On Upper *k*-Record Values from the Generalized Linear Exponential Distribution

M. Alam, M. A. Khan[*], R. U. Khan

Department of Statistics and Operations Research, Aligarh Muslim University, Aligarh, 202 002, India

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ABSTRACT

In this paper, we derive the exact expressions as well as recurrence relations for single and product moment of generalized upper record values from the four-parameter generalized linear exponential distribution. Further, we characterize the given distribution through conditional expectation and recurrence relations.

1. INTRODUCTION

The exponential, Rayleigh, Weibull, and the linear exponential distribution (LED) are widely used to model lifetime data. These distribution have several desirable properties which are frequently used in analyzing lifetime data. In particular, the generalized linear exponential distribution (GLED) having linearly increasing failure rates is very useful for modeling lifetime data. The exponential and Rayleigh distributions are special cases of LED. Since the LED does not provide a reasonable parametric fit for bathtub shape (decreasing, nonlinear increasing, nonmonotone failure rates). Therefore, a lot of work is devoted to GLED and propose some new distribution which are more flexible. So, in this continuation Mahmoud and Alam [1] introduced the four-parameter GLED, which defines the different distributional properties of GLED. Lee and Tsai [2] modify the moment of the GLED in a different way, which is more robust. The GLED shows the increasing, decreasing, and bathtub-shaped hazard rate functions which are quite useful in reliability and biological studies.

A random variable *X* is said to have a GLED with four parameters θ, $\alpha > 0$ and β, $\lambda \geq 0$, if its probability density function (*pdf*) is of the form

$$f(x) = \alpha(\beta + \theta x)\left(\frac{\theta}{2}x^2 + \beta x - \lambda\right)^{\alpha-1} exp\left\{-\left(\frac{\theta}{2}x^2 + \beta x - \lambda\right)^{\alpha}\right\} I_{(\Psi,\infty)}(x), \quad x > 0, \tag{1.1}$$

where

$$I_{(\Psi,\infty)}(x) = \begin{cases} 1 & x > \Psi \\ 0 & \text{otherwise} \end{cases}$$

is the indicator function, and

$$\Psi = \frac{-\beta + \sqrt{\beta^2 + 2\theta\lambda}}{\theta}$$

and the distribution function (*df*) is

$$F(x) = \left[1 - exp\left\{-\left(\frac{\theta}{2}x^2 + \beta x - \lambda\right)^{\alpha}\right\}\right] I_{(\Psi,\infty)}(x), \quad x > 0. \tag{1.2}$$

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[*]*Corresponding author. Email: khanazam2808@gmail.com*

It is easy to see that

$$\left(\frac{\theta}{2}x^2 + \beta x - \lambda\right) f(x) = \alpha(\beta + \theta x)[-\ln \overline{F}(x)][\overline{F}(x)], \tag{1.3}$$

where

$$\overline{F}(x) = 1 - F(x).$$

The concept of record values was introduced by Chandler [3]. An observation is called a record if its value is greater (or less) than all the previous observations. Record values are used in a wide variety of practical situations, such as industrial stress testing, meteorological analysis, hydrology, seismology, oil, mining surveys, sports, and athletic events. For a survey on important results in this area one may refer to Ahsanullah [4], Kamps [5], Arnold *et al.* [6] and Ahsanullah and Nevzorov [7]. Dziubdziela and Kopcoiński [8] have generalized the concept of record values of Chandler [3] by random variables of a more generalized nature and called them the *k*-th record values. Later, Minimol and Thomas ([9], p. 487) called the record values defined by Dziubdziela and Kopcoiński [8] also as the generalized record values, since the *r*-th member of the sequence of the classical record values is also known as the *r*-th record value. By setting $k = 1$, we obtain ordinary record statistics.

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identical distributed (*iid*) continuous random variables with *df* $F(x)$ and *pdf* $f(x)$. Then for a fixed positive integer $k \geq 1$, the sequence of *k*-th upper record times $\left\{U_n^{(k)}, n \geq 1\right\}$ is defined as Nevzorov [10]:

$$U_1^{(k)} = k$$

and for $n \geq 1$,

$$U_{n+1}^{(k)} = \min \left\{ j : j > U_n^{(k)}, X_j > X_{U_n^{(k)}-k+1:U_n^{(k)}} \right\}.$$

The sequence $\left\{Y_n^{(k)}, n \geq 1\right\}$, where $Y_n^{(k)} = X_{U_n^{(k)}:U_n^{(k)}+k-1}$ is called the sequence of generalized upper record values (*k*-th upper record values) of $\{X_n, n \geq 1\}$. Note that for $k = 1$, we have $Y_n^{(1)} = X_{U_n}, n \geq 1$, which are the record values of $\{X_n, n \geq 1\}$ as defined in Ahsanullah [4].

The *pdf* of $Y_n^{(k)}$ and the joint *pdf* of $Y_m^{(k)}$ and $Y_n^{(k)}$ are given by (Dziubdziela and Kopcoiński [8], Grudzień [11])

$$f_{Y_n^{(k)}}(x) = \frac{k^n}{(n-1)!}\left[-\ln \overline{F}(x)\right]^{n-1}\left[\overline{F}(x)\right]^{k-1} f(x), \quad n \geq 1, \tag{1.4}$$

$$\begin{aligned} f_{Y_m^{(k)}, Y_n^{(k)}}(x, y) = {} & \frac{k^n}{(m-1)!(n-m-1)!}\left[-\ln \overline{F}(x)\right]^{m-1}\frac{f(x)}{\overline{F}(x)} \\ & \times \left[\ln \overline{F}(x) - \ln \overline{F}(y)\right]^{n-m-1}\left[\overline{F}(y)\right]^{k-1} f(y), \quad x < y, 1 \leq m < n, \end{aligned} \tag{1.5}$$

and the conditional *pdf* of $Y_n^{(k)}$ given $Y_m^{(k)} = x$, is

$$f_{Y_n^{(k)}|Y_m^{(k)}}(y|x) = \frac{k^{n-m}}{(n-m-1)!}[\ln \overline{F}(x) - \ln \overline{F}(y)]^{n-m-1}\left[\frac{\overline{F}(y)}{\overline{F}(x)}\right]^{k-1}\frac{f(y)}{\overline{F}(x)}, \quad x < y. \tag{1.6}$$

For some recent developments on generalized record values with special reference to those arising from exponential, Gumbel, Pareto, generalized Pareto, Burr, Weibull, Gompertz, Makeham, modified-Weibull, exponential-Weibull, additive Weibull, and Kumaraswamy log-logistic distributions, see Grudzień and Szynal [12,13], Pawlas and Szynal [14–1516], Minimol and Thomas [9,17], Khan and Khan [18], Khan *et al.* [19,20], and Singh *et al.* [21], respectively. In this paper, we mainly focus on the study of generalized upper record values arising from the GLED and discussed exact explicit expressions as well as several recurrence relations satisfied by single and product moments. In addition, conditional expectations and recurrence relations of generalized record values are used to characterize this distribution.

