\alpha(\zeta_{x_i}) - \alpha D_t(\tau) {}_aD_t^\alpha(x_i) + \sum_{n=1}^{\infty} \binom{\alpha}{n} \left(\frac{n-\alpha}{n+1}\right) {}_aD_t^{\alpha-n}(x_i) D_t^{n+1}(\tau), \tag{2.28}$$

$i = 1, 2, \dots, n$. Making use of transformation (2.27) in (2.26) and equating the coefficients ϵ and omitting the terms of higher powers of ϵ , we obtain the invariant equations for (2.26) as follows

$$\left(\zeta_{x_i}^{(\alpha)} - \tau \frac{\partial f_i}{\partial t} - \zeta_{x_1} \frac{\partial f_i}{\partial x_1} - \dots - \zeta_{x_n} \frac{\partial f_i}{\partial x_n} \right) \Big|_{\substack{d^\alpha x_1 = f_1, \\ d^\alpha x_2 = f_2, \dots, \\ d^\alpha x_n = f_n}} = 0, \quad i = 1, 2, \dots, n.$$

Therefore, the infinitesimal generator X reads

$$\begin{aligned} X &= \tau(t, x_1, x_2, \dots, x_n) \frac{\partial}{\partial t} + \zeta_{x_1}(t, x_1, x_2, \dots, x_n) \frac{\partial}{\partial x_1} \\ &\quad + \zeta_{x_2}(t, x_1, x_2, \dots, x_n) \frac{\partial}{\partial x_2} + \dots + \zeta_{x_n}(t, x_1, x_2, \dots, x_n) \frac{\partial}{\partial x_n}, \end{aligned} \tag{2.29}$$

where

$$\begin{aligned} \tau(t, x_1, x_2, \dots, x_n) &= \left. \frac{d\bar{t}}{d\epsilon} \right|_{\epsilon=0}, \\ \zeta_{x_i}(t, x_1, x_2, \dots, x_n) &= \left. \frac{d\bar{x}_i}{d\epsilon} \right|_{\epsilon=0}, \quad i = 1, 2, \dots, n. \end{aligned}$$

Therefore, the obtained prolongation formula for (2.26) is as follows

$$Pr^{(\alpha)}X(H) \Big|_{H=0} = 0, \quad \text{where } H = {}_aD_t^\alpha x_i - f_i, \quad i = 1, 2, \dots, n,$$

which can be written as

$$\zeta_{x_i}^{(\alpha)} - Xf_i = 0, \quad i = 1, 2, \dots, n,$$

where the infinitesimals $\zeta_{x_i}^{(\alpha)}$ are given in (2.28).

3. SYMMETRIES AND EXACT SOLUTIONS FOR SCALAR AND COUPLED FODEs

3.1. Fractional Thomas-Fermi Equation

Consider the fractional Thomas-Fermi equation [12,17] of the form

$${}_0D_t^{\alpha+1}x = t^{-\frac{\alpha}{2}}x^{\frac{3}{2}}, \quad \alpha \in (0, 1), \quad t > 0. \tag{3.1}$$

Note that, when $\alpha = 1$, the above equation is the well-known classical Thomas-Fermi equation that describes the charge distribution of a neutral atom which is a function of radius t . The analytical and numerical solutions of the classical Thomas-Fermi equation were discussed in [1,72].

3.1.1. Lie point symmetries

We assume that the Thomas-Fermi equation (3.1) is invariant under one-parameter Lie group of infinitesimal point transformations given in (2.6). Hence, we obtain the following invariant equation

$$\zeta^{(\alpha+1)} - \frac{3}{2}x^{\frac{1}{2}}t^{-\frac{\alpha}{2}}\zeta(t, x) + \frac{\alpha}{2}x^{\frac{3}{2}}t^{-\frac{\alpha}{2}-1}\tau(t, x) = 0. \tag{3.2}$$

Substituting the expression for the infinitesimal $\zeta^{(\alpha+1)}$ given in (2.8) into the above equation (3.2), we obtain

$$\begin{aligned} &{}_0D_t^{\alpha+1}(\zeta) + \sum_{n=0}^{\infty} \binom{\alpha+1}{n} \left(\frac{n-\alpha-1}{n+1} \right) {}_0D_t^{\alpha+1-n}(x) D_t^{n+1}(\tau) \\ &- \frac{3}{2}x^{\frac{1}{2}}t^{-\frac{\alpha}{2}}\zeta(t, x) + \frac{\alpha}{2}x^{\frac{3}{2}}t^{-\frac{\alpha}{2}-1}\tau(t, x) = 0. \end{aligned} \tag{3.3}$$

The above equation (3.3) is not solvable for the arbitrary infinitesimals $\tau(t, x)$ and $\zeta(t, x)$ in general. In order to find the infinitesimals in equation (3.3), we assume

$$\tau = \tau(t), \quad \zeta = a_1(t)x + a_2(t),$$

where $a_1(t)$ and $a_2(t)$ are the unknown functions to be determined. Using the above expressions in (3.3), we obtain

$$\begin{aligned} &a_1(t) {}_0D_t^{\alpha+1}x + {}_0D_t^{\alpha+1}a_2(t) - (\alpha+1)t^{-\frac{\alpha}{2}}x^{\frac{3}{2}}D_t(\tau) - \frac{3}{2}x^{\frac{1}{2}}t^{-\frac{\alpha}{2}}(a_1(t)x + a_2(t)) \\ &+ \frac{\alpha}{2}x^{\frac{3}{2}}t^{-\frac{\alpha}{2}-1}\tau + \sum_{n=1}^{\infty} \binom{\alpha+1}{n} \left[\frac{d^n a_1(t)}{dt^n} + \left(\frac{n-\alpha-1}{n+1} \right) D_t^{n+1}(\tau) \right] {}_0D_t^{\alpha+1-n}(x) = 0. \end{aligned} \tag{3.4}$$

Making use of fractional Thomas-Fermi equation (3.1) in (3.4), we obtain

$$\begin{aligned} &\sum_{n=1}^{\infty} \binom{\alpha+1}{n} \left[\frac{d^n a_1(t)}{dt^n} + \left(\frac{n-\alpha-1}{n+1} \right) D_t^{n+1}(\tau) \right] {}_0D_t^{\alpha+1-n}(x) \\ &+ \left[-\frac{1}{2}t^{-\frac{\alpha}{2}}a_1(t) - (\alpha+1)t^{-\frac{\alpha}{2}}D_t(\tau) + \frac{\alpha}{2}t^{-\frac{\alpha}{2}-1}\tau \right] x^{\frac{3}{2}} - \frac{3}{2}t^{-\frac{\alpha}{2}}x^{\frac{1}{2}}a_2(t) + {}_0D_t^{\alpha+1}a_2(t) = 0 \end{aligned} \tag{3.5}$$

which reduces to the following overdetermined system of equations

$$\frac{d^n a_1(t)}{dt^n} + \left(\frac{n-\alpha-1}{n+1} \right) D_t^{n+1}(\tau) = 0, \tag{3.6}$$

$$-\frac{1}{2}a_1(t)t^{-\frac{\alpha}{2}} - (\alpha+1)t^{-\frac{\alpha}{2}}D_t(\tau) + \frac{\alpha}{2}t^{-\frac{\alpha}{2}-1}\tau = 0, \tag{3.7}$$

$$a_2(t) = 0. \tag{3.8}$$

Solving the above equations (3.6)-(3.8) subject to $\tau(0) = 0$, we obtain the explicit form of infinitesimals as

$$\tau(t) = c_1 t, \quad \zeta(t, x) = -c_1(\alpha+2)x, \quad c_1 \in \mathbb{R}.$$

Thus, the fractional Thomas-Fermi equation (3.1) is invariant under the following one-parameter Lie group transformation

$$\begin{aligned} \bar{t} &= t + \epsilon c_1 t, \\ \bar{x} &= x - \epsilon c_1(\alpha+2)x \end{aligned}$$

with the infinitesimal operator $X = c_1 X_1$, where $X_1 = t \frac{\partial}{\partial t} - (\alpha+2) \frac{\partial}{\partial x}$.

3.1.2. Construction of exact solution

In this subsection, we explain how to construct an exact solution for equation (3.1). The obtained infinitesimals of (3.1) are

$$\tau(t) = c_1 t, \quad \zeta(t, x) = -c_1(\alpha + 2)x.$$

Let us assume that $F(x, t)$ is an invariant of fractional Thomas-Fermi equation (3.1), that is

$$X_1 F = t \frac{\partial F}{\partial t} - (\alpha + 2)x \frac{\partial F}{\partial x} = 0,$$

which gives the characteristic equation as

$$\frac{dt}{t} = \frac{dx}{-(\alpha + 2)x}.$$

On solving the above equation, we obtain the invariant function

$$F(x, t) = k = xt^{(\alpha+2)}, \quad k \in \mathbb{R}.$$

Hence this yields an invariant solution as

$$x(t) = kt^{-(\alpha+2)}. \tag{3.9}$$

Substituting the above invariant solution in equation (3.1), we have $k = \left(\frac{\Gamma(-(\alpha+1))}{\Gamma(-2(\alpha+1))}\right)^2$. Hence, we obtain an exact solution of Thomas-Fermi equation (3.1) as

$$x(t) = \left(\frac{\Gamma(-(\alpha + 1))}{\Gamma(-2(\alpha + 1))}\right)^2 t^{-(\alpha+2)}, \quad \alpha \in (0, 1). \tag{3.10}$$

The gamma coefficient $\left(\frac{\Gamma(-(\alpha+1))}{\Gamma(-2(\alpha+1))}\right)$ in the above equation (3.30) becomes zero when $\alpha = \frac{1}{2}$ and hence the equation (3.1) has a trivial solution $x(t) = 0$. The exact solution (3.30) of fractional Thomas-Fermi equation (3.1) for different values of α is shown in Figure 1.

3.2. Bagley-Torvik Equation

Let us consider the Bagley-Torvik equation [3]

$$A\ddot{x}(t) + B {}_0D_t^{\frac{3}{2}}x(t) + Cx(t) = f, \quad t > 0, \tag{3.11}$$

where $A \neq 0, B, C$ are arbitrary constants and f is a function of t . It describes the motion of a rigid plate immersed in a Newtonian fluid [3]. The analytical and approximate solutions of the Bagley-Torvik equation were discussed through various methods [14,50,66].

