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New Discrete Lifetime Distribution with Applications to Count Data

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ABSTRACT

In this paper, we present a new class of distribution called generalized Hermite–Genocchi distribution (GHGD). This model is obtained by compounding generalized Hermite–Genocchi polynomials given by Gould and Hopper with powers series distribution. Statistical properties and reliability characteristics are studied. The model has been applied to several real data. Finally, a simulation study is performed to assess the performance of the model.

1. INTRODUCTION

In this paper, we introduced a new discrete distribution based on the generalized Hermite polynomials given by, see [1]

$$H_{n,m}(x,y) = n! \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{y^k\, x^{n-mk}}{k!(n-mk)!}, \quad (n \geq 0,\ m \in \mathbb{N}).$$

For more details, see [2], [3] and [5].

Gupta and Jain [9] extended the Hermite distribution (HD) of the generalized HD defined by

$$P(Y=n) = e^{-(a+b)} \sum_{j=0}^{\left[\frac{n}{m}\right]} \frac{a^{(n-mj)} b^j}{(n-mj)!j!},$$

where $a \geq 0,\ b \geq 0$ and $m \in \mathbb{N}$.

The distribution has been applied to the frequency of bacteria in leucoytes and frequency of larvae in corn plants [6].

Moreover, there are a lot of popular statistical distributions that have specific applications, but sometimes, observable data contain distinct features not shown by these classic distributions. So to overcome these limitations, researchers often develop new distributions so that these new distributions can be used in these cases where the classical distributions don't provide any suitable fit. There are many techniques with which we can get new distribution, for more details see [7–9].

Recently, El-Desouky *et al.* [10] introduced a new generalized Hermite–Genocchi distribution (GHGD). By compounding (1) and powers series distribution defined new multivariate distribution called GHGD.

$$P(\underline{X}) = \mathbf{B} \prod_{i=1}^{r} (\alpha_{_{i-1}})^{x_i}\ \ H_{n,m}\left(\sum_{i=1}^{r} x_i + \gamma, \beta \right). \tag{1.1}$$

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$$\beta, \gamma \geq 0;\ r \geq 1,\ m \in \mathbb{N},$$

where

$$\frac{1}{\mathbf{B}} = {}_H M_n^{(r)}(\gamma, \beta; \overline{\alpha}_r) = \sum_{\ell_1,\ell_2,\ldots,\ell_r=0}^{\infty} \prod_{i=1}^{r} (\alpha_{i-1})^{\ell_i} H_{n,m}\left(\sum_{i=1}^{r}, \ell_i, +, \gamma, , , \beta\right),$$

${}_H M_n^{(r)}(\gamma, \beta; \overline{\alpha}_r)$ is convergent and positive for $\overline{\alpha}_r = (\alpha_0, \alpha_1, \ldots, \alpha_{r-1}), 0 < \alpha_r < 1$.

The paper is organized as follows: In Section 2, when set $r = 1$ in (1.1), we introduce a new univariate discrete distribution and discuss mathematical and statistical properties of the model. In Section 3, we introduce monotonic properties. In Section 4, reliability characteristics are obtained. In Section 5, moment and maximum likelihood estimates of unknown parameters are presented and simulation study is performed. In Section 6, we apply the new model to real data sets to illustrate the usefulness and applicability of the model. Graphical assesment of goodness of fit of the model based on empirical probability generating function is presented. Finally, in Section 7, conclusion and remarks are given.

2. GENERALIZED HERMITE–GENOCCHI DISTRIBUTION

Definition 2.1. A discrete random variable X taking value in the set $\mathbb{Z}^+ \cup \{0\}$ is said to follow GHGD with three parameters, that is $GHG(\alpha; \beta, \gamma)$, if its probability mass function can be written as

$$P(X = x; \alpha, \beta, \gamma) = \begin{cases} \dfrac{H_{n,m}(\gamma, \beta)}{G(\alpha; \beta, \gamma)}, & x = 0 \\ \\ \dfrac{\alpha^x \ H_{n,m}(x + \gamma, \beta)}{G(\alpha; \beta, \gamma)}, & x > 0, \end{cases} \tag{2.1}$$

where $\beta \geq 0$ is scale parameter, $\gamma \geq 0$ is shape parameter, $0 < \alpha < 1$ is shape parameter, $m \in \mathbb{N}$,

$$G(\alpha; \beta, \gamma) = \sum_{\ell=0}^{\infty} \alpha^{\ell} \quad H_{n,m}(\ell + \gamma, \beta),$$

and

$$H_{n,m}(x + \gamma, \beta) = \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{\beta^k}{k!} \frac{n!}{(n - mk)!} (x + \gamma)^{n-mk}.$$

$G(\alpha; \beta, \gamma)$ is convergent and positive for $0 < \alpha < 1$.