2. RELATIONS FOR SINGLE MOMENTS

In this section, we derive the exact expressions for single moments of generalized upper record values and recurrence relations in the following theorems:

Theorem 2.1. For the distribution given in (1.1). Fix a positive integer $k \geq 1$, for $n \geq 1$, $(\beta^2 + 2\theta\lambda) > 0$ and $j = 0, 1, \ldots$

$$
E\left(Y_n^{(k)}\right)^j = \frac{1}{(n-1)!} \left[\sum_{u=0}^{j} \sum_{v=0}^{\infty} \binom{j}{u} \binom{\frac{j-u}{2}}{v} (-\beta)^u 2^v \frac{(\beta^2 + 2\theta\lambda)^{\frac{j-u}{2}-v}}{\theta^{j-v} k^{\frac{v}{\alpha}}} \gamma\left\{ \frac{v}{\alpha} + n, k\left(\frac{\beta^2 + 2\theta\lambda}{2\theta}\right)^\alpha \right\} \right.
$$
$$
\left. + \sum_{u=0}^{j} \sum_{v=0}^{\infty} \binom{j}{u} \binom{\frac{j-u}{2}}{v} (-\beta)^u 2^{\frac{j-u}{2}-v} \frac{(\beta^2 + 2\theta\lambda)^v}{\theta^{\frac{j+u}{2}+v} k^{\frac{j-u}{2\alpha} - \frac{v}{\alpha}}} \Gamma\left\{ \frac{j-u}{2\alpha} - \frac{v}{\alpha} + n, k\left(\frac{\beta^2 + 2\theta\lambda}{2\theta}\right)^\alpha \right\} \right],
$$

(2.1)

where $\Gamma(s, t) = \int_t^\infty x^{s-1} e^{-x} dx$ and $\gamma(s, t) = \int_0^t x^{s-1} e^{-x} dx$ are the upper and lower incomplete gamma functions, respectively.

Proof. In the view of (1.4), we have

$$
E(Y_n^{(k)})^j = \frac{k^n}{(n-1)!} \int_{-\infty}^{\infty} x^j \left[-\ln \overline{F}(x) \right]^{n-1} \left[\overline{F}(x) \right]^{k-1} f(x) dx, \quad n \geq 1.
$$

Now using (1.3), we get

$$
E(Y_n^{(k)})^j = \frac{\alpha k^n}{(n-1)!} \int_{\infty}^{\Psi} \frac{x^j(\beta + \theta x)}{\left(\frac{\theta}{2}x^2 + \beta x - \lambda\right)} \left[-\ln \overline{F}(x) \right]^n \left[\overline{F}(x) \right]^k dx
$$
$$
= \frac{\alpha k^n}{(n-1)!} \int_{\infty}^{\Psi} \frac{x^j(\beta + \theta x)}{\left(\frac{\theta}{2}x^2 + \beta x - \lambda\right)} \left[\left(\frac{\theta}{2}x^2 + \beta x - \lambda\right)^\alpha \right]^n \left[e^{-\left(\frac{\theta}{2}x^2 + \beta x - \lambda\right)^\alpha} \right]^k dx,
$$

(2.2)

after substituting $t = \left(\frac{\theta}{2}x^2 + \beta x - \lambda\right)^\alpha$ in (2.2), we get

$$
E\left(Y_n^{(k)}\right)^j = \frac{k^n}{(n-1)!} \int_0^\infty \left[\frac{-\beta + \sqrt{2\theta t^{1/\alpha} + (\beta^2 + 2\theta\lambda)}}{\theta} \right]^j t^{n-1} e^{-kt} dt.
$$

(2.3)

On using binomial expansion in (2.3) and we have

$$
E\left(Y_n^{(k)}\right)^j = \frac{k^n}{(n-1)!} \sum_{u=0}^{j} \binom{j}{u} \frac{(-\beta)^u}{\theta^j} \int_{\infty}^0 \left[2\theta t^{1/\alpha} + (\beta^2 + 2\theta\lambda) \right]^{\frac{j-u}{2}} t^{n-1} e^{-kt} dt.
$$
$$
= \frac{k^n}{(n-1)!} \sum_{u=0}^{j} \binom{j}{u} \frac{(-\beta)^u}{\theta^j} \int_{\infty}^0 \left[1 + \frac{\beta^2 + 2\theta\lambda}{2\theta t^{1/\alpha}} \right]^{\frac{j-u}{2}} (2\theta t^{1/\alpha})^{\frac{j-u}{2}} t^{n-1} e^{-kt} dt.
$$
$$
= \frac{k^n}{(n-1)!} \sum_{u=0}^{j} \binom{j}{u} \frac{(-\beta)^u}{\theta^j} \int_{\infty}^0 \left[1 + \frac{1}{\psi t^{1/\alpha}} \right]^{\frac{j-u}{2}} (2\theta t^{1/\alpha})^{\frac{j-u}{2}} t^{n-1} e^{-kt} dt.
$$

(2.4)

where $\psi = \frac{2\theta}{\beta^2 + 2\theta\lambda}$, by the expression $|\frac{\beta^2 + 2\theta\lambda}{2\theta t^{1/\alpha}}| < 1$ only if $\left(\frac{\beta^2 + 2\theta\lambda}{2\theta}\right)^\alpha < t < \infty$. We can verify that $|\psi\, t^{1/\alpha}| < 1$ if $0 < t < \left(\frac{1}{\psi}\right)^\alpha$ and $|\frac{1}{\psi t^{1/\alpha}}| < 1$ if $\left(\frac{1}{\psi}\right)^\alpha < t < \infty$, thus the integral in (2.4) should be separated into two parts as:

$$
\begin{aligned}
E\left(Y_n^{(k)}\right)^j &= \frac{k^n}{(n-1)!} \sum_{u=0}^{j} \binom{j}{u} \frac{(-\beta)^u}{\theta^j} \left[\int_0^{\left(\frac{1}{\psi}\right)^\alpha} \left(1 + \frac{1}{\psi t^{1/\alpha}}\right)^{\frac{j-u}{2}} (2\theta t^{1/\alpha})^{\frac{j-u}{2}} \right. \\
&\quad \left. \times t^{n-1} e^{-kt} dt + \int_{\left(\frac{1}{\psi}\right)^\alpha}^{\infty} \left[1 + \frac{1}{\psi t^{1/\alpha}}\right]^{\frac{j-u}{2}} (2\theta t^{1/\alpha})^{\frac{j-u}{2}} t^{n-1} e^{-kt} dt \right] \\
&= \frac{k^n}{(n-1)!} \sum_{u=0}^{j} \binom{j}{u} \frac{(-\beta)^u}{\theta^j} (\beta^2 + 2\theta\lambda)^{\frac{j-u}{2}} \left[\int_0^{\left(\frac{1}{\psi}\right)^\alpha} \left[1 + \psi t^{1/\alpha}\right]^{\frac{j-u}{2}} \right. \\
&\quad \left. \times t^{n-1} e^{-kt} dt + \int_{\left(\frac{1}{\psi}\right)^\alpha}^{\infty} \left[1 + \frac{1}{\psi t^{1/\alpha}}\right]^{\frac{j-u}{2}} (\psi t^{1/\alpha})^{\frac{j-u}{2}} t^{n-1} e^{-kt} dt \right].
\end{aligned}
\tag{2.5}
$$

On using the binomial expansion in (2.5), we get:

$$
\begin{aligned}
E\left(Y_n^{(k)}\right)^j &= \frac{k^n}{(n-1)!} \sum_{u=0}^{j} \binom{j}{u} \frac{(-\beta)^u}{\theta^j} (\beta^2 + 2\theta\lambda)^{\frac{j-u}{2}} \left[\sum_{v=0}^{\infty} \binom{\frac{j-u}{2}}{v} \psi^v \times \int_0^{\left(\frac{1}{\psi}\right)^\alpha} t^{\frac{v}{\alpha}+n-1} e^{-kt} dt \right. \\
&\quad \left. + \sum_{v=0}^{\infty} \binom{\frac{j-u}{2}}{v} \psi^{\frac{j-u}{2}-v} \int_\infty^{\left(\frac{1}{\psi}\right)^\alpha} t^{\frac{j-u}{2\alpha}-\frac{v}{\alpha}+n-1} e^{-kt} dt \right]. \\
&= \frac{k^n}{(n-1)!} \sum_{u=0}^{j} \binom{j}{u} \frac{(-\beta)^u}{\theta^j} (\beta^2 + 2\theta\lambda)^{\frac{j-u}{2}} \left[\sum_{v=0}^{\infty} \binom{\frac{j-u}{2}}{v} \left(\frac{2\theta}{\beta^2 + 2\theta\lambda}\right)^v \times \int_0^{\left(\frac{\beta^2+2\theta\lambda}{2\theta}\right)^\alpha} t^{\frac{v}{\alpha}+n-1} e^{-kt} dt \right. \\
&\quad \left. + \sum_{v=0}^{\infty} \binom{\frac{j-u}{2}}{v} \left(\frac{2\theta}{\beta^2+2\theta\lambda}\right)^{\frac{j-u}{2}-v} \int_\infty^{\left(\frac{\beta^2+2\theta\lambda}{2\theta}\right)^\alpha} t^{\frac{j-u}{2\alpha}-\frac{v}{\alpha}+n-1} e^{-kt} dt \right],
\end{aligned}
\tag{2.6}
$$

using the incomplete gamma function in (2.6), we get the result as given in (2.1).