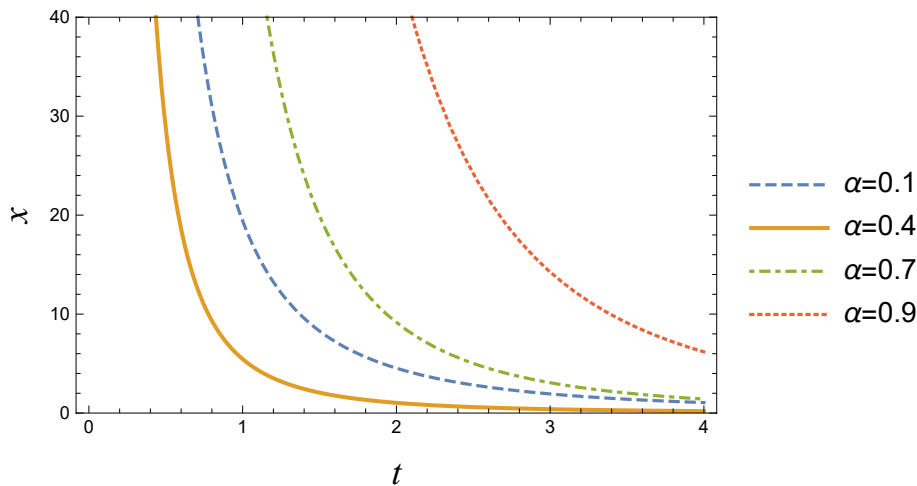


Figure 1 | Graphical representation of solutions of fractional Thomas-Fermi equation (3.1) for various values of α .

3.2.1. Lie point symmetries

Here, we assume that the Bagley-Torvik equation (3.11) is invariant under the one-parameter Lie group of continuous point transformations (2.6). Thus, the obtained invariant equation is of the form

$$A\zeta^{(2)} + B\zeta^{(\frac{3}{2})} + C\zeta = \tau f_t. \tag{3.12}$$

Substituting $\alpha = \frac{3}{2}$ in (2.8), we obtain the expression for $\zeta^{(\frac{3}{2})}$ which in turn can be used in the above equation (3.12) to obtain

$$A \left[\zeta_{tt} + (2\zeta_{tx} - \tau_{tt}) \dot{x} + (\zeta_{xx} - 2\tau_{tx}) \dot{x}^2 - \tau_{xx} \dot{x}^3 + (\zeta_x - 2\tau_t - 3\dot{x}\tau_x) \ddot{x} \right] + B \left[{}_0D_t^{(\frac{3}{2})}(\zeta) + \sum_{n=0}^{\infty} \binom{\frac{3}{2}}{n} \left(\frac{n - \frac{3}{2}}{n+1} \right) D_t^{n+1} \tau {}_0D_t^{\frac{3}{2}-n}(x) \right] - \tau f_t + C\zeta = 0. \tag{3.13}$$

The above equation (3.13) is unsolvable for the arbitrary infinitesimals $\tau(t, x)$ and $\zeta(t, x)$. In order to find the infinitesimals in equation (3.13), we assume

$$\tau = \tau(t), \quad \zeta = a_1(t)x + a_2(t),$$

where $a_1(t)$ and $a_2(t)$ are the unknown arbitrary functions to be determined. Substituting the above expressions along with

$${}_0D_t^{\frac{3}{2}}(a_1(t)x) = \sum_{n=0}^{\infty} \binom{\frac{3}{2}}{n} \frac{d^n a_1(t)}{dt^n} {}_0D_t^{\frac{3}{2}-n} x$$

and substituting equation (3.11) in (3.13) and then rearranging term by term, we have

$$\begin{aligned} &-\frac{1}{2}A\tau_t\ddot{x}(t) + \left(2A\frac{da_1}{dt} - A\tau_{tt} \right) \dot{x}(t) + \left(A\frac{d^2a_1}{dt^2} + C\tau_t \right) x(t) \\ &+ B \sum_{n=1}^{\infty} \binom{\frac{3}{2}}{n} \left[\frac{d^n a_1}{dt^n} + \left(\frac{n - \frac{3}{2}}{n+1} \right) D_t^{n+1}(\tau) \right] {}_0D_t^{\frac{3}{2}-n} x + Ca_2(t) - \tau f_t \\ &+ A\frac{d^2a_2}{dt^2} + \left(a_1(t) - \frac{3}{2}\tau_t \right) f(t) + B {}_0D_t^{\frac{3}{2}} a_2(t) = 0 \end{aligned}$$

which reduces to the following overdetermined system

$$Ca_2(t) - \tau f_t + A\frac{d^2a_2}{dt^2} + \left(a_1(t) - \frac{3}{2}\tau_t \right) f(t) + B {}_0D_t^{\frac{3}{2}} a_2(t) = 0, \tag{3.14}$$

$$\frac{d^n}{dt^n} a_1(t) + \left(\frac{n - \frac{3}{2}}{n+1} \right) D_t^{n+1} \tau = 0, \tag{3.15}$$

$$A\frac{d^2a_1}{dt^2} + \frac{3}{2}C\tau_t = 0, \tag{3.16}$$

$$2A\frac{da_1}{dt} - A\tau_{tt} = 0, \tag{3.17}$$

$$A\tau_t = 0. \tag{3.18}$$

On solving the last three equations with $A \neq 0$ and using $\tau(0) = 0$, we obtain

$$\tau(t) = 0, \quad a_1(t) = k_1, \quad \text{and} \quad \zeta(t, x) = k_1x + a_2(t), \quad k_1, k_2 \in \mathbb{R}. \tag{3.19}$$

Substituting (3.19) in (3.14), we obtain

$$A\ddot{a}_2(t) + B {}_0D_t^{\frac{3}{2}} a_2(t) + k_1f(t) + Ca_2(t) = 0. \tag{3.20}$$

Making use of Laplace transformation on both sides of the equation (3.20) and regrouping the terms, we obtain the following equation

$$\begin{aligned} \bar{a}_2(s) = & \frac{\left(\frac{A}{B}\right) a_2(0)s}{s^{\frac{3}{2}} + \left(\frac{A}{B}\right) s^2 + \left(\frac{C}{B}\right)} + \frac{\left(\frac{A}{B}\right) \dot{a}_2(0)}{s^{\frac{3}{2}} + \left(\frac{A}{B}\right) s^2 + \left(\frac{C}{B}\right)} + \frac{D^{\frac{1}{2}} a_2(0)}{s^{\frac{3}{2}} + \left(\frac{A}{B}\right) s^2 + \left(\frac{C}{B}\right)} \\ & + \frac{sD^{\frac{1}{2}} a_2(0)}{s^{\frac{3}{2}} + \left(\frac{A}{B}\right) s^2 + \left(\frac{C}{B}\right)} - \frac{\left(\frac{1}{B}\right) k_1 \bar{f}(s)}{s^{\frac{3}{2}} + \left(\frac{A}{B}\right) s^2 + \left(\frac{C}{B}\right)} \end{aligned} \tag{3.21}$$

whose inverse Laplace transformation along convolution is

$$\begin{aligned}
 a_2(t) = & k_2 \left(\frac{A}{B}\right) t^{-\frac{1}{2}} \sum_{r=0}^{\infty} \left(-\frac{A}{B}\right)^r t^{-\frac{1}{2}r} \mathbf{E}_{\frac{3}{2}, \frac{1}{2}(1-r)}^{r+1} \left(-\frac{C}{B} t^{\frac{3}{2}}\right) \\
 & + k_3 \left(\frac{A}{B}\right) t^{\frac{1}{2}} \sum_{r=0}^{\infty} \left(-\frac{A}{B}\right)^r t^{-\frac{1}{2}r} \mathbf{E}_{\frac{3}{2}, \frac{1}{2}(3-r)}^{r+1} \left(-\frac{C}{B} t^{\frac{3}{2}}\right) \\
 & + k_4 t^{\frac{1}{2}} \sum_{r=0}^{\infty} \left(-\frac{A}{B}\right)^r t^{-\frac{1}{2}r} \mathbf{E}_{\frac{3}{2}, \frac{1}{2}(3-r)}^{r+1} \left(-\frac{C}{B} t^{\frac{3}{2}}\right) \\
 & + k_5 t^{-\frac{1}{2}} \sum_{r=0}^{\infty} \left(-\frac{A}{B}\right)^r t^{-\frac{1}{2}r} \mathbf{E}_{\frac{3}{2}, \frac{1}{2}(1-r)}^{r+1} \left(-\frac{C}{B} t^{\frac{3}{2}}\right) \\
 & - k_1 \left(\frac{1}{B}\right) \int_0^t f(t-\xi) \xi^{\frac{1}{2}} \sum_{r=0}^{\infty} \left(-\frac{A}{B}\right)^r \xi^{-\frac{1}{2}r} \mathbf{E}_{\frac{3}{2}, \frac{1}{2}(1-r)}^{r+1} \left(-\frac{C}{B} \xi^{\frac{3}{2}}\right) d\xi,
 \end{aligned} \tag{3.22}$$

where the constants $k_2 = a_2(0)$, $k_3 = \dot{a}_2(0)$, $k_4 = D^{\frac{1}{2}} a_2(0)$ and $k_5 = D^{-\frac{1}{2}} a_2(0)$. Thus, the Bagley-Torvik equation (3.11) is invariant under

$$\begin{aligned}
 \bar{t} &= t, \\
 \bar{x} &= x + \epsilon (k_1 x + a_2(t)),
 \end{aligned}$$

where $a_2(t)$ is given in (3.22). The infinitesimal operator takes the following form

$$X = \sum_{i=1}^5 k_i X_i, \tag{3.23}$$

where

$$\begin{aligned}
 X_1 &= \left[x - \left(\frac{1}{B}\right) \int_0^t f(t-\xi) \xi^{\frac{1}{2}} \sum_{r=0}^{\infty} \left(-\frac{A}{B}\right)^r \xi^{-\frac{1}{2}r} \mathbf{E}_{\frac{3}{2}, \frac{1}{2}(1-r)}^{r+1} \left(-\frac{C}{B} \xi^{\frac{3}{2}}\right) d\xi \right] \frac{\partial}{\partial x}, \\
 X_2 &= \left[\left(\frac{A}{B}\right) t^{-\frac{1}{2}} \sum_{r=0}^{\infty} \left(-\frac{A}{B}\right)^r t^{-\frac{1}{2}r} \mathbf{E}_{\frac{3}{2}, \frac{1}{2}(1-r)}^{r+1} \left(-\frac{C}{B} t^{\frac{3}{2}}\right) \right] \frac{\partial}{\partial x}, \\
 X_3 &= \left[\left(\frac{A}{B}\right) t^{\frac{1}{2}} \sum_{r=0}^{\infty} \left(-\frac{A}{B}\right)^r t^{-\frac{1}{2}r} \mathbf{E}_{\frac{3}{2}, \frac{1}{2}(3-r)}^{r+1} \left(-\frac{C}{B} t^{\frac{3}{2}}\right) \right] \frac{\partial}{\partial x}, \\
 X_4 &= \left[t^{\frac{1}{2}} \sum_{r=0}^{\infty} \left(-\frac{A}{B}\right)^r t^{-\frac{1}{2}r} \mathbf{E}_{\frac{3}{2}, \frac{1}{2}(3-r)}^{r+1} \left(-\frac{C}{B} t^{\frac{3}{2}}\right) \right] \frac{\partial}{\partial x}, \\
 X_5 &= \left[t^{-\frac{1}{2}} \sum_{r=0}^{\infty} \left(-\frac{A}{B}\right)^r t^{-\frac{1}{2}r} \mathbf{E}_{\frac{3}{2}, \frac{1}{2}(1-r)}^{r+1} \left(-\frac{C}{B} t^{\frac{3}{2}}\right) \right] \frac{\partial}{\partial x}.
 \end{aligned}$$

Note that when $A \neq 0$, $B \neq 0$, $C \neq 0$ and $f(t) \neq 0$, the infinitesimal generator cannot be written as a linear combination of the space coordinate and the time coordinate. For this reason, the invariant and exact solutions of (3.11) cannot be obtained.