2.1. Structural Properties of GHGD Model

2.1.1. Shape and behavior of pmf plots of GHG distribution with serval values of parameters α, β and γ are present in Figure 1

Three examples in Figure 1 showing effects of scale and shape parameters.

2.1.2. Cumulative distribution function

The cumulative distribution function (cdf) of GHGD is given by

$$\begin{aligned} F(x) &= P(X \leq x) \\ &= 1 - P(X > x) \\ &= 1 - \frac{\alpha^{x+1} G(\alpha; \beta, x + \gamma + 1)}{G(\alpha; \beta, \gamma)}. \end{aligned}$$

Figure 2 showing shape and behavior of Cdf plots of GHG distribution with several values of parameters α, β and γ.

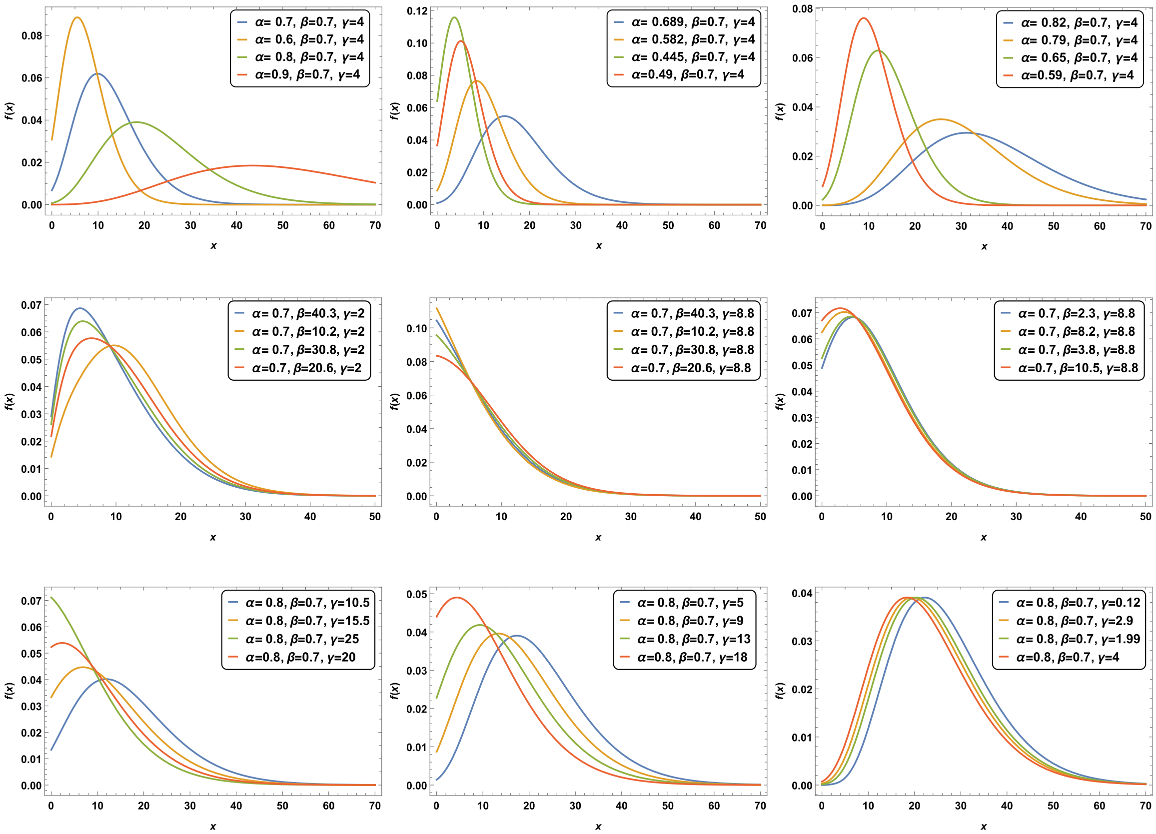


Figure 1 | Shape and behavior of pmf plots of GHGD with serval values of parameters α, β and γ.