Corollary 2.1. The j-th moment of the k-th record values from GLED has another form:

$$
\begin{aligned}
E(Y_n^{(k)})^j &= \frac{1}{(n-1)!} \left[\frac{(\beta^2 + 2\theta\lambda)^{\frac{j}{2}}}{\theta^j} \gamma\left\{n, k\left(\frac{\beta^2+2\theta\lambda}{2\theta}\right)^\alpha\right\} + \frac{2^{\frac{j}{2}}}{\theta^{\frac{j}{2}} k^{\frac{j}{2\alpha}}} \Gamma\left\{\frac{j}{2\alpha} + n, k\left(\frac{\beta^2+2\theta\lambda}{2\theta}\right)^\alpha\right\} \right. \\
&\quad + \sum_{u=1}^{j} \sum_{v=1}^{\infty} \binom{j}{u} \binom{\frac{j-u}{2}}{v} (-\beta)^u 2^v \frac{(\beta^2+2\theta\lambda)^{\frac{j-u}{2}-v}}{\theta^{j-v} k^{\frac{v}{\alpha}}} \gamma\left\{\frac{v}{\alpha} + n, k\left(\frac{\beta^2+2\theta\lambda}{2\theta}\right)^\alpha\right\} \\
&\quad \left. + \sum_{u=1}^{j} \sum_{v=1}^{\infty} \binom{j}{u} \binom{\frac{j-u}{2}}{v} (-\beta)^u 2^{\frac{j-u}{2}-v} \frac{(\beta^2+2\theta\lambda)^v}{\theta^{\frac{j+u}{2}+v} k^{\frac{j-u}{2\alpha}-\frac{v}{\alpha}}} \Gamma\left\{\frac{j-u}{2\alpha} - \frac{v}{\alpha} + n, k\left(\frac{\beta^2+2\theta\lambda}{2\theta}\right)^\alpha\right\} \right].
\end{aligned}
\tag{2.7}
$$

Proof. On expanding (2.1), at $u=0$ and $v=0$. Hence, the result.

Remark 2.1.

i) Setting $j = 0$ in (2.1), then

$$RHS = \frac{1}{(n-1)!} \left[\gamma \left\{ n, k \left(\frac{\beta^2 + 2\theta\lambda}{2\theta} \right)^\alpha \right\} + \Gamma \left\{ n, k \left(\frac{\beta^2 + 2\theta\lambda}{2\theta} \right)^\alpha \right\} \right],$$

by using the formula $[\Gamma(s,t) + \gamma(s,t)] = \Gamma(s) = \int_0^\infty t^{s-1} e^{-t} \, dt$, we get

$$RHS = \frac{1}{(n-1)!} \Gamma(n) = 1 = LHS \quad \text{as} \quad \Gamma(n) = (n-1)!$$

ii) Setting $k = 1$ and $n = 1$ in (2.7), we get the exact expression for the j-th ordinary moment of GLED as obtained by Lee and Tsai [2].

iii) Setting $\beta = \lambda = 0$ in (2.7), we get the exact expression for single moments of generalized upper records from Weibull distribution

$$E(Y_n^{(k)})^j = \frac{1}{(n-1)!} \left[\frac{2^{\frac{j}{2}}}{\theta^{\frac{j}{2}} k^{\frac{j}{2\alpha}}} \Gamma \left\{ \frac{j}{2\alpha} + n, 0 \right\} \right] = \left(\sqrt{\frac{2}{k^{1/\alpha}\theta}} \right)^j \frac{\Gamma\left(\frac{j}{2\alpha} + n\right)}{(n-1)!}, \tag{2.8}$$

as obtained by Kamps [5] and at $k = 1$, we get the result as given in Nagaraja [22].

iv) Setting $\alpha = \frac{1}{2}$ in (2.8), we get the exact expression for single moments of generalized upper records from the exponential distribution

$$E\left(Y_n^{(k)}\right)^j = \frac{1}{\left(k\sqrt{\frac{\theta}{2}}\right)^j} \frac{\Gamma(j+n)}{(n-1)!},$$

as obtained by Kamps [5] and at $k = 1$ in Ahsanullah [23].

v) Setting $\alpha = 1$ in (2.8), we get the exact expression for single moments of generalized upper records from Rayleigh distribution

$$E\left(Y_n^{(k)}\right)^j = \left(\sqrt{\frac{2}{k\theta}} \right)^j \frac{\Gamma\left(\frac{j}{2} + n\right)}{(n-1)!},$$

at $k = 1$ we get the similar result as given in Ahsanullah and Shakil [24].

vi) Setting $\lambda = 0$ and $\alpha = 1$ in (2.1), we get the exact expression for single moments of generalized upper records from the LED

$$E(Y_n^{(k)})^j = \frac{1}{(n-1)!} \left[\sum_{u=0}^{j} \sum_{v=0}^{\infty} \binom{j}{u} \binom{\frac{j-u}{2}}{v} (-1)^u 2^v \frac{\beta^{j-2v}}{\theta^{j-v} k^v} \gamma \left\{ v + n, k \left(\frac{\beta^2}{2\theta} \right) \right\} \right.$$
$$\left. + \sum_{u=0}^{j} \sum_{v=0}^{\infty} \binom{j}{u} \binom{\frac{j-u}{2}}{v} (-1)^u 2^{\frac{j-u}{2}-v} \frac{\beta^{u+2v}}{\theta^{\frac{j+u}{2}+v} k^{\frac{j-u}{2}-v}} \Gamma \left\{ \frac{j-u}{2} - v + n, k \left(\frac{\beta^2}{2\theta} \right) \right\} \right].$$

vii) Setting $\beta = 0$ in (2.7), we get the exact expression for single moments of generalized upper records from the generalized Rayleigh distribution

$$E\left(Y_n^{(k)}\right)^j = \frac{1}{(n-1)!} \left[\frac{(2\theta\lambda)^{\frac{j}{2}}}{\theta^j} \gamma \{n, k\lambda^\alpha\} + \frac{2^{\frac{j}{2}}}{\theta^{\frac{j}{2}} k^{\frac{j}{2\alpha}}} \Gamma \left\{ \frac{j}{2\alpha} + n, k\lambda^\alpha \right\} \right]$$
$$= \left(\sqrt{\frac{2}{\theta}} \right)^j \frac{1}{(n-1)!} \left[\lambda^{\frac{j}{2}} \gamma \{n, k\lambda^\alpha\} + \frac{1}{k^{\frac{j}{2\alpha}}} \Gamma \left\{ \frac{j}{2\alpha} + n, k\lambda^\alpha \right\} \right].$$

Theorem 2.2. For the distribution given in (1.1), fix a positive integer $k \geq 1$, for $n \geq 1$ and $j = 0, 1, \ldots$

$$
E\left(Y_n^{(k)}\right)^j = \frac{n\alpha\theta}{(j+2)\lambda} \left[\left(1 + \frac{j+2}{2n\alpha}\right) E\left(Y_n^{(k)}\right)^{j+2} - E\left(Y_{n+1}^{(k)}\right)^{j+2}\right]
$$

$$
+ \frac{n\alpha\beta}{(j+1)\lambda} \left[\left(1 + \frac{j+1}{n\alpha}\right) E\left(Y_n^{(k)}\right)^{j+1} - E\left(Y_{n+1}^{(k)}\right)^{j+1}\right].
$$