3.2.2. One-dimensional optimal system

In this subsection, we construct a one-dimensional optimal system for the symmetry algebra generated by X_1, X_2, X_3, X_4 and X_5 . In order to obtain the optimal system, we follow the algorithm provided by Olver [48] and Ibragimov [28]. The commutator table for the symmetry generators $X_i, i = 1, 2, \dots, 5$ is given in Table 1.

Table 1 | Commutator table for infinitesimal generators of equation (3.11)

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5
X_1	0	$-X_2$	$-X_3$	$-X_4$	$-X_5$
X_2	X_2	0	0	0	0
X_3	X_3	0	0	0	0
X_4	X_4	0	0	0	0
X_5	X_5	0	0	0	0

Table 2 | Adjoint representation for infinitesimal generators of equation (3.11)

$\text{Ad}(\exp(\epsilon X_i)) X_j$	X_1	X_2	X_3	X_4	X_5
X_1	X_1	$e^\epsilon X_2$	$e^\epsilon X_3$	$e^\epsilon X_4$	$e^\epsilon X_5$
X_2	$X_1 - \epsilon X_2$	X_2	X_3	X_4	X_5
X_3	$X_1 - \epsilon X_3$	X_2	X_3	X_4	X_5
X_4	$X_1 - \epsilon X_4$	X_2	X_3	X_4	X_5
X_5	$X_1 - \epsilon X_5$	X_2	X_3	X_4	X_5

From the table we observe that the symmetry generators of Bagley-Torvik equation (3.11) form a closed Lie algebra \mathcal{L} of dimension five. Now we determine optimal system of one-dimensional subalgebras for (3.11). The adjoint representation of the infinitesimal generators is obtained using the following series

$$\text{Ad}(\exp(\epsilon X_i)) X_j = X_j - \epsilon [X_i, X_j] + \frac{1}{2!} \epsilon^2 [X_i, [X_i, X_j]] - \dots \tag{3.24}$$

Hence we obtain the adjoint representation of form given in Table 2.

Theorem 3.1. The one-dimensional optimal system of Lie algebra \mathcal{L} for the Bagley-Torvik equation (3.11) is X_1 .

Proof. Define the maps $\mathcal{F}_i^\epsilon : \mathcal{L} \rightarrow \mathcal{L}$ as $\mathcal{F}_i^\epsilon(X) = \text{Ad}(\exp(\epsilon X_i)) X$, for $X \in \mathcal{L}$ and $i = 1, 2, \dots, 5$. Let $X \in \mathcal{L}$ be arbitrary. Then $X = \sum_{i=1}^5 a_i X_i$. The composition of maps on X takes the form

$$\mathcal{F}_1^\epsilon \circ \mathcal{F}_2^\epsilon \circ \mathcal{F}_3^\epsilon \circ \mathcal{F}_4^\epsilon \circ \mathcal{F}_5^\epsilon(X) = a_1 X_1 + \sum_{i=2}^5 (a_i - a_1 \epsilon) e^\epsilon X_i \tag{3.25}$$

If $a_1 \neq 0$, then by taking $\epsilon = \frac{a_i}{a_1}$ for each $i = 1, 2, \dots, 5$, the arbitrary vector X is the scaling of X_1 . This completes the proof. □

3.2.3. Special cases

The special cases for Bagley-Torvik equation (3.11) are discussed below.

Case 1: Let $B = 0$ and $f(t) = 0, t > 0$. Then the equation (3.11) reads

$$A\ddot{x}(t) + Cx(t) = 0 \tag{3.26}$$

which can be written as

$$\ddot{x}(t) + kx(t) = 0, \tag{3.27}$$

where $k = \frac{C}{A}$. For $k > 1$, the above equation (3.27) models the motion of a simple harmonic oscillator [67]. Let (3.27) be invariant under the Lie group of transformation (2.6). Then the invariant equation is

$$\zeta_{tt} + (2\zeta_{tx} - \tau_{tt} + 3k\tau_x x(t))\dot{x} + (\zeta_{xx} - 2\tau_{tx})\dot{x}^2 - \tau_{xx}\dot{x}^3 - k\zeta_x x(t) + 2k\tau_t x(t) + k\zeta = 0. \tag{3.28}$$

Proceeding in the above similar manner, the above equation (3.26) is invariant under

$$\begin{aligned} \bar{t} &= t + \epsilon \tau(t, x), \\ \bar{x} &= x + \epsilon \zeta(t, x), \end{aligned}$$

where

$$\begin{aligned} \tau(t, x) &= \left[k_3 \cos(\sqrt{k}t) + k_4 \sin(\sqrt{k}t) \right] x + \frac{k_5 \sin(2\sqrt{k}t)}{2\sqrt{k}} - \frac{k_6 \cos(2\sqrt{k}t)}{2\sqrt{k}} + k_7, \\ \zeta(t, x) &= \left[-k_3 \sqrt{k} \sin(\sqrt{k}t) + k_4 \sqrt{k} \cos(\sqrt{k}t) \right] x^2 + \frac{1}{2} \left[k_5 \cos(2\sqrt{k}t) + k_6 \sin(2\sqrt{k}t) \right] x \\ &\quad + k_8 x + k_1 \cos(\sqrt{k}t) + k_2 \sin(\sqrt{k}t), \end{aligned}$$

$k_i, i = 1, 2, 3, \dots, 8$ are constants of integration and $k = \frac{C}{A}, A \neq 0$. Hence the infinitesimal operator reads

$$X = \sum_{i=1}^8 k_i X_i,$$

Table 3 | Commutator table for infinitesimal generators of equation (3.27)

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_1	0	0	$\frac{1}{2}X_7 - \sqrt{k}X_6$	$\sqrt{k}(X_5 + \frac{3}{2}X_8)$	$\frac{1}{2}X_1$	$\frac{1}{2}X_2$	$\sqrt{k}X_2$	X_1
X_2	0	0	$\sqrt{k}X_5 - \frac{3}{2}X_8$	$\frac{1}{2}X_7 + \sqrt{k}X_6$	$-\frac{1}{2}X_2$	$\frac{1}{2}X_1$	$-\sqrt{k}X_1$	X_2
X_3	$\sqrt{k}X_6 - \frac{1}{2}X_7$	$\frac{3}{2}X_8 - \sqrt{k}X_5$	0	0	$\frac{1}{2}X_3$	$\frac{1}{2}X_4$	$\sqrt{k}X_4$	$-X_3$
X_4	$-\sqrt{k}(X_5 + \frac{3}{2}X_8)$	$-\frac{1}{2}X_7 + \sqrt{k}X_6$	0	0	$-\frac{1}{2}X_4$	$\frac{1}{2}X_3$	$-\sqrt{k}X_3$	$-X_4$
X_5	$-\frac{1}{2}X_1$	$\frac{1}{2}X_2$	$-\frac{1}{2}X_3$	$\frac{1}{2}X_4$	0	$\frac{1}{2\sqrt{k}}X_7$	$2\sqrt{k}X_6$	0
X_6	$-\frac{1}{2}X_2$	$-\frac{1}{2}X_1$	$-\frac{1}{2}X_4$	$-\frac{1}{2}X_3$	$-\frac{1}{2\sqrt{k}}X_7$	0	$-2\sqrt{k}X_5$	0
X_7	$-\sqrt{k}X_2$	$\sqrt{k}X_1$	$-\sqrt{k}X_4$	$\sqrt{k}X_3$	$-2\sqrt{k}X_6$	$2\sqrt{k}X_5$	0	0
X_8	$-X_1$	$-X_2$	X_3	X_4	0	0	0	0

where

$$\begin{aligned}
 X_1 &= \cos(\sqrt{kt}) \frac{\partial}{\partial x}, \\
 X_2 &= \sin(\sqrt{kt}) \frac{\partial}{\partial x}, \\
 X_3 &= x \cos(\sqrt{kt}) \frac{\partial}{\partial t} - x^2 \sqrt{k} \sin(\sqrt{kt}) \frac{\partial}{\partial x}, \\
 X_4 &= x \sin(\sqrt{kt}) \frac{\partial}{\partial t} + x^2 \sqrt{k} \cos(\sqrt{kt}) \frac{\partial}{\partial x}, \\
 X_5 &= \frac{\sin(2\sqrt{kt})}{2\sqrt{k}} \frac{\partial}{\partial t} + \frac{1}{2}x \cos(2\sqrt{kt}) \frac{\partial}{\partial x}, \\
 X_6 &= -\frac{\cos(2\sqrt{kt})}{2\sqrt{k}} \frac{\partial}{\partial t} + \frac{1}{2}x \sin(2\sqrt{kt}) \frac{\partial}{\partial x}, \\
 X_7 &= \frac{\partial}{\partial t} \text{ and } X_8 = x \frac{\partial}{\partial x}.
 \end{aligned}$$

The commutator table for the infinitesimal generators X_i , for $i = 1, 2, \dots, 8$, is given in Table 3. By taking $k = 1$, the linear combinations $X_2 - X_3, X_2 + X_3, X_1 + X_4, X_1 - X_4, X_7, X_8, 2X_5$ and $-2X_6$ coincide with the operators obtained for one-dimensional harmonic oscillator equation in [67]. Also, as discussed in [67] the combinations $X_2 - X_3, X_1 + X_4$ and X_7 form a compact subalgebra. The Lie symmetries of the second order ODEs are discussed in [2,8,27,49,67].