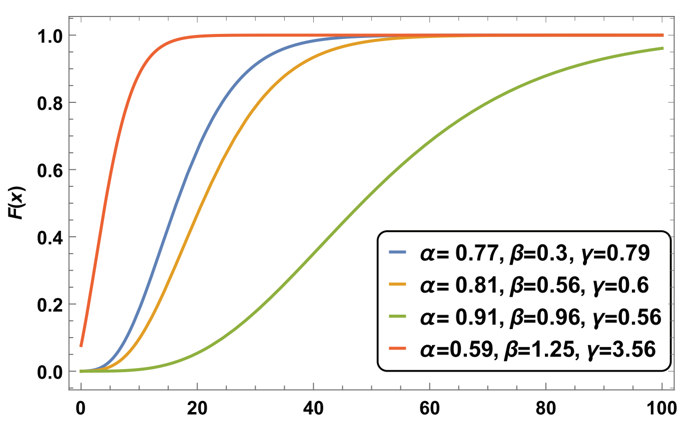


Figure 2 | Cdf of GHGD for different values of α, β and γ.

2.1.3. Moments and related measures

The moment-generating function of GHGD is given by

$$
\begin{aligned}
M_X(t) &= E(e^{tx}) \\
&= \frac{G(\alpha e^t; \beta, \gamma)}{G(\alpha; \beta, \gamma)}.
\end{aligned}
$$

The $r-th$ factorial moments $\mu_{[r]}$ is given by

$$\mu_{[r]} = E[(X)_r] = \frac{\sum_{x=0}^{\infty} (x)_{\underline{r}} \ \alpha^x \ H_{n,m}(x+\gamma,\beta)}{G(\alpha;\beta,\gamma)}.$$

The $r-th$ moments μ_r' is given by

$$\mu_r' = E(X^r) = \frac{\sum_{x=0}^{\infty} x^r \ \alpha^x \ H_{n,m}(x+\gamma,\beta)}{G(\alpha;\beta,\gamma)}.$$

The mean and variance are given, respectively, by

$$E(X) = \frac{\sum_{x=0}^{\infty} x \ \alpha^x \ H_{n,m}(x+\gamma,\beta)}{G(\alpha;\beta,\gamma)}.$$

$$Var(X) = \left[\frac{\sum_{x=0}^{\infty} x^2 \ \alpha^x \ H_{n,m}(x+\gamma,\beta)}{G(\alpha;\beta,\gamma)}\right] - \left[\frac{\sum_{x=0}^{\infty} x \ \alpha^x \ H_{n,m}(x+\gamma,\beta)}{G(\alpha;\beta,\gamma)}\right]^2.$$

The plots in Figure 3, it is apparent that both mean and variance of GHGD have bounds.

2.1.4. Over-dispersion

The over-dispersion (OD) index of GHGD is given by

$$OD = \frac{\sigma^2}{\mu} = \frac{\sum_{x=0}^{\infty} x^2 \ \alpha^x \ H_{n,m}(x+\gamma,\beta)}{\sum_{x=0}^{\infty} x \ \alpha^x \ H_{n,m}(x+\gamma,\beta)} - \frac{\sum_{x=0}^{\infty} x \ \alpha^x \ H_{n,m}(x+\gamma,\beta)}{G(\alpha;\beta,\gamma)}. \tag{2.2}$$

From Figure 3 and Eq. (2.2), we can obtain the following corollary:

Corollary 2.2.

1. $OD = (>)(<)\ 1$if and only if$\alpha = (>)(<)\ 0.4$, $\beta = (>)(<)\ 0.3$and$\gamma = (>)(<)\ 1$.
2. GHGD is no over-dispersion, over-dispersion and under-dispersion for $\alpha = (>)(<)\ 0.4$, $\beta = (>)(<)\ 0.3$ and $\gamma = (>)(<)\ 1$, respectively.

We obtained that numerically.