(2.9)

Proof. From (1.4) for $n \geq 1$ and $j = 0, 1, \ldots$, we have

$$
\frac{\theta}{2} \left[E\left(Y_n^{(k)}\right)^{j+2}\right] + \beta \left[E\left(Y_n^{(k)}\right)^{j+1}\right] - \lambda \left[E\left(Y_n^{(k)}\right)^j\right] = \frac{k^n}{(n-1)!} \times \int_\Psi^\infty \left(\frac{\theta}{2}x^2 + \beta x - \lambda\right) x^j \left[-\ln \overline{F}(x)\right]^{n-1} \left(\overline{F}(x)\right)^{k-1} f(x)dx,
$$

(2.10)

on using (1.3) in (2.10), we get

$$
\frac{\theta}{2} \left[E(Y_n^{(k)})^{j+2}\right] + \beta \left[E(Y_n^{(k)})^{j+1}\right] - \lambda \left[E(Y_n^{(k)})^j\right]
$$

$$
= \frac{k^n}{(n-1)!} \int_\Psi^\infty \alpha(\beta + \theta x)x^j \left[-\ln \overline{F}(x)\right]^n \left[\overline{F}(x)\right]^k dx
$$

$$
= \frac{\alpha\beta k^n}{(n-1)!} \int_\Psi^\infty x^j \left[-\ln \overline{F}(x)\right]^n \left[\overline{F}(x)\right]^k dx + \frac{\alpha\theta k^n}{(n-1)!} \int_\Psi^\infty x^{j+1} \left[-\ln \overline{F}(x)\right]^n \left[\overline{F}(x)\right]^k dx.
$$

(2.11)

Now applying Lemma 2.1 stated in Khan *et al.* [19] to (2.11), we have

$$
\frac{\theta}{2} \left[E(Y_n^{(k)})^{j+2}\right] + \beta \left[E(Y_n^{(k)})^{j+1}\right] - \lambda \left[E(Y_n^{(k)})^j\right]
$$

$$
= \frac{n\alpha\theta}{j+2} \left[E\left(Y_{n+1}^{(k)}\right)^{j+2} - E(Y_n^{(k)})^{j+2}\right] + \frac{n\alpha\beta}{j+1} \left[E\left(Y_{n+1}^{(k)}\right)^{j+1} - E(Y_n^{(k)})^{j+1}\right],
$$

(2.12)

after arranging (2.12) we get the yields given in (2.9).

Remark 2.2. If we put

i) $\beta = \lambda = 0$ in (2.11) we get the recurrence relations for single moment of generalized upper records from Weibull distribution

$$
E\left(Y_{n+1}^{(k)}\right)^{j+2} = \left(1 + \frac{j+2}{2n\alpha}\right) E(Y_n^{(k)})^{j+2},
$$

at $j' = j + 2$, we get similar result as obtained by Khan and Khan [18].

ii) $\beta = \lambda = 0$ and $\alpha = \frac{1}{2}$ in (2.11) we get the recurrence relations for single moment of generalized upper records from the exponential distribution

$$
E\left(Y_{n+1}^{(k)}\right)^{j+2} = \left(1 + \frac{j+2}{n}\right) E(Y_n^{(k)})^{j+2},
$$

at $j' = j + 2$, we get similar result as obtained by Khan and Khan [18].

iii) $\beta = \lambda = 0$ and $\alpha = 1$ in (2.11) we get the recurrence relations for single moment of generalized upper records from Rayleigh distribution

$$
E\left(Y_{n+1}^{(k)}\right)^{j+2} = \left(1 + \frac{j+2}{2n}\right) E(Y_n^{(k)})^{j+2},
$$

at $j' = j + 2$, we get similar result as obtained by Khan and Khan [18].

iv) $\lambda = 0$ and $\alpha = 1$ in (2.11) we get the recurrence relations for single moment of generalized upper records from the LED

$$E\left(Y_{n+1}^{(k)}\right)^{j+2} = \left(1 + \frac{j+2}{2n}\right) E\left(Y_n^{(k)}\right)^{j+2} + \frac{(j+2)\beta}{(j+1)\theta}\left[\left(1 + \frac{j+1}{n}\right) E\left(Y_n^{(k)}\right)^{j+1} - E\left(Y_{n+1}^{(k)}\right)^{j+1}\right].$$

v) $\beta = 0$ in (2.8) we get the recurrence relations for single moment of generalized upper records from the generalized Rayleigh distribution

$$E(Y_n^{(k)})^j = \frac{n\alpha\theta}{(j+2)\lambda}\left[\left(1 + \frac{j+2}{2n\alpha}\right) E\left(Y_n^{(k)}\right)^{j+2} - E\left(Y_{n+1}^{(k)}\right)^{j+2}\right].$$

Corollary 2.2. The recurrence relation for single moments of the upper record values from GLED has the form

$$\lambda E\left(X_{U_n}\right)^j = \frac{n\alpha\theta}{j+2}\left[\left(1 + \frac{j+2}{2n\alpha}\right) E\left(X_{U_n}\right)^{j+2} - E\left(X_{U_{n+1}}\right)^{j+2}\right]$$

$$+ \frac{n\alpha\beta}{j+1}\left[\left(1 + \frac{j+1}{n\alpha}\right) E\left(X_{U_n}\right)^{j+1} - E\left(X_{U_{n+1}}\right)^{j+1}\right]. \tag{2.13}$$

Remark 2.3. If we put

i) $\beta = \lambda = 0$ in (2.12), we get the recurrence relations for single moment of upper records from Weibull distribution, which is similar to the result as obtained by Balakrishnan and Chan [25] for $j' = j + 2$.

ii) $\beta = \lambda = 0$ and $\alpha = \frac{1}{2}$ and (2.12) in, the results for the single moments of upper records obtained by Balakrishnan and Chan [25] at $j' = j + 2$ for the exponential distribution is deduced.

iii) $\beta = \lambda = 0$ and $\alpha = 1$ in (2.12), the results for the single moments of upper records obtained by Balakrishnan and Chan [25] at $j' = j+2$ for Rayleigh distribution is deduced.

iv) $\lambda = 0$ and $\alpha = 1$ in (2.12), we get the recurrence relations for single moments of upper records from the LED

$$E\left(X_{U_{n+1}}\right)^{j+2} = \left(1 + \frac{j+2}{2n}\right) E(X_{U_n})^{j+2} + \frac{(j+2)\beta}{(j+|1)\theta}\left[\left(1 + \frac{j+1}{n}\right) E(X_{U_n})^{j+1} - E(X_{U_{n+1}})^{j+1}\right].$$

v) Setting $\beta = 0$ in (2.12), we get the recurrence relations for single moments of upper records from the generalized Rayleigh distribution

$$E\left(X_{U_n}\right)^j = \frac{n\alpha\theta}{(j+2)\lambda}\left[\left(1 + \frac{j+2}{2n\alpha}\right) E\left(X_{U_n}\right)^{j+2} - E\left(X_{U_{n+1}}\right)^{j+2}\right].$$

Numerical computations for moments of k-th upper record values from the GLED for arbitrary chosen values of α, β, θ. Setting $\lambda = 0$ without loss of generality and various sample size $n = 1, 2, \ldots 10$ are given in Tables 1 and 2.

Table 1 | Moments of the generalized linear exponential distribution(GLED).