Construction of exact solution

Consider the infinitesimal generator $X = X_7 + X_1 + X_2$. Let us assume that $F(x, t)$ is an invariant of equation (3.27), that is

$$XF = \frac{\partial F}{\partial t} + [\cos(\sqrt{kt}) + \sin(\sqrt{kt})] \frac{\partial F}{\partial x} = 0,$$

which gives the characteristic equation as

$$\frac{dt}{1} = \frac{dx}{\cos(\sqrt{kt}) + \sin(\sqrt{kt})}.$$

On solving the above equation, we obtain the invariant function

$$F(x, t) = c_1 = x - \frac{1}{\sqrt{k}} (\sin(\sqrt{kt}) - \cos(\sqrt{kt})), \quad c_1 \in \mathbb{R}.$$

Hence this yields an invariant solution as

$$x(t) = \frac{1}{\sqrt{k}} (\sin(\sqrt{kt}) - \cos(\sqrt{kt})) + c_1. \tag{3.29}$$

Substituting the above invariant solution (3.29) in equation (3.27), we have $c_1 = 0$. Hence, we obtain an exact solution of equation (3.27) as

$$x(t) = \frac{1}{\sqrt{k}} (\sin(\sqrt{kt}) - \cos(\sqrt{kt})). \tag{3.30}$$

Case 2: For $C = 0$ and $f(t) = 0, t > 0$, the equation (3.11) takes the form

$$A\ddot{x}(t) + B_0 D_t^{\frac{3}{2}} x(t) = 0. \tag{3.31}$$

The above equation (3.31) is invariant under

$$\begin{aligned} \bar{t} &= t, \\ \bar{x} &= x + \epsilon(k_1x + a_2(t)), \end{aligned}$$

where

$$a_2(t) = k_2 \mathbf{E}_{\frac{1}{2},1} \left(-\frac{B}{A} t^{\frac{1}{2}} \right) + k_3 t \mathbf{E}_{\frac{1}{2},2} \left(-\frac{B}{A} t^{\frac{1}{2}} \right) + \left(\frac{B}{A} \right) k_4 t \mathbf{E}_{\frac{1}{2},2} \left(-\frac{B}{A} t^{\frac{1}{2}} \right) + \left(\frac{B}{A} \right) k_5 t \mathbf{E}_{\frac{1}{2},1} \left(-\frac{B}{A} t^{\frac{1}{2}} \right),$$

$k_2 = a_2(0)$, $k_3 = \dot{a}_2(0)$, $k_4 = D^{\frac{1}{2}} a_2(0)$ and $k_5 = D^{-\frac{1}{2}} a_2(0)$. Hence the infinitesimal operator reads

$$X = \sum_{i=1}^5 k_i X_i,$$

where

$$\begin{aligned} X_1 &= x \frac{\partial}{\partial x}, \\ X_2 &= \mathbf{E}_{\frac{1}{2},1} \left(-\frac{B}{A} t^{\frac{1}{2}} \right) \frac{\partial}{\partial x}, \\ X_3 &= t \mathbf{E}_{\frac{1}{2},2} \left(-\frac{B}{A} t^{\frac{1}{2}} \right) \frac{\partial}{\partial x}, \\ X_4 &= \left(\frac{B}{A} \right) t \mathbf{E}_{\frac{1}{2},2} \left(-\frac{B}{A} t^{\frac{1}{2}} \right) \frac{\partial}{\partial x} \text{ and} \\ X_5 &= \left(\frac{B}{A} \right) \mathbf{E}_{\frac{1}{2},1} \left(-\frac{B}{A} t^{\frac{1}{2}} \right) \frac{\partial}{\partial x}. \end{aligned}$$

The Lie brackets for the above case are $[X_1, X_j] = -X_j$ for $j = 2, 3, 4, 5$, $[X_i, X_j] = 0$ for $i, j = 2, 3, 4, 5 \dots$, and $[X_1, X_1] = 0$. In this case the commutator table and adjoint representation are same as that of Tables 1 and 2 respectively.

Note 7. Since the adjoint representation of the infinitesimal operators are same as that of Bagley-Torvik equation (3.11), the one-dimensional optimal system of Lie algebra is X_1 .

Case 3: Let $C = 0$, $f(t) = \frac{3}{4} [At^{-\frac{1}{2}} + B\sqrt{\pi}]$. Then (3.11) becomes

$$A\ddot{x}(t) + B_0 D_t^{\frac{3}{2}} x(t) = \frac{3}{4} [At^{-\frac{1}{2}} + B\sqrt{\pi}] \tag{3.32}$$

which is invariant under

$$\begin{aligned} \bar{t} &= t, \\ \bar{x} &= x + \epsilon(k_1x + a_2(t)), \end{aligned}$$

where

$$\begin{aligned} a_2(t) &= k_2 \mathbf{E}_{\frac{1}{2},1} \left(-\frac{B}{A} t^{\frac{1}{2}} \right) + k_3 t \mathbf{E}_{\frac{1}{2},2} \left(-\frac{B}{A} t^{\frac{1}{2}} \right) + \left(\frac{B}{A} \right) k_4 t \mathbf{E}_{\frac{1}{2},2} \left(-\frac{B}{A} t^{\frac{1}{2}} \right) + \left(\frac{B}{A} \right) k_5 t \mathbf{E}_{\frac{1}{2},1} \left(-\frac{B}{A} t^{\frac{1}{2}} \right) \\ &+ \left(\frac{3}{4} \right) k_1 \sqrt{\pi} t^{\frac{3}{2}} \mathbf{E}_{\frac{1}{2},\frac{5}{2}} \left(-\frac{B}{A} t^{\frac{1}{2}} \right) + \left(\frac{3B}{4A} \right) k_1 \sqrt{\pi} t^2 \mathbf{E}_{\frac{1}{2},3} \left(-\frac{B}{A} t^{\frac{1}{2}} \right), \end{aligned}$$

where $k_2 = a_2(0)$, $k_3 = \dot{a}_2(0)$, $k_4 = D^{\frac{1}{2}} a_2(0)$ and $k_5 = D^{-\frac{1}{2}} a_2(0)$. Hence the infinitesimal operator reads

$$X = \sum_{i=1}^5 k_i X_i,$$

where

$$\begin{aligned} X_1 &= \left[x + \left(\frac{3}{4} \right) \sqrt{\pi} t^{\frac{3}{2}} \mathbf{E}_{\frac{1}{2},\frac{5}{2}} \left(-\frac{B}{A} t^{\frac{1}{2}} \right) + \left(\frac{3B}{4A} \right) \sqrt{\pi} t^2 \mathbf{E}_{\frac{1}{2},3} \left(-\frac{B}{A} t^{\frac{1}{2}} \right) \right] \frac{\partial}{\partial x}, \\ X_2 &= \mathbf{E}_{\frac{1}{2},1} \left(-\frac{B}{A} t^{\frac{1}{2}} \right) \frac{\partial}{\partial x}, \end{aligned}$$

$$\begin{aligned}
X_3 &= tE_{\frac{1}{2},2} \left(-\frac{B}{A} t^{-\frac{1}{2}} \right) \frac{\partial}{\partial x}, \\
X_4 &= \left(\frac{B}{A} \right) tE_{\frac{1}{2},2} \left(-\frac{B}{A} t^{-\frac{1}{2}} \right) \frac{\partial}{\partial x} \text{ and} \\
X_5 &= \left(\frac{B}{A} \right) E_{\frac{1}{2},1} \left(-\frac{B}{A} t^{-\frac{1}{2}} \right) \frac{\partial}{\partial x}.
\end{aligned}$$

The commutator table and the adjoint representation of the equation (3.32) take the form same as that of Tables 1 and 2 respectively and hence the one-dimensional optimal system of Lie algebra is X_1 .

Note 8. We would like to point out that in the above case 2 and case 3, the infinitesimal generator cannot be written as a linear combination of the space coordinate and the time coordinate. For this reason, the invariant and exact solutions of (3.31) and (3.32) cannot be obtained.

3.3. Two-Coupled System of Fractional Quartic Oscillator

Consider the two-coupled system of fractional quartic oscillator as follows

$$\begin{aligned}
{}_0D_t^{\alpha+1} x_1 &= 4k_3 x_1^3 + 2k_5 x_1 x_2^2, \\
{}_0D_t^{\alpha+1} x_2 &= 4k_4 x_2^3 + 2k_5 x_1^2 x_2, \quad \alpha \in (0, 1), \quad t > 0.
\end{aligned} \tag{3.33}$$

Note that, when $\alpha = 1$, $k_3 = 1$, $k_4 = 8$ and $k_5 = 6$, the above system (3.33) is known as classical quartic oscillator. The integrability of the classical system was discussed in [35]. It is noteworthy to mention that when $\alpha = 1$, $k_3 = -B$, $k_4 = -1$, $k_5 = -A$, the above system (3.33) becomes the equations of motion for quartic potential and its detailed discussion can be found in [33].

3.3.1. Lie point symmetries

If the system given by equation (3.33) is invariant under one-parameter Lie group of transformation as given in (2.27), then the invariant equations are as follows

$$\zeta_{x_1}^{(\alpha+1)} = 12k_3 x_1^2 \zeta_{x_1} + 2k_5 x_2^2 \zeta_{x_1} + 4k_5 x_1 x_2 \zeta_{x_2}, \tag{3.34}$$

$$\zeta_{x_2}^{(\alpha+1)} = 12k_4 x_2^2 \zeta_{x_2} + 2k_5 x_1^2 \zeta_{x_2} + 4k_5 x_1 x_2 \zeta_{x_1}. \tag{3.35}$$

The above invariant system (3.34)-(3.35) is not solvable for arbitrary infinitesimals $\tau(t, x)$ and $\zeta(t, x)$. Thus, we assume the infinitesimals of the form $\zeta_{x_1} = a(t)x_1 + b(t)x_2 + c(t)$ and $\zeta_{x_2} = p(t)x_1 + q(t)x_2 + r(t)$. Hence, the equation (3.34) can be written as

$$\begin{aligned}
\zeta_{x_1}^{(\alpha+1)} &= {}_aD_t^{\alpha+1}(\zeta_{x_1}) - (\alpha + 1)D_t(\tau) {}_aD_t^{\alpha+1}(x_1) \\
&\quad + \sum_{n=1}^{\infty} \binom{\alpha+1}{n} \binom{n-\alpha-1}{n+1} {}_aD_t^{\alpha+1-n}(x_1) D_t^{n+1}(\tau).
\end{aligned} \tag{3.36}$$

Substituting the expression for the infinitesimal $\zeta_{x_1}^{(\alpha+1)}$ in the above equation, we obtain

$$\begin{aligned}
&{}_0D_t^{\alpha+1}(a(t)x_1 + b(t)x_2 + c(t)) - (\alpha + 1)D_t(\tau) {}_0D_t^{\alpha+1}(a(t)x_1 + b(t)x_2 + c(t)) \\
&\quad + \sum_{n=1}^{\infty} \binom{\alpha+1}{n} \binom{n-\alpha-1}{n+1} {}_0D_t^{\alpha+1-n}(x_1) D_t^{n+1}(\tau) \\
&= 2k_1(a(t)x_1 + b(t)x_2 + c(t)) + 12k_3 x_1^2(a(t)x_1 + b(t)x_2 + c(t)) \\
&\quad + 2k_5 x_2^2(a(t)x_1 + b(t)x_2 + c(t)) + 4k_5 x_1 x_2(p(t)x_1 + q(t)x_2 + r(t)).
\end{aligned} \tag{3.37}$$

We recall that

$${}_0D_t^{\alpha+1} a(t)x_1 = \sum_{n=0}^{\infty} \binom{\alpha+1}{n} \frac{d^n}{dt^n} a(t) {}_0D_t^{\alpha+1-n} x_1, \tag{3.38}$$

$${}_0D_t^{\alpha+1} b(t)x_2 = \sum_{n=0}^{\infty} \binom{\alpha+1}{n} \frac{d^n}{dt^n} b(t) {}_0D_t^{\alpha+1-n} x_2. \tag{3.39}$$