2.1.5. Surprise index

The surprise index (SI) of GHGD is given by

$$SI = \frac{E(P(X = x, \alpha; \beta, \gamma))}{P(X = x, \alpha; \beta, \gamma)}$$
$$= \left(\sum_{x=0}^{\infty} \frac{\alpha^{2x} \ \ (H_{n,m}(x + \gamma, \beta))^2}{G(\alpha; \beta, \gamma)} \right) / \left(\alpha^x \ \ H_{n,m}(x + \gamma, \beta) \right).$$

From Figure 4 for various value of α, β *and* γ, where α, β *and* γ, decreases, large values of x become more surprising.

2.1.6. Generating function

The probability-generating function of GHGD is given by

$$G_X(t) = E(t^X) = \sum_{x=0} P(X = x) \ t^x$$
$$= \frac{G(\alpha t; \beta, \gamma)}{G(\alpha; \beta, \gamma)}.$$

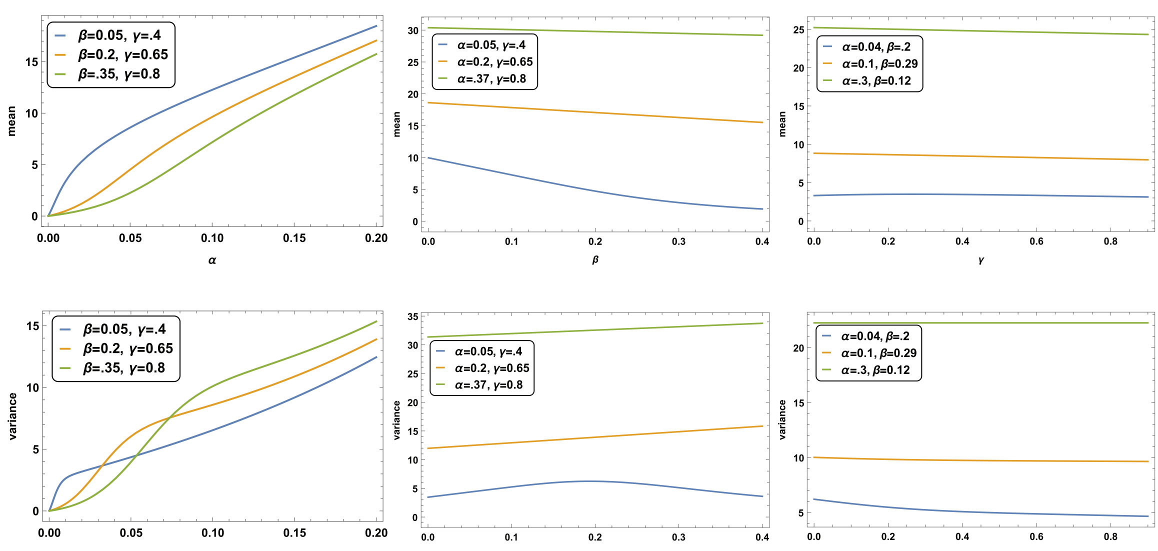


Figure 3 | Plots the mean and variance of GHGD with serval values of parameters α, β and γ.

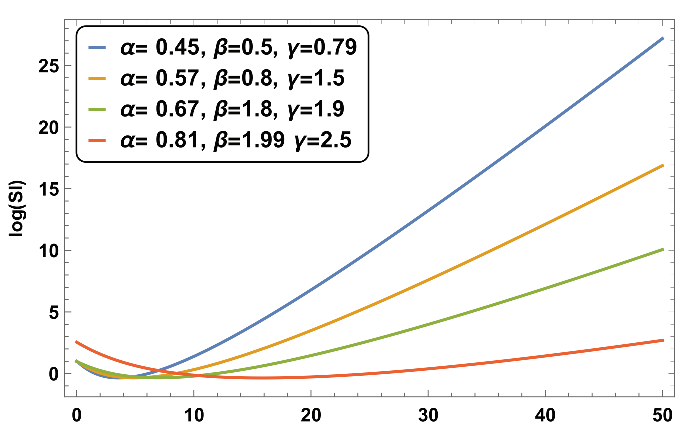


Figure 4 | log(SI)'s for GHGD.

3. MONOTONIC PROPERTIES

Log-concavity is an essential property of the probability distribution. Characteristics such as reliability function , failure rate, mean residual and moment of log-concave probability have specific properties see [11–14].

Theorem 