	$E(X)$			$E(X)^2$		
	$\alpha = 0.5, \theta = 1, \beta = 2$			$\alpha = 0.5, \theta = 1, \beta = 2$		
n	$k = 1$	$k = 2$	$k = 3$	$k = 1$	$k = 2$	$k = 3$
1	0.736241	0.2210499	0.1040861	2.110072	0.2316011	0.0561998
2	1.947826	0.6230462	0.3010611	8.417389	1.015631	0.2581783
3	3.449258	1.16617	0.5780411	20.40594	2.670637	0.7090046
4	5.12462	1.816237	0.9222053	39.00304	5.470104	1.511246
5	6.905298	2.546131	1.321817	64.75762	9.630955	2.758801
6	8.751285	3.335303	1.766684	97.98972	15.31757	4.533193
7	10.63908	4.168577	2.248265	138.8874	22.65138	6.90277
8	12.55456	5.03488	2.759559	187.5635	31.72096	9.923529
9	14.48904	5.926162	3.294916	244.0877	42.59071	13.64067
10	16.43696	6.836557	3.849815	308.5043	55.30755	18.09037

Table 2 | Moments of the generalized linear exponential distribution(GLED).

	$E(X)^3$			$E(X)^4$		
	$\alpha = 0.5, \theta = 1, \beta = 2$			$\alpha = 0.5, \theta = 1, \beta = 2$		
n	k = 1	k = 2	k = 3	k = 1	k = 2	k = 3
1	10.71348	0.479532	0.064216	77.53965	1.505019	0.116498
2	55.65176	2.772377	0.393788	490.8589	10.64161	0.879686
3	168.2068	9.18847	1.377652	1762.711	42.06369	3.692394
4	388.0785	22.94524	3.613889	4734.55	122.7896	11.38394
5	758.6281	48.00968	7.906454	10597.46	295.4639	28.7853
6	1325.249	88.92437	15.24622	20908.68	620.8854	63.26942
7	2134.446	150.654	26.78464	37600.07	1179.848	125.224
8	3233.338	238.4644	43.80586	62982.53	2074.428	228.4448
9	4669.385	357.8329	67.7008	99748.23	3428.868	390.451
10	6490.247	514.3848	99.94468	150971.8	5390.161	632.7313

3. RELATIONS FOR PRODUCT MOMENTS

In this section, we derived the recurrence relations for product moments of generalized upper record values. Before proving the main result, we shall prove the following lemma:

Lemma 3.1. Fix a positive integer $k \geq 1$, for $1 \leq m \leq n - 2$ and $i, j = 0, 1, \ldots,$

$$
E\left[\left(Y_{m+1}^{(k)}\right)^i \left(Y_n^{(k)}\right)^j\right] - E\left[\left(Y_m^{(k)}\right)^i \left(Y_n^{(k)}\right)^j\right] = \frac{ik^n}{m!(n-m-1)!} \int_{-\infty}^{\infty} \int_{-\infty}^{y} x^{i-1} y^j
$$
$$
\times [-\ln \overline{F}(x)]^m [\ln \overline{F}(x) - \ln \overline{F}(y)]^{n-m-1} [\overline{F}(y)]^{k-1} f(y) dx\, dy.
$$

(3.1)

Proof. From (1.5), we have

$$
E\left[\left(Y_{m+1}^{(k)}\right)^i \left(Y_n^{(k)}\right)^j\right] - E\left[\left(Y_m^{(k)}\right)^i \left(Y_n^{(k)}\right)^j\right] = \frac{k^n}{(m-1)!(n-m-1)!}
$$
$$
\times \int_{-\infty}^{\infty} \int_{-\infty}^{y} x^i y^j \left[-\ln \overline{F}(x)\right]^{m-1} \frac{f(x)}{\overline{F}(x)} \left[\ln \overline{F}(x) - \ln \overline{F}(y)\right]^{n-m-2}
$$
$$
\times \left[\overline{F}(y)\right]^{k-1} f(y) \left\{ \left[-\ln \overline{F}(x)\right] \frac{(n-m-1)}{m} - \left[\ln \overline{F}(x) - \ln \overline{F}(y)\right] \right\} dx\, dy.
$$

(3.2)

Let

$$
h(x, y) = \frac{1}{m} \left[\ln \overline{F}(x) - \ln \overline{F}(y)\right]^{n-m-1} \left[-\ln \overline{F}(x)\right]^m
$$

(3.3)

$$
\frac{\partial}{\partial x} h(x, y) = -\left[-\ln \overline{F}(x)\right]^{m-1} \frac{f(x)}{\overline{F}(x)} \left[\ln \overline{F}(x) - \ln \overline{F}(y)\right]^{n-m-2}
$$
$$
\times \left\{ \left[-\ln \overline{F}(x)\right] \frac{(n-m-1)}{m} - \left[\ln \overline{F}(x) - \ln \overline{F}(y)\right] \right\}.
$$

(3.4)

On using the value of (3.4) in (3.2), we find that

$$
E\left[\left(Y_{m+1}^{(k)}\right)^i \left(Y_n^{(k)}\right)^j\right] - E\left[\left(Y_m^{(k)}\right)^i \left(Y_n^{(k)}\right)^j\right] = \frac{-k^n}{(m-1)!(n-m-1)!}
$$
$$
\times \int_{-\infty}^{\infty} y^j \left[\overline{F}(y)\right]^{k-1} f(y) \left\{ \int_{-\infty}^{y} x^i \frac{\partial}{\partial x} h(x, y) dx \right\} dy.
$$

(3.5)

Now, in view of (3.3)

$$
\int_{-\infty}^{y} x^i \frac{\partial}{\partial x} h(x, y) dx = -\frac{i}{m} \int_{-\infty}^{y} x^{i-1} \left[-\ln \overline{F}(x)\right]^m \left[\ln \overline{F}(x) - \ln \overline{F}(y)\right]^{n-m-1} dx.
$$

(3.6)

On substituting (3.6) in (3.5) and simplifying, we get the required result.

Theorem 3.1. For the distribution given in (1.1) and $k \geq 1$, $m \geq 1$ and $i, j = 0, 1, ...$

$$
\begin{aligned}
E\left[\left(Y_m^{(k)}\right)^i \left(Y_{m+1}^{(k)}\right)^j\right] &= \frac{m\alpha\theta}{(i+2)\lambda}\left[\left(1 + \frac{i+2}{2m\alpha}\right)E\left[\left(Y_m^{(k)}\right)^{i+2}\left(Y_{m+1}^{(k)}\right)^j\right] - E\left[(Y_{m+1}^{(k)})^{i+j+2}\right]\right] \\
&\quad + \frac{m\alpha\beta}{(i+1)\lambda}\left[\left(1 + \frac{i+1}{m\alpha}\right)E\left[\left(Y_m^{(k)}\right)^{i+1}\left(Y_{m+1}^{(k)}\right)^j\right] - E\left[\left(Y_{m+1}^{(k)}\right)^{i+j+1}\right]\right]
\end{aligned}
\tag{3.7}
$$

and for $1 \leq m \leq n-2$, and $i, j = 0, 1, ...$

$$
\begin{aligned}
E\left[\left(Y_m^{(k)}\right)^i \left(Y_n^{(k)}\right)^j\right] &= \frac{m\alpha\theta}{(i+2)\lambda}\left[\left(1 + \frac{i+2}{2m\alpha}\right)E\left[\left(Y_m^{(k)}\right)^{i+2}(Y_n^{(k)})^j\right] - E\left[\left(Y_{m+1}^{(k)}\right)^{i+2}\left(Y_n^{(k)}\right)^j\right]\right] \\
&\quad + \frac{m\alpha\beta}{(i+1)\lambda}\left[\left(1 + \frac{i+1}{m\alpha}\right)E\left[\left(Y_m^{(k)}\right)^{i+1}(Y_n^{(k)})^j\right] - E\left[\left(Y_{m+1}^{(k)}\right)^{i+1}\left(Y_n^{(k)}\right)^j\right]\right].
\end{aligned}
\tag{3.8}
$$