Using equations (3.38) and (3.39) in equation (3.37), we obtain

$$\begin{aligned}
 & a(t) {}_0D_t^{\alpha+1} x_1 + \sum_{n=1}^{\infty} \binom{\alpha+1}{n} \frac{d^n}{dt^n} a(t) {}_0D_t^{\alpha+1-n} x_1 + b(t) {}_0D_t^{\alpha+1} x_2 \\
 & + \sum_{n=1}^{\infty} \binom{\alpha+1}{n} \frac{d^n}{dt^n} b(t) {}_0D_t^{\alpha+1-n} x_2 + {}_0D_t^{\alpha+1} c(t) - 2k_1(\alpha+1) D_t(\tau) x_1 \\
 & - 4k_3(\alpha+1) D_t(\tau) x_1^3 - 2k_5(\alpha+1) D_t(\tau) x_1 x_2^2 \\
 & + \sum_{n=1}^{\infty} \binom{\alpha+1}{n} \left(\frac{n-\alpha-1}{n+1} \right) {}_0D_t^{\alpha+1-n} (x_1) D_t^{n+1}(\tau) \\
 & = 2k_1 a(t) x_1 + 2k_1 b(t) x_2 + 2k_1 c(t) + 12k_3 a(t) x_1^3 \\
 & + 12k_3 b(t) x_1^2 x_2 + 12k_3 c(t) x_1^2 + 2k_5 a(t) x_2^2 x_1 + 2k_5 b(t) x_2^2 + 2k_5 c(t) x_2^2 \\
 & + 4k_5 p(t) x_1^2 x_2 + 4k_5 q(t) x_1 x_2^2 + 4k_5 r(t) x_1 x_2.
 \end{aligned} \tag{3.40}$$

Making use of fractional quartic oscillator equation (3.33) in the above equation (3.40), we obtain

$$\begin{aligned}
 & - [8k_3 a(t) + 4k_3(\alpha+1) D_t(\tau)] x_1^3 + [2k_2 b(t) - 2k_1 b(t)] x_2 \\
 & + [4k_4 b(t) - 2k_5 b(t)] x_2^3 + [2k_5 b(t) - 12k_3 b(t) - 4k_5 p(t)] x_1^2 x_2 \\
 & - [2k_1(\alpha+1) D_t(\tau) x_1 - [2k_5(\alpha+1) D_t(\tau) + 4k_5 q(t)] x_1 x_2^2 \\
 & - 12k_3 c(t) x_1^2 - 2k_5 c(t) x_2^2 - 4k_5 r(t) x_1 x_2 \\
 & + \sum_{n=1}^{\infty} \binom{\alpha+1}{n} \left[\frac{d^n}{dt^n} a(t) + \left(\frac{n-\alpha-1}{n+1} \right) D_t^{n+1}(\tau) \right] {}_0D_t^{\alpha+1-n} x_1 \\
 & + D_t^{\alpha+1} c(t) - 2k_1 c(t) + \sum_{n=1}^{\infty} \binom{\alpha+1}{n} \frac{d^n}{dt^n} b(t) {}_0D_t^{\alpha+1-n} x_2 = 0.
 \end{aligned} \tag{3.41}$$

In a similar manner, equation (3.35) can be reduced to

$$\begin{aligned}
 & [2k_1 p(t) - 2k_2 p(t)] x_1 + [4k_3 p(t) - 2k_5 p(t)] x_1^3 + [2k_5 p(t) - 12k_4 p(t) - 4k_5 b(t)] x_1 x_2^2 \\
 & + [4k_4 q(t) - 4k_4(\alpha+1) D_t(\tau) - 12k_4 q(t)] x_2^3 - 2k_2(\alpha+1) D_t(\tau) x_2 \\
 & - [2k_5(\alpha+1) D_t(\tau) + 4k_5 a(t)] x_1^2 x_1 - 12k_4 r(t) x_2^2 - 2k_5 r(t) x_1^2 + {}_0D_t^{\alpha+1} r(t) \\
 & - 2k_2 r(t) - 4k_5 c(t) x_1 x_2 + \sum_{n=1}^{\infty} \binom{\alpha+1}{n} \frac{d^n}{dt^n} p(t) {}_0D_t^{\alpha+1-n} x_1 \\
 & + \sum_{n=1}^{\infty} \binom{\alpha+1}{n} \left[\frac{d^n}{dt^n} q(t) + \left(\frac{n-\alpha-1}{n+1} \right) D_t^{n+1}(\tau) \right] {}_0D_t^{\alpha+1-n} x_2 = 0.
 \end{aligned} \tag{3.42}$$

Equations (3.41) and (3.42) reduces to the following overdetermined system of equations

$$\frac{d^n}{dt^n} a(t) + \left(\frac{n-\alpha-1}{n+1} \right) D_t^{n+1}(\tau) = 0, \quad n \in \mathbb{N}, \tag{3.43}$$

$$\frac{d^n}{dt^n} q(t) + \left(\frac{n-\alpha-1}{n+1} \right) D_t^{n+1}(\tau) = 0, \quad n \in \mathbb{N}, \tag{3.44}$$

$$k_5 b(t) - 6k_3 b(t) - 2k_5 p(t) = 0, \tag{3.45}$$

$$k_5 p(t) - 6k_4 p(t) - 2k_5 b(t) = 0, \tag{3.46}$$

$$k_5(\alpha+1) D_t(\tau) + 2k_5 q(t) = 0, \tag{3.47}$$

$$k_5(\alpha+1) D_t(\tau) + 2k_5 a(t) = 0, \tag{3.48}$$

$$2k_3 a(t) + k_3(\alpha+1) D_t(\tau) = 0, \tag{3.49}$$

$$2k_4 q(t) + k_4(\alpha+1) D_t(\tau) = 0, \tag{3.50}$$

$$b(t) = c(t) = 0, \tag{3.51}$$

$$p(t) = r(t) = 0. \tag{3.52}$$

Equation (3.43) subjected to the condition $\tau(0) = 0$ gives

$$\tau(t) = c_1 t, \quad a(t) = -\frac{(\alpha+1)c_1}{2},$$

where c_1 is an arbitrary constant. Hence we obtain infinitesimal

$$\zeta_{x_1} = -\frac{(\alpha + 1)c_1x_1}{2}.$$

Similarly, from equation (3.44) we obtain

$$\zeta_{x_2} = -\frac{(\alpha + 1)c_1x_2}{2}.$$

Thus, the two-coupled system of fractional quartic oscillator (3.33) is invariant under

$$\begin{aligned} \bar{t} &= t + \epsilon c_1 t, \\ \bar{x}_1 &= x_1 - \frac{\epsilon(\alpha + 1)c_1x_1}{2}, \\ \bar{x}_2 &= x_2 - \frac{\epsilon(\alpha + 1)c_1x_2}{2} \end{aligned}$$

with infinitesimal generator as

$$X = c_1 t \frac{\partial}{\partial t} - \frac{(\alpha + 1)c_1}{2} x_1 \frac{\partial}{\partial x_1} - \frac{(\alpha + 1)c_1}{2} x_2 \frac{\partial}{\partial x_2}. \tag{3.53}$$

3.3.2. Construction of exact solution

Let $F(t, x)$ be an invariant of equation (3.33). Then

$$XF = t \frac{\partial F}{\partial t} - \frac{(\alpha + 1)}{2} x_1 \frac{\partial F}{\partial x_1} - \frac{(\alpha + 1)}{2} x_2 \frac{\partial F}{\partial x_2}.$$

The characteristic equation of the above equation becomes

$$\frac{dt}{t} = \frac{dx_1}{-\frac{(\alpha + 1)}{2}x_1} = \frac{dx_2}{-\frac{(\alpha + 1)}{2}x_2}.$$

First we consider

$$\frac{dt}{t} = \frac{dx_1}{-\frac{(\alpha + 1)}{2}x_1},$$

which gives the invariant solution $x_1(t) = kt^{-\frac{(\alpha+1)}{2}}$, where k is the constant of integration and is given by

$$k = \pm \left[\frac{1}{4k_3} \left(\frac{4k_3k_4 - 2k_3k_5}{4k_3k_4 - k_5^2} \right) \frac{\Gamma\left(\frac{1}{2}(1 - \alpha)\right)}{\Gamma\left(-\frac{1}{2}(1 + 3\alpha)\right)} \right]^{\frac{1}{2}}.$$

Hence, we obtain the exact solution of the coupled system of fractional quartic oscillator equations (3.33) as

$$x_1(t) = \pm \left[\frac{1}{4k_3} \left(\frac{4k_3k_4 - 2k_3k_5}{4k_3k_4 - k_5^2} \right) \frac{\Gamma\left(\frac{1}{2}(1 - \alpha)\right)}{\Gamma\left(-\frac{1}{2}(1 + 3\alpha)\right)} \right]^{\frac{1}{2}} t^{-\frac{(\alpha+1)}{2}}$$

and by proceeding in the similar manner, we obtain

$$x_2(t) = \pm \left[\left(\frac{2k_3 - k_5}{8k_4k_3 - 2k_5^2} \right) \frac{\Gamma\left(\frac{1}{2}(1 - \alpha)\right)}{\Gamma\left(-\frac{1}{2}(1 + 3\alpha)\right)} \right]^{\frac{1}{2}} t^{-\frac{(\alpha+1)}{2}}, \quad k_3, k_4, k_5 \in \mathbb{R}, \quad \alpha \in (0, 1).$$

For $\alpha = \frac{1}{3}$, the system (3.33) has a trivial solution $x_i = 0$ for $i = 1, 2$ because of singularity of Gamma function at nonpositive integers. The graphical representations of solutions of fractional system of two-coupled quartic oscillator equation (3.33) for various values of α with $k_3 = 1, k_4 = 8$ and $k_5 = -5$ are shown in Figure 2.

3.4. Fractional Type Coupled Equation of Motion

Consider the following fractional type coupled equation of motion having the form

$$\begin{aligned} {}_0D_t^{\alpha+1}x_1 &= -2Bx_1x_2 - 3Cx_1^2, \\ {}_0D_t^{\alpha+1}x_2 &= -3x_2^2 - Bx_1^2, \quad \alpha \in (0, 1), \end{aligned} \tag{3.54}$$

where B, C are constants. Note that, when $\alpha = 1$, the system (3.54) is referred to as equations of motion for a cubic potential and its integrability is discussed in [33].