Proof. From (1.5), for $m \leq n-1$, we have

$$
\frac{\theta}{2}E\left[\left(Y_m^{(k)}\right)^{i+2}\left(Y_n^{(k)}\right)^j\right] + \beta E\left[\left(Y_m^{(k)}\right)^{i+1}\left(Y_n^{(k)}\right)^j\right] - \lambda E\left[\left(Y_m^{(k)}\right)^i\left(Y_n^{(k)}\right)^j\right]
$$

$$
\begin{aligned}
&= \frac{k^n}{(m-1)!(n-m-1)!}\int_\Psi^\Psi \int_y^\Psi \left(\frac{\theta}{2}x^2 + \beta x - \lambda\right)x^i y^j\left[-\ln\overline{F}(x)\right]^{m-1}\frac{f(x)}{\overline{F}(x)} \\
&\quad \times \left[\ln\overline{F}(x) - \ln\overline{F}(y)\right]^{n-m-1}\left[\overline{F}(y)\right]^{k-1}f(y)dx\,dy.
\end{aligned}
$$

In view of (1.3),

$$
\frac{\theta}{2}E\left[\left(Y_m^{(k)}\right)^{i+2}\left(Y_n^{(k)}\right)^j\right] + \beta E\left[\left(Y_m^{(k)}\right)^{i+1}\left(Y_n^{(k)}\right)^j\right] - \lambda E\left[\left(Y_m^{(k)}\right)^i\left(Y_n^{(k)}\right)^j\right]
$$

$$
\begin{aligned}
&= \frac{\alpha k^n}{(m-1)!(n-m-1)!}\int_\Psi^\infty \int_\Psi^y (\beta + \theta x)x^i y^j[-\ln\overline{F}(x)]^m \\
&\quad \times [\ln\overline{F}(x) - \ln\overline{F}(y)]^{n-m-1}[\overline{F}(y)]^{k-1}f(y)dx\,dy.
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\alpha\beta k^n}{(m-1)!(n-m-1)!}\int_\Psi^\infty \int_\Psi^y x^i y^j[-\ln\overline{F}(x)]^m[\ln\overline{F}(x) - \ln\overline{F}(y)]^{n-m-1} \\
&\quad \times [\overline{F}(y)]^{k-1}f(y)dx\,dy + \frac{\alpha\theta k^n}{(m-1)!(n-m-1)!}\int_\Psi^\infty \int_\Psi^y x^{i+1} y^j[-\ln\overline{F}(x)]^m \\
&\quad \times [\ln\overline{F}(x) - \ln\overline{F}(y)]^{n-m-1}[\overline{F}(y)]^{k-1}f(y)dx\,dy.
\end{aligned}
$$

Using Lemma 3.1, we get

$$
\frac{\theta}{2}E\left[\left(Y_m^{(k)}\right)^{i+2}\left(Y_n^{(k)}\right)^j\right] + \beta E\left[\left(Y_m^{(k)}\right)^{i+1}\left(Y_n^{(k)}\right)^j\right] - \lambda E\left[\left(Y_m^{(k)}\right)^i\left(Y_n^{(k)}\right)^j\right]
$$

$$
\begin{aligned}
&= \frac{m\alpha\theta}{i+2}\left[E[(Y_{m+1}^{(k)})^{i+2}(Y_n^{(k)})^j] - E[(Y_m^{(k)})^{i+2}(Y_n^{(k)})^j]\right] \\
&\quad + \frac{m\alpha\beta}{i+1}\left[E[(Y_{m+1}^{(k)})^{i+1}(Y_n^{(k)})^j] - E[(Y_m^{(k)})^{i+1}(Y_n^{(k)})^j],\right]
\end{aligned}
\tag{3.9}
$$

arranging (3.9) we get the result as given in (3.8).

Now putting $n = m + 1$ and noting that $E[(Y_m^{(k)})^i(Y_m^{(k)})^j] = E[(Y_m^{(k)})^{i+j}]$, the recurrence relation given in (3.7) can be easily obtained on the same line of proof (3.8).

Remark 3.1. If we put

i): $\beta = \lambda = 0$ in (3.9), we get the recurrence relations for product moment of generalized upper records from Weibull distribution

$$E\left[\left(Y_{m+1}^{(k)}\right)^{i+2}\left(Y_n^{(k)}\right)^j\right] = \left(1 + \frac{i+2}{2\alpha m}\right)E\left[\left(Y_m^{(k)}\right)^{i+2}\left(Y_n^{(k)}\right)^j\right],$$

at $i' = i + 2$, we get similar result as obtained by Khan and Khan [18].

ii): $\beta = \lambda = 0$ and $\alpha = \frac{1}{2}$ in (3.9), we get the recurrence relations for product moment of generalized upper records from the exponential distribution

$$E\left[\left(Y_{m+1}^{(k)}\right)^{i+2}\left(Y_n^{(k)}\right)^j\right] = \left(1 + \frac{i+2}{m}\right)E\left[\left(Y_m^{(k)}\right)^{i+2}\left(Y_n^{(k)}\right)^j\right],$$

at $i' = i + 2$, we get similar result as obtained by Khan and Khan [18].

iii): $\beta = \lambda = 0$ and $\alpha = 1$ in (3.9), we get the recurrence relations for product moment of generalized upper records from Rayleigh distribution

$$E\left[\left(Y_{m+1}^{(k)}\right)^{i+2}\left(Y_n^{(k)}\right)^j\right] = \left(1 + \frac{i+2}{2m}\right)E\left[\left(Y_m^{(k)}\right)^{i+2}\left(Y_n^{(k)}\right)^j\right],$$

at $i' = i + 2$, we get similar result as obtained by Khan and Khan [18].

iv): $\lambda = 0$ and $\alpha = 1$ in (3.9), we get the recurrence relations for product moment of generalized upper records from the LED

$$E\left[\left(Y_{m+1}^{(k)}\right)^{i+2}\left(Y_n^{(k)}\right)^j\right] = \left(1 + \frac{i+2}{2m}\right)E\left[\left(Y_m^{(k)}\right)^{i+2}\left(Y_n^{(k)}\right)^j\right]$$
$$+ \frac{(i+2)\beta}{(i+1)\theta}\left[\left(1 + \frac{i+1}{m}\right)E\left[\left(Y_m^{(k)}\right)^{i+1}\left(Y_n^{(k)}\right)^j\right] - E\left[\left(Y_{m+1}^{(k)}\right)^{i+1}\left(Y_n^{(k)}\right)^j\right]\right].$$

v):   Setting $\beta = 0$ in (3.8) we get the recurrence relations for product moment of generalized upper records from the generalized Rayleigh distribution

$$E\left[\left(Y_m^{(k)}\right)^i\left(Y_n^{(k)}\right)^j\right] = \frac{m\alpha\theta}{(i+2)\lambda}\left[\left(1 + \frac{i+2}{2m\alpha}\right)E\left[\left(Y_m^{(k)}\right)^{i+2}\left(Y_n^{(k)}\right)^j\right] - E\left[\left(Y_{m+1}^{(k)}\right)^{i+2}\left(Y_n^{(k)}\right)^j\right]\right].$$

vi):   Putting $j = 0$ in (3.8) we get the recurrence relation for single moment of generalized record value from the GLED.

Corollary 3.1. The recurrence relation for product moments of the upper record values from GLED has the form

$$\lambda E\left[\left(X_{U_m}\right)^i\left(X_{U_n}\right)^j\right] = \frac{m\alpha\theta}{i+2}\left[\left(1 + \frac{i+2}{2m\alpha}\right)E\left[\left(X_{U_m}\right)^{i+2}\left(X_{U_n}\right)^j\right] - E\left[\left(X_{U_{m+1}}\right)^{i+2}\left(X_{U_n}\right)^j\right]\right]$$
$$+ \frac{m\alpha\beta}{i+1}\left[\left(1 + \frac{i+1}{m\alpha}\right)E\left[\left(X_{U_m}\right)^{i+1}\left(X_{U_n}\right)^j\right] - E\left[\left(X_{U_{m+1}}\right)^{i+1}\left(X_{U_n}\right)^j\right]\right]. \tag{3.10}$$

Remark 3.2. If we put

i): $\beta = \lambda = 0$ in (3.10), we get the recurrence relation of upper records from Weibull distribution, which is similar to the result as given by Balakrishnan and Chan [25] for $i' = i + 2$.

ii): $\beta = \lambda = 0$ and $\alpha = \frac{1}{2}$ in (3.10), we get the recurrence relation of upper records from the exponential distribution, which is similar to the result as given by Balakrishnan and Chan [25] for $i' = i + 2$.

iii): $\beta = \lambda = 0$ and $\alpha = 1$ in (3.10), we get the recurrence relation of upper records from Rayleigh distribution, which is similar to the result as given by Balakrishnan and Chan [25] for $i' = i + 2$.

iv): $\lambda = 0$ and $\alpha = 1$ in (3.10), we get the recurrence relations for upper records from the LED

$$E\left[\left(X_{U_{m+1}}\right)^{i+2}\left(X_{U_n}\right)^j\right] = \left(1 + \frac{i+2}{2m}\right) E\left[\left(X_{U_m}\right)^{i+2}\left(X_{U_n}\right)^j\right]$$
$$+ \frac{(i+2)\beta}{(i+1)\theta}\left[\left(1 + \frac{i+1}{m}\right) E\left[\left(X_{U_m}\right)^{i+i}\left(X_{U_n}\right)^j\right] - E\left[\left(X_{U_{m+1}}\right)^{i+1}\left(X_{U_n}\right)^j\right]\right].$$

v):   Setting $\beta = 0$ in (3.10) we get the recurrence relations of upper records from the generalized Rayleigh distribution

$$E\left[\left(X_{U_m}\right)^i\left(X_{U_n}\right)^j\right]$$
$$= \frac{m\alpha\theta}{(i+2)\lambda}\left[\left(1 + \frac{i+2}{2m\alpha}\right) E\left[\left(X_{U_m}\right)^{i+2}\left(X_{U_n}\right)^j\right] - E\left[\left(X_{U_{m+1}}\right)^{i+2}\left(X_{U_n}\right)^j\right]\right].$$

vi):   Putting $j = 0$ in (3.10) we get (2.12), the recurrence relation for single moment of upper record value from the GLED.

4. CHARACTERIZATION

This section contains the characterizations of GLED, we start with the following result of Lin [26].

PROPOSITION. Let n_0 be any fixed non-negative integer, $-\infty \le a < b \le \infty$ and $g(x) \ge 0$ an absolutely continuous function with $g'(x) \ne 0$ a.e. on (a,b). Then the sequence of functions $\left\{(g(x))^n e^{-g(x)}, n \ge n_0\right\}$ is complete in $L(a,b)$ if $g(x)$ is strictly monotone on (a,b).

Theorem 4.1.   Fix a positive integer $k \ge 1$ and let j be any non-negative integer. A necessary and sufficient condition for a random variable X to be distributed with *pdf* given by (1.1) is that

$$E\left(Y_n^{(k)}\right)^j = \frac{n\alpha\theta}{(j+2)\lambda}\left[\left(1 + \frac{j+2}{2n\alpha}\right) E\left(Y_n^{(k)}\right)^{j+2} - E\left(Y_{n+1}^{(k)}\right)^{j+2}\right]$$
$$+ \frac{n\alpha\beta}{(j+1)\lambda}\left[\left(1 + \frac{j+1}{n\alpha}\right) E\left(Y_n^{(k)}\right)^{j+1} - E\left(Y_{n+1}^{(k)}\right)^{j+1}\right]$$

(4.1)

for $n = 1, 2, \ldots$.

Proof.   The necessary part follows from (2.9). On the other hand if the recurrence relation (4.1) is satisfied, then on rearranging (4.1)

$$\frac{\theta}{2}\left[E\left(Y_n^{(k)}\right)^{j+2}\right] + \beta\left[E\left(Y_n^{(k)}\right)^{j+1}\right] - \lambda\left[E\left(Y_n^{(k)}\right)^j\right]$$
$$= \frac{n\alpha\theta}{j+2}\left[E\left(Y_{n+1}^{(k)}\right)^{j+2} - E\left(Y_n^{(k)}\right)^{j+2}\right] + \frac{n\alpha\beta}{j+1}\left[E\left(Y_{n+1}^{(k)}\right)^{j+1} - E\left(Y_n^{(k)}\right)^{j+1}\right]$$

in view of Khan *et al.* [19], we have

$$\frac{k^n}{(n-1)!}\int_\Psi^\infty \left(\frac{\theta}{2}x^2 + \beta x - \lambda\right) x^j\left[-\ln\overline{F}(x)\right]^{n-1}\left[\overline{F}(x)\right]^{k-1} f(x)dx$$

$$= \frac{n\alpha\theta}{j+2}\left[\frac{(j+2)k^n}{n!}\int_\Psi^\infty x^{j+1}\left[-\ln\overline{F}(x)\right]^n\left[\overline{F}(x)\right]^k dx\right]$$
$$+ \frac{n\alpha\beta}{j+1}\left[\frac{(j+1)k^n}{n!}\int_\Psi^\infty x^j\left[-\ln\overline{F}(x)\right]^n\left[\overline{F}(x)\right]^k dx\right],$$

which implies

$$\frac{k^n}{(n-1)!}\int_\Psi^\infty x^j\left[-\ln\overline{F}(x)\right]^{n-1}\left[\overline{F}(x)\right]^{k-1}$$
$$\times\left\{\left(\frac{\theta}{2}x^2 + \beta x - \lambda\right) f(x) - \alpha(\theta x + \beta)\left[-\ln\overline{F}(x)\right]\left[\overline{F}(x)\right]\right\} dx = 0$$

Now follow from the above proposition, we get

$$\left(\frac{\theta}{2}x^2 + \beta x - \lambda\right)f(x) = \alpha(\beta + \theta x)\left[-\ln \overline{F}(x)\right]\left[\overline{F}(x)\right]$$

which proves that $f(x)$ has the form as given in (1.3).

Theorem 4.2. For a positive integer k, i and j be a non-negative integer, a necessary and sufficient condition for a random variable X to be distributed with *pdf* given by (1.1) and for $1 \le m \le n - 2$, is that

$$E\left[\left(Y_m^{(k)}\right)^i\left(Y_n^{(k)}\right)^j\right] = \frac{m\alpha\theta}{(i+2)\lambda}\left[\left(1 + \frac{i+2}{2m\alpha}\right)E\left[\left(Y_m^{(k)}\right)^{i+2}\left(Y_n^{(k)}\right)^j\right] - E\left[\left(Y_{m+1}^{(k)}\right)^{i+2}\left(Y_n^{(k)}\right)^j\right]\right]$$
$$+ \frac{m\alpha\beta}{(i+1)\lambda}\left[\left(1 + \frac{i+1}{m\alpha}\right)E\left[\left(Y_m^{(k)}\right)^{i+1}\left(Y_n^{(k)}\right)^j\right] - E\left[\left(Y_{m+1}^{(k)}\right)^{i+1}\left(Y_n^{(k)}\right)^j\right]\right]. \tag{4.2}$$