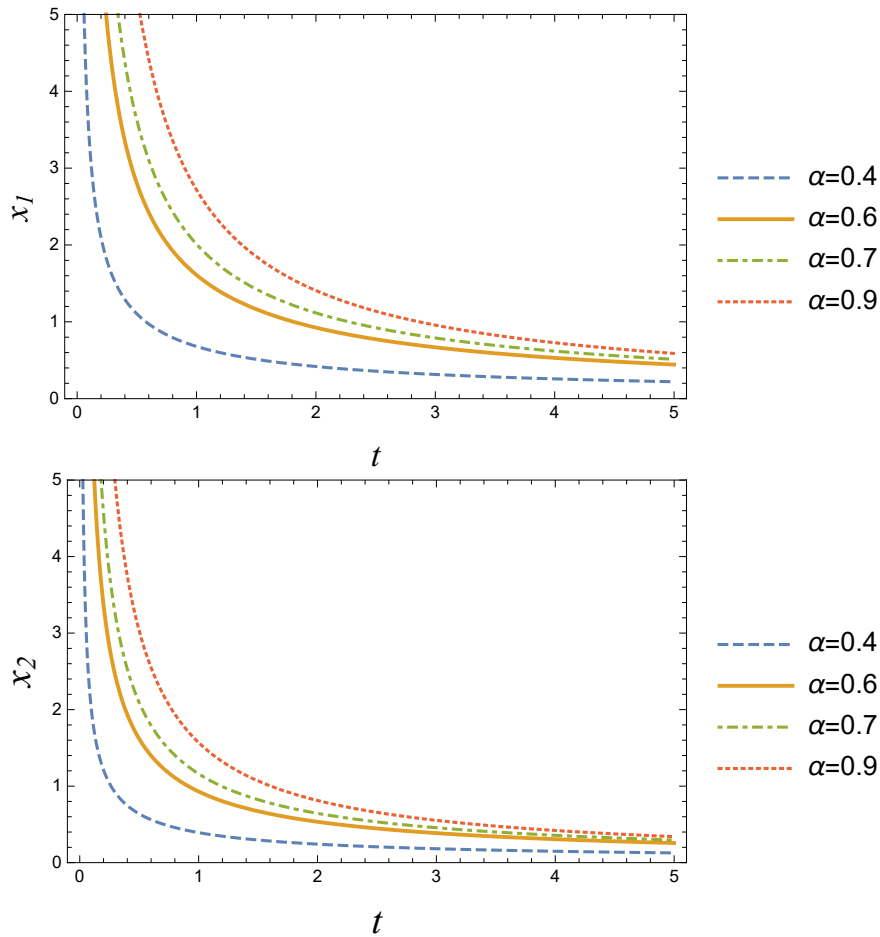


Figure 2 | Graphical representations of solutions of fractional system of two-coupled quartic oscillator equation (3.33) for various values of α .

3.4.1. Lie point symmetries

Assume that the fractional type coupled equation of motion (3.54) is invariant under one-parameter Lie group of transformation (2.27). Then, we obtain the invariant system as

$$\begin{aligned} \zeta_{x_1}^{(\alpha+1)} &= -2Bx_1\zeta_{x_2} - 2Bx_2\zeta_{x_1} - 6Cx_1\zeta_{x_1}, \\ \zeta_{x_2}^{(\alpha+1)} &= -6x_2\zeta_{x_2} - 2Bx_1\zeta_{x_1}. \end{aligned} \tag{3.55}$$

The above system (3.55) is not solvable for τ and ζ_{x_i} , $i = 1, 2, 3$ in general. Then by using the following assumptions for ζ_{x_1} , ζ_{x_2} and ζ_{x_3} ,

$$\begin{aligned} \zeta_{x_1} &= a_1(t)x_1 + a_2(t)x_2 + a_3(t), \\ \zeta_{x_2} &= b_1(t)x_1 + b_2(t)x_2 + b_3(t) \end{aligned}$$

and also by using the series representation given in (2.28), we obtain the following overdetermined system

$$\begin{aligned} \frac{d^n}{dt^n} a_1(t) + \left(\frac{n-\alpha}{n+1}\right) D_t^{n+1}(\tau) &= 0, \quad n \in \mathbb{N}, \\ \frac{d^n}{dt^n} b_2(t) + \left(\frac{n-\alpha}{n+1}\right) D_t^{n+1}(\tau) &= 0, \quad n \in \mathbb{N}, \\ a_1(t) + (\alpha+1)D_t(\tau) &= 0, \\ b_2(t) + (\alpha+1)D_t(\tau) &= 0, \\ -b_2(t) + (\alpha+1)D_t(\tau) + 2a_1(t) &= 0, \\ a_2(t) = a_3(t) &= 0, \\ b_1(t) = b_3(t) &= 0. \end{aligned} \tag{3.56}$$

Solving the above system (3.56) with $\tau(0) = 0$, we obtain the infinitesimals as

$$\tau = ct, \quad \zeta_{x_1} = -(\alpha + 1)cx_1, \quad \zeta_{x_2} = -(\alpha + 1)cx_2.$$

Thus, the fractional type coupled equation of motion (3.54) is invariant under one-parameter Lie group of transformations

$$\begin{aligned} \bar{t} &= t + \epsilon ct, \\ \bar{x}_1 &= x_1 - \epsilon(\alpha + 1)cx_1, \\ \bar{x}_2 &= x_2 - \epsilon(\alpha + 1)cx_2 \end{aligned}$$

with infinitesimal generator

$$X = ct \frac{\partial}{\partial t} - (\alpha + 1)cx_1 \frac{\partial}{\partial x_1} - (\alpha + 1)cx_2 \frac{\partial}{\partial x_2},$$

which can be written as $X = cX_1$, where

$$X_1 = t \frac{\partial}{\partial t} - (\alpha + 1) \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \right).$$

3.4.2. Construction of exact solution

The invariant function $F(x, t)$ of (3.54) reads

$$X_1 F = t \frac{\partial F}{\partial t} - (\alpha + 1) \left(x_1 \frac{\partial F}{\partial x_1} + x_2 \frac{\partial F}{\partial x_2} \right) = 0,$$

which gives as follows

$$\frac{dt}{t} = \frac{dx_1}{-(\alpha + 1)x_1} = \frac{dx_2}{-(\alpha + 1)x_2}.$$

Thus, we obtain an exact solution of the fractional type coupled equation of motion (3.54) as follows

$$\begin{aligned} x_1(t) &= \left[\frac{9C \pm \sqrt{81C^2 - (16B^3 - 24B^2) \frac{\Gamma(-\alpha)}{\Gamma(-2\alpha+1)}}}{4B^2} \right] t^{-(\alpha+1)}, \\ x_2(t) &= -\frac{1}{2B} \left[\frac{27C^2 \pm 3C \sqrt{81C^2 - (16B^3 - 24B^2) \frac{\Gamma(-\alpha)}{\Gamma(-2\alpha+1)}}}{4B^2} + \frac{\Gamma(-\alpha)}{\Gamma(-2\alpha + 1)} \right] t^{-(\alpha+1)}, \end{aligned} \tag{3.57}$$

where $\alpha \in (0, 1)$ and $81C^2\Gamma(-2\alpha + 1) \geq (16B^3 - 24B^2)\Gamma(-\alpha)$ must hold for the existence of real solutions to the system (3.54). In the Figure 3, the geometrical representations of solutions of fractional type coupled equation of motion for various values of α are given, in which the constants are assumed to be $B = 2$ and $C = 8$.

3.5. Fractional Lotka-Volterra ABC System

Consider the following fractional form of Lotka-Volterra ABC system [65]

$$\begin{aligned} {}_0D_t^\alpha x_1 &= x_1(Cx_2 + x_3), \\ {}_0D_t^\alpha x_2 &= x_2(x_1 + Ax_3), \\ {}_0D_t^\alpha x_3 &= x_3(Bx_1 + x_2), \quad \alpha \in (0, 1), \end{aligned} \tag{3.58}$$

where A, B, C are constants, x_1, x_2 and x_3 are functions of t . The approximate analytic solution of fractional two-dimensional Lotka-Volterra system was investigated in [13]. Note that, when $\alpha = 1$, the system (3.58) is referred to as the classical Lotka-Volterra system or the Lotka-Volterra ABC system. The integrability of (3.58) with $\alpha = 1$ was discussed in [22,35,41].

3.5.1. Lie point symmetries

Assuming that the system (3.58) is invariant under Lie group transformations given in equation (2.27), we get

$$\begin{aligned} \zeta_{x_1}^{(\alpha)} &= C\zeta_{x_2}x_1 + C\zeta_{x_1}x_2 + \zeta_{x_3}x_1 + \zeta_{x_1}x_3, \\ \zeta_{x_2}^{(\alpha)} &= \zeta_{x_1}x_2 + \zeta_{x_2}x_1 + A\zeta_{x_3}x_2 + A\zeta_{x_2}x_3, \\ \zeta_{x_3}^{(\alpha)} &= B\zeta_{x_1}x_3 + B\zeta_{x_3}x_1 + \zeta_{x_2}x_3 + \zeta_{x_3}x_2. \end{aligned}$$

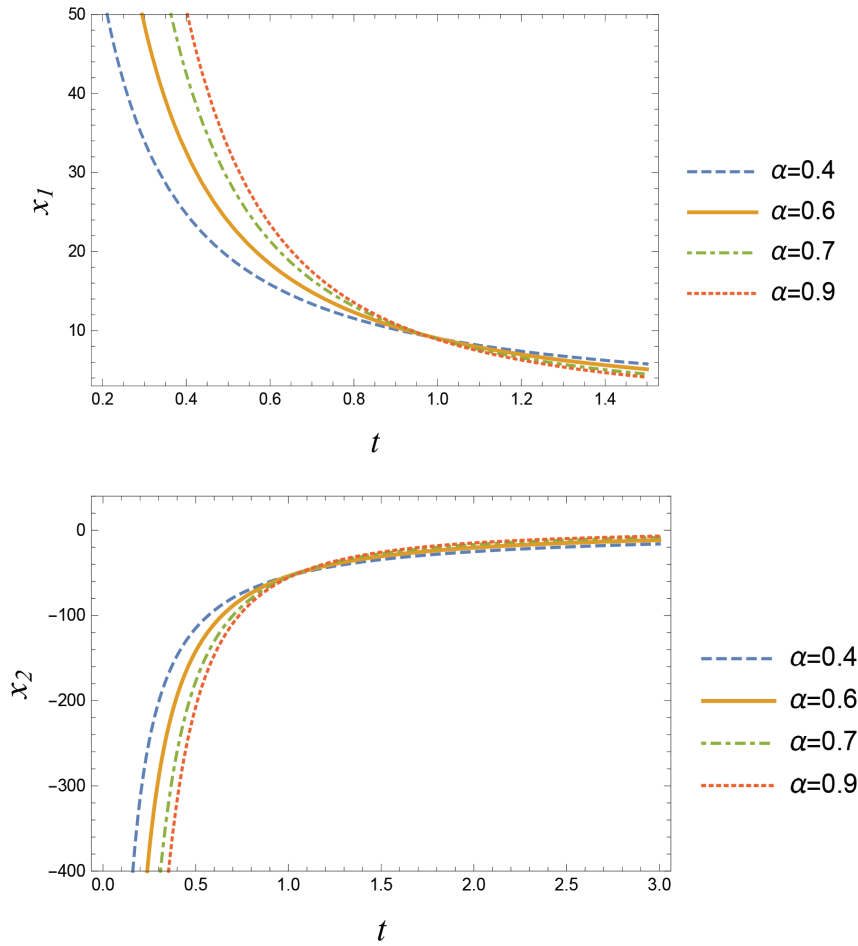


Figure 3 | Graphical representations of solutions of fractional type coupled equation of motion for various values of α .

Let us assume that the infinitesimals ζ_{x_1} , ζ_{x_2} and ζ_{x_3} are of the form

$$\zeta_{x_1} = a_1(t)x_1 + a_2(t)x_2 + a_3(t)x_3 + a_4(t),$$

$$\zeta_{x_2} = b_1(t)x_1 + b_2(t)x_2 + b_3(t)x_3 + b_4(t),$$

$$\zeta_{x_3} = c_1(t)x_1 + c_2(t)x_2 + c_3(t)x_3 + c_4(t).$$

Proceeding in the above similar manner, we obtain

$$\tau = ct, \quad \zeta_{x_1} = -\alpha cx_1, \quad \zeta_{x_2} = -\alpha cx_2, \quad \zeta_{x_3} = -\alpha cx_3.$$

Hence, the fractional Lotka-Volterra ABC system (3.58) is invariant under

$$\begin{aligned} \bar{t} &= t + c\epsilon t, \\ \bar{x}_1 &= x_1 - \epsilon\alpha cx_1, \\ \bar{x}_2 &= x_2 - \epsilon\alpha cx_2, \\ \bar{x}_3 &= x_3 - \epsilon\alpha cx_3. \end{aligned}$$

Hence, the infinitesimal generator is

$$X = ct \frac{\partial}{\partial t} - \alpha cx_1 \frac{\partial}{\partial x_1} - \alpha cx_2 \frac{\partial}{\partial x_2} - \alpha cx_3 \frac{\partial}{\partial x_3}.$$

3.5.2. Construction of exact solution

The invariant function F of system (3.58) satisfies the equation $XF = 0$. Hence its characteristic equation becomes

$$\frac{dt}{t} = \frac{dx_1}{-\alpha x_1} = \frac{dx_2}{-\alpha x_2} = \frac{dx_3}{-\alpha x_3}.$$

On solving the above characteristic equation, we obtain an exact solution of fractional Lotka-Volterra ABC system (3.58) as

$$\begin{aligned}
 x_1(t) &= \left[\left(\frac{AC - A + 1}{ABC + 1} \right) \frac{\Gamma(1 - \alpha)}{\Gamma(1 - 2\alpha)} \right] t^{-\alpha}, \\
 x_2(t) &= \left[\left(\frac{AB - B + 1}{ABC + 1} \right) \frac{\Gamma(1 - \alpha)}{\Gamma(1 - 2\alpha)} \right] t^{-\alpha}, \\
 x_3(t) &= \left[\left(\frac{BC - C + 1}{ABC + 1} \right) \frac{\Gamma(1 - \alpha)}{\Gamma(1 - 2\alpha)} \right] t^{-\alpha},
 \end{aligned}
 \tag{3.59}$$

where $\alpha \in (0, 1)$. Note that for $\alpha = \frac{1}{2}$, the solutions of the system (3.58) takes the form $x_1(t) = x_2(t) = x_3(t) = 0$ which is trivial. In the Figure 4, for $A = 2, B = 1$ and $C = -1$ the geometrical representations of solutions of the fractional Lotka-Volterra system (3.58) for various values of α are given.

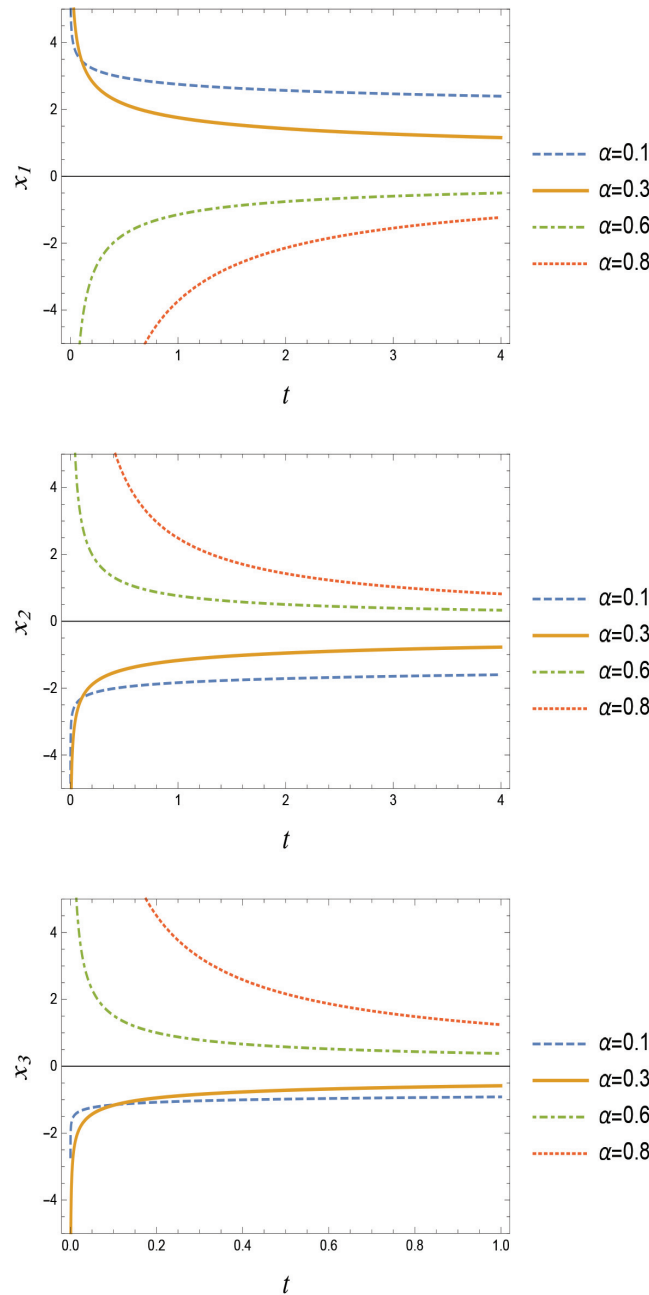


Figure 4 | Graphical representation of solutions of fractional Lotka-Volterra ABC system (3.58) for various values of α .

Note 9. Observe that the obtained invariant solutions of the form $x(t) \propto t^{-\alpha}$ holds only for the R-L fractional derivative because

$$\frac{d^\alpha}{dt^\alpha} (t^{-\alpha}) = \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)} t^{-2\alpha}, \quad \alpha \in (0, 1), \quad t > 0.$$

These type of solutions are not available in the sense of Caputo fractional derivative [32,52].

4. CONCLUDING REMARKS AND DISCUSSION

In this article, we have shown how we can extend the Lie symmetry analysis method for n -coupled system of FODEs with R-L fractional derivative. Also, we systematically presented how to derive the Lie point symmetries of scalar and coupled FODEs namely (i) fractional Thomas-Fermi equation (3.1), (ii) Bagley-Torvik equation (3.12), (iii) two-coupled system of fractional quartic oscillator (3.33), (iv) fractional type coupled equation of motion (3.54) and (v) fractional Lotka-Volterra ABC system (3.58). Furthermore, we explicitly derived the exact solutions of the above mentioned FODEs wherever possible using the obtained Lie point symmetries. The numerical solutions of fractional Thomas-Fermi equation with Caputo derivative was discussed in [12]. In the presented work, we derived the invariant solution of Thomas-Fermi equation (3.1) in the sense of R-L fractional derivative of the form $x(t) \propto t^{-(\alpha+2)}$. Since the dimensions of the symmetry algebras for Bagley-Torvik equation and its various cases are greater than 2, we constructed the optimal system of one-dimensional subalgebras. We know that the maximal dimension of symmetry algebra for second-order classical ODE is eight. For the second-order Bagley-Torvik equation (3.11), the dimension of symmetry algebra is five due to the presence of R-L fractional derivative of order $\frac{3}{2}$. We also systematically found the optimal system of Lie algebra for (3.11) with various cases.

We not only extended the Lie symmetry analysis method for n -coupled system of FODEs with R-L fractional derivative but also established the efficiency of the method by solving the two-coupled system of fractional quartic oscillator (3.33), fractional type coupled equation of motion (3.54) and fractional Lotka-Volterra ABC system (3.58). For the system (3.33) the invariant solution $x_i \propto t^{-\frac{(\alpha+1)}{2}}$ for $i = 1, 2$. The fractional type coupled equation of motion (3.54) admits the invariant solution of the form $x_i \propto t^{-(\alpha+1)}$ for $i = 1, 2$. Similarly, we observe that the invariant solution of the fractional Lotka-Volterra ABC system (3.58) is of singular type and is of the form $x_i \propto t^{-\alpha}$ for $i = 1, 2, 3$. These type of obtained exact solutions are possible in R-L fractional derivative but not in Caputo sense [32,52]. The exact solutions of the given scalar and coupled FODEs were graphically shown for various values of α .

CONFLICTS OF INTEREST

The authors declare they have no conflicts of interest.

ACKNOWLEDGMENT

The authors wish to thank the editors and all the anonymous reviewers for their fruitful comments and suggestions for the significant improvement of the manuscript.

REFERENCES

- [1] G. Adomian, Solution of the Thomas-Fermi equation, *App. Math. Lett.* 11 (1998), 131–133.
- [2] D.J. Arrigo, *Symmetry Analysis of Differential Equations: An Introduction*, John Wiley & Sons, New Jersey, 2015, p. 192.
- [3] R.L. Bagley, P.J. Torvik, On the appearance of the fractional derivative in the behavior of real materials, *ASME J. Appl. Mech.* 51 (1984), 294–298.
- [4] T. Bakkyaraj, Lie symmetry analysis of system of nonlinear fractional partial differential equations with Caputo fractional derivative, *Eur. Phys. J. Plus* 135 (2020), 126.
- [5] T. Bakkyaraj, R. Sahadevan, Group formalism of Lie transformations to time-fractional partial differential equations, *Pramana - J. Phys.* 85 (2015), 849–860.
- [6] T. Bakkyaraj, R. Sahadevan, Invariant analysis of nonlinear fractional ordinary differential equations with Riemann–Liouville fractional derivative, *Nonlinear Dyn.* 80 (2015), 447–455.
- [7] D. Baleanu, M. Inc, A. Yusuf, A.I. Aliyu, Lie symmetry analysis, exact solutions and conservation laws for the time fractional Caudrey–Dodd–Gibbon–Sawada–Kotera equation, *Commun. Nonlinear Sci. Numer. Simulat.* 59 (2018), 222–234.
- [8] G.W. Bluman, S.C. Anco, *Symmetry and Integration Methods for Differential Equations*, Springer, New York, 2002.
- [9] A. Chauhan, R. Arora, Time fractional Kupershmidt equation: symmetry analysis and explicit series solution with convergence analysis, *Commun. Math.* 27 (2019), 171–185.
- [10] A. Chauhan, K. Sharma, R. Arora, Lie symmetry analysis, optimal system, and generalized group invariant solutions of the (2 + 1)-dimensional Date–Jimbo–Kashiwara–Miwa equation, *Math. Meth. Appl. Sci.* 43 (2020), 8823–8840.