Proof. The necessary part follows from (3.8). On the other hand if the relation in (4.2) is satisfied, then arranging the (4.2)

$$\frac{\theta}{2}E\left[\left(Y_m^{(k)}\right)^{i+2}\left(Y_n^{(k)}\right)^j\right] + \beta E\left[\left(Y_m^{(k)}\right)^{i+1}\left(Y_n^{(k)}\right)^j\right] - \lambda E\left[\left(Y_m^{(k)}\right)^i\left(Y_n^{(k)}\right)^j\right]$$

$$= \frac{m\alpha\theta}{i+2}\left[E\left[\left(Y_{m+1}^{(k)}\right)^{i+2}\left(Y_n^{(k)}\right)^j\right] - E\left[\left(Y_m^{(k)}\right)^{i+2}\left(Y_n^{(k)}\right)^j\right]\right]$$

$$+ \frac{m\alpha\beta}{i+1}\left[E\left[\left(Y_{m+1}^{(k)}\right)^{i+1}\left(Y_n^{(k)}\right)^j\right] - E\left[\left(Y_m^{(k)}\right)^{i+1}\left(Y_n^{(k)}\right)^j\right]\right].$$

On using Lemma 3.1, we have

$$\frac{k^n}{(m-1)!(n-m-1)!}\int_\Psi^\infty \int_\Psi^y \left(\frac{\theta}{2}x^2 + \beta x - \lambda\right)x^i y^j \left[-\ln \overline{F}(x)\right]^{m-1}\frac{f(x)}{\overline{F}(x)}$$

$$\times [\ln \overline{F}(x) - \ln \overline{F}(y)]^{n-m-1}[\overline{F}(y)]^{k-1}f(y)dx\,dy$$

$$= \frac{\alpha\theta k^n}{(m-1)!(n-m-1)!} \times \int_\Psi^\infty \int_\Psi^y x^{i+1}y^j[-\ln \overline{F}(x)]^m[\ln \overline{F}(x) - \ln \overline{F}(y)]^{n-m-1}[\overline{F}(y)]^{k-1}f(y)dx\,dy$$

$$+ \frac{\alpha\beta k^n}{(m-1)!(n-m-1)!} + \int_\Psi^\infty \int_\Psi^y x^i y^j[-\ln \overline{F}(x)]^m[\ln \overline{F}(x) - \ln \overline{F}(y)]^{n-m-1}[\overline{F}(y)]^{k-1}f(y)dx\,dy.$$

which implies

$$\int_\Psi^\infty \int_\Psi^y x^i y^j\left[-\ln \overline{F}(x)\right]^{m-1}\left[\ln \overline{F}(x) - \ln \overline{F}(y)\right]^{n-m-1}\left[\overline{F}(y)\right]^{k-1}f(y)$$

$$\times \left\{\left(\frac{\theta}{2}x^2 + \beta x - \lambda\right)\frac{f(x)}{[\overline{F}(x)]} - \alpha(\theta x + \beta)\left[-\ln \overline{F}(x)\right]\right\}dx\,dy = 0.$$

Now, follow from the above proposition with

$$\left(\frac{\theta}{2}x^2 + \beta x - \lambda\right)f(x) = \alpha(\beta + \theta x)\left[-\ln \overline{F}(x)\right]\left[\overline{F}(x)\right]$$

which proves that $f(x)$ has the form as given in (1.3).

Theorem 4.3. Let X be a non-negative random variable having an absolutely continuous df $F(x)$ and $F(0) = 0$ and $0 \leq F(x) \leq 1$ for all $x > 0$, then

$$E\left[\xi\left(Y_n^{(k)}\right) \mid Y_m^{(k)} = x\right] = \xi(x)\left(\frac{k}{k+1}\right)^{n-m}, \tag{4.3}$$

if and only if

$$\overline{F}(x) = exp\left\{-\left(\frac{\theta}{2}x^2 + \beta x - \lambda\right)^\alpha\right\}, \quad x > \Psi.$$

where

$$\xi(t) = exp\left\{-\left(\frac{\theta}{2}t^2 + \beta t - \lambda\right)^\alpha\right\}, \quad t > \Psi.$$

Proof. From (1.6), we have

$$E[\xi(Y_n^{(k)}) \mid Y_m^{(k)} = x] = \frac{k^{n-m}}{(n-m-1)!}$$

$$\times \int_x^\infty \xi(y)\left[\ln \overline{F}(x) - \ln \overline{F}(y)\right]^{n-m-1}\left[\frac{\overline{F}(y)}{\overline{F}(x)}\right]^{k-1}\frac{f(y)}{\overline{F}(x)}dy. \tag{4.4}$$

By setting $u = \frac{\overline{F}(y)}{\overline{F}(x)} = \frac{exp\left\{-\left(\frac{\theta}{2}y^2+\beta y-\lambda\right)^\alpha\right\}}{exp\left\{-\left(\frac{\theta}{2}x^2+\beta x-\lambda\right)^\alpha\right\}}$ from (1.2) in (4.4), we have

$$E\left[\xi\left(Y_n^{(k)}\right) \mid \left(Y_m^{(k)}\right) = x\right] = \frac{k^{n-m}}{(n-m-1)!}\xi(x)\int_0^1 u^k[-\ln u]^{n-m-1}du. \tag{4.5}$$

Therefore we have (see Gradshteyn and Ryzhik [27], p. 551)

$$\int_0^1 [-\ln x]^{\mu-1}x^{\nu-1}dx = \frac{\Gamma\mu}{\nu^\mu}, \quad \mu > 0, \nu > 0. \tag{4.6}$$

On using (4.6) in (4.5), we have the result given in (4.3).

To prove sufficient part, we have

$$\frac{k^{n-m}}{(n-m-1)!}\int_x^\infty \xi(y)[-\ln \overline{F}(y) + \ln \overline{F}(x)]^{n-m-1}\left[\overline{F}(y)\right]^{k-1}f(y)dy$$

$$= [\overline{F}(x)]^k g_{n|m}(x), \tag{4.7}$$

where

$$g_{n|m}(x) = \xi(x)\left(\frac{k}{k+1}\right)^{n-m}.$$

Differentiating (4.7) both sides with respect to x, we get

$$-\frac{k^{n-m}}{(n-m-2)!}\frac{f(x)}{\overline{F}(x)}\int_x^\infty \xi(y)\left[-\ln \overline{F}(y) + \ln \overline{F}(x)\right]^{n-m-2}\left[\overline{F}(y)\right]^{k-1}f(y)dy$$

$$= g'_{n|m}(x)\left[\overline{F}(x)\right]^k - kg_{n|m}(x)\left[\overline{F}(x)\right]^{k-1}f(x)$$

or

$$-k\,g_{n|m+1}(x)[\overline{F}(x)]^{k-1}f(x) = g'_{n|m}(x)[\overline{F}(x)]^k - k\,g_{n|m}(x)\,[\overline{F}(x)]^{k-1}f(x).$$

Therefore,

$$\frac{f(x)}{\overline{F}(x)} = -\frac{g'_{n|m}(x)}{k[g_{n|m+1}(x) - g_{n|m}(x)]} = -\frac{\xi'(x)}{\xi(x)}, \tag{4.8}$$

where

$$g'_{n|m}(x) = \xi'(x)\left(\frac{k}{k+1}\right)^{n-m},$$
$$g_{n|m+1}(x) - g_{n|m}(x) = \frac{1}{k}\xi(x)\left(\frac{k}{k+1}\right)^{n-m},$$

Integrating both the sides (4.8) with respect to x between (Ψ, y), which gives $\overline{F}(x) = \xi(x)$, the sufficiency part is proved.

5. CONCLUSION

In this study, we demonstrate the explicit expression as well as recurrence relation for the moments of k-th upper record values from the GLED. These relations can be used to reduce the amount of direct computation and moments of any order can be calculated easily. To verify the designed models which is required in probability distribution, we used the results of the characterization. At the different values of parameters, we reduced some well-known results. We can explore our study for generalized order statistics which contains several models of order random variates.

CONFLICTS OF INTEREST

The authors are declare no competing interests.

AUTHORS' CONTRIBUTIONS

Mahfooz Alam proposed the concept the paper and calculate the existing results. M.A. Khan contributed in writing—original draft preparation. R.U. Khan performed revision and improve the quality of the draft. All authors have read and agreed to the published version of the manuscript.

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