- [11] S. Choudhary, P. Prakash, V. Daftardar-Gejji, Invariant subspaces and exact solutions for a system of fractional PDEs in higher dimensions, *Comp. Appl. Math.* 38 (2019), 126.
- [12] Z. Dahmani, A. Anber, Two numerical methods for solving the fractional Thomas-Fermi equation, *J. Interdisciplinary Math.* 18 (2015), 35–41.
- [13] S. Das, P.K. Gupta, A mathematical model on fractional Lotka–Volterra equations, *J. Theor. Biol.* 277 (2011), 1–6.
- [14] K. Diethelm, N.J. Ford, Numerical solution of the Bagley–Torvik equation, *BIT Numer. Math.* 42 (2002), 490–507.
- [15] X.X. Du, B. Tian, Q.X. Qu, Y.Q. Yuan, X.H. Zhao, Lie group analysis, solitons, self-adjointness and conservation laws of the modified Zakharov–Kuznetsov equation in an electron-positron-ion magnetoplasma. *Chaos Solitons Fractals* 134 (2020), 109709.
- [16] R.A. El-Nabulsi, The fractional Boltzmann transport equation, *Comput. Math. Appl.* 62 (2011), 1568–1575.
- [17] W. Feng, S. Sun, Y. Sun, Existence of positive solutions for a generalized and fractional ordered Thomas-Fermi theory of neutral atoms, *Adv. Differ. Equ.* 2015 (2015), 350.
- [18] R.K. Gazizov, A.A. Kasatkin, S.Y. Lukashchuk, Continuous transformation groups of fractional differential equations, *Vestnik USATU* 9 (2007), 125–135. (in Russian)
- [19] R.K. Gazizov, A.A. Kasatkin, S.Y. Lukashchuk, Group-invariant solutions of fractional differential equations, in: J.A.T. Machado, A.C.J. Luo, R.S. Barbosa, M.F. Silva, L.B. Figueiredo, (eds.), *Nonlinear Science and Complexity*, Springer, Dordrecht, Heidelberg, 2011, pp. 51–59.
- [20] R.K. Gazizov, A.A. Kasatkin, S.Y. Lukashchuk, Symmetry properties of fractional diffusion equations, *Phys. Scr.* 136 (2009), 014016.
- [21] R.K. Gazizov, A.A. Kasatkin, Construction of exact solutions for fractional order differential equations by the invariant subspace method, *Comput. Math. Appl.* 66 (2013), 576–584.
- [22] A. Goriely, *Integrability and Nonintegrability of Dynamical Systems*, World Scientific, Singapore, 2001, p. 436.
- [23] P. Górká, H. Prado, J. Trujillo, The time fractional Schrödinger equation on Hilbert space, *Integr. Equ. Oper. Theory* 87 (2017), 1–14.
- [24] I. Grigorenko, E. Grigorenko, Chaotic dynamics of the fractional Lorenz system, *Phys. Rev. Lett.* 91 (2003), 034101.
- [25] R. Herrmann, Infrared spectroscopy of diatomic molecules — a fractional calculus approach, *Int. J. Mod. Phys. B* 27 (2013), 1350019.
- [26] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000, p. 472.
- [27] P.E. Hydon, *Symmetry Methods for Differential Equations*, Cambridge University Press, Cambridge, 2000.
- [28] N.H. Ibragimov, *CRC Handbook of Lie group Analysis of Differential Equations*, vol. 1: Symmetries, Exact Solutions and Conservation Laws, CRC Press, Boca Raton, FL, USA, 1994.
- [29] G.F. Jefferson, J. Carminati, *FracSym: automated symbolic computation of Lie symmetries of fractional differential equations*, *Comput. Phys. Commun.* 185 (2014), 430–441.
- [30] A.A. Kasatkin, Symmetry properties for systems of two ordinary fractional differential equations, *Ufa Mathematical J.* 4 (2012), 65–75.
- [31] B. Kaur, R.K. Gupta, Multiple types of exact solutions and conservation laws of new coupled $(2 + 1)$ -dimensional Zakharov–Kuznetsov system with time-dependent coefficients, *Pramana - J. Phys.* 93 (2019), 59.
- [32] A.A. Kilbas, J.J. Trujillo, H.M. Srivastava, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006, p. 540.
- [33] Y. Kosmann-Schwarzbach, B. Grammaticos, K.M. Tamizhmani, *Integrability of Nonlinear systems*, Springer-Verlag, Berlin, 2004.
- [34] M. Lakshmanan, P. Kaliappan, Lie transformations, nonlinear evolution equations, and Painlevé forms, *J. Math. Phys.* 24 (1983), 795–806.
- [35] M. Lakshmanan, S. Rajasekar, *Nonlinear Dynamics: Integrability, Chaos, and Patterns*, Springer-Verlag, Berlin, 2003.
- [36] N. Laskin, *Fractional quantum mechanics*, World Scientific, Singapore, 2018, p. 360.
- [37] C. Li, J. Zhang, Lie symmetry analysis and exact solutions of generalized fractional Zakharov–Kuznetsov equations, *Symmetry* 11 (2019), 601.
- [38] W.X. Ma, Conservation laws by symmetries and adjoint symmetries, *Discrete and Continuous Dynamical Systems-Series S* 11 (2018), 707–721.
- [39] W.X. Ma, Y. Liu, Invariant subspaces and exact solutions of a class of dispersive evolution equations, *Commun. Nonlinear Sci. Numer. Simul.* 17 (2012), 3795–3801.
- [40] W.X. Ma, M.M. Mousa, M.R. Ali, Application of a new hybrid method for solving singular fractional Lane–Emden-type equations in astrophysics, *Mod. Phys. Lett. B* 34 (2020), 2050049.
- [41] R.S. Maier, The integration of three-dimensional Lotka–Volterra systems, *Proc. R. Soc. A* 469 (2013), 20120693.
- [42] F. Mainardi, Fractional calculus: some basic problems in continuum and statistical mechanics, in: F. Mainardi (ed.), *Fractals and fractional calculus in continuum mechanics*, International Centre for Mechanical Sciences, Springer-Verlag, New York, 1997, pp. 291–348.
- [43] A.M. Mathai, H.J. Haubold, *Special Functions for Applied Scientists*, Springer, New York, 2008.
- [44] S. Momani, Z. Odibat, Analytical approach to linear fractional partial differential equations arising in fluid mechanics, *Phys. Lett. A* 355 (2006), 271–279.
- [45] S. Momani, Z. Odibat, Homotopy perturbation method for nonlinear partial differential equations of fractional order, *Phys. Lett. A* 365 (2007), 345–350.
- [46] A.M. Nass, Symmetry analysis of space-time fractional Poisson equation with a delay, *Quaestiones Mathematicae* 42 (2019), 1221–1235.
- [47] A.M. Nass, Lie symmetry analysis and exact solutions of fractional ordinary differential equations with neutral delay, *Appl. Math. Comput.* 347 (2019), 370–380.
- [48] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Springer, Heidelberg, 1986.
- [49] L.V. Ovsiannikov, *Group Analysis of Differential Equations*, Academic Press, New York, 1982, p. 432.
- [50] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, 1999.
- [51] P. Prakash, New exact solutions of generalized convection–reaction–diffusion equation, *Eur. Phys. J. Plus* 134 (2019), 261.
- [52] P. Prakash, Invariant subspaces and exact solutions for some types of scalar and coupled time-space fractional diffusion equations, *Pramana - J. Phys.* 94 (2020), 103.
- [53] P. Prakash, S. Choudhary, V. Daftardar-Gejji, Exact solutions of generalized nonlinear time-fractional reaction–diffusion equations with time delay, *Eur. Phys. J. Plus* 135 (2020), 490.
- [54] P. Prakash, R. Sahadevan, Lie symmetry analysis and exact solution of certain fractional ordinary differential equations, *Nonlinear Dyn.* 89 (2017), 305–319.
- [55] A. Saadatmandi, M. Dehghan, A new operational matrix for solving fractional-order differential equations, *Comput. Math. Appl.* 59 (2010), 1326–1336.
- [56] R. Sadat, M.M. Kaseem, W.X. Ma, Families of analytic solutions for $(2 + 1)$ model in unbounded domain via optimal Lie vectors with integrating factors, *Mod. Phys. Lett. B* 33 (2019), 1950229.
- [57] R. Sahadevan, P. Prakash, Exact solution of certain time fractional nonlinear partial differential equations, *Nonlinear Dyn.* 85 (2016), 659–673.
- [58] R. Sahadevan, P. Prakash, Exact solutions and maximal dimension of invariant subspaces of time fractional coupled nonlinear partial differential equations, *Commun. Nonlinear Sci. Numer. Simulat.* 42 (2017), 158–177.

- [59] R. Sahadevan, T. Bakkyaraj, Invariant analysis of time fractional generalized Burgers and Korteweg–de Vries equations, *J. Math. Anal. Appl.* 393 (2012), 341–347.
- [60] R. Sahadevan, P. Prakash, On Lie symmetry analysis and invariant subspace methods of coupled time fractional partial differential equations, *Chaos Solitons Fractals* 104 (2017), 107–120.
- [61] R. Sahadevan, P. Prakash, Lie symmetry analysis and conservation laws of certain time fractional partial differential equations, *Int. J. Dynamical Systems and Differential Equations* 9 (2019), 44–64.
- [62] M. Senthilvelan, V.K. Chandrasekar, R. Mohanasubha, Symmetries of nonlinear ordinary differential equations: the modified Emden equation as a case study, *Pramana - J. Phys.* 85 (2015), 755–787.
- [63] M.F. Silva, J.A.T. Machado, A.M. Lopes, Fractional order control of a hexapod robot, *Nonlinear Dyn.* 38 (2004), 417–433.
- [64] V.E. Tarasov, Fractional hydrodynamic equations for fractal media, *Ann. Phys.* 318 (2005), 286–307.
- [65] J. Tian, Y. Yu, H. Wang, Stability and bifurcation of two kinds of three-dimensional fractional Lotka-Volterra systems, *Math. Prob. Eng.* 2014 (2014), 695871.
- [66] Z.H. Wang, X. Wang, General solution of the Bagley–Torvik equation with fractional-order derivative, *Commun. Nonlinear Sci. Numer. Simulat.* 15 (2010), 1279–1285.
- [67] C.E. Wulfman, B.G. Wybourne, The Lie group of Newton's and Lagrange's equations for the harmonic oscillator, *J. Phys. A: Math. Gen.* 9 (1976), 507–518.
- [68] E. Yaşar, İ.B. Giresunlu, The $(G'/G, 1/G)$ -expansion method for solving nonlinear space–time fractional differential equations, *Pramana - J. Phys.* 87 (2016), 17.
- [69] E. Yaşar, Y. Yıldırım, E. Yaşar, New optical solitons of space-time conformable fractional perturbed Gerdjikov-Ivanov equation by sine-Gordon equation method, *Results Phys.* 9 (2018), 1666–1672.
- [70] E. Yaşar, Y. Yıldırım, C.M. Khalique, Lie symmetry analysis, conservation laws and exact solutions of the seventh-order time fractional Sawada–Kotera–Ito equation, *Results Phys.* 6 (2016), 322–328.
- [71] Y. Ye, W.X. Ma, S. Shen, D. Zhang, A class of third-order nonlinear evolution equations admitting invariant subspaces and associated reductions, *J. Nonlinear Math. Phys.* 21 (2014), 132–148.
- [72] N.A. Zaitsev, I.V. Matyushkin, D.V. Shamonov, Numerical solution of Thomas–Fermi equation for the centrally symmetric atom, *Russ. Microelectron.* 33 (2004), 303–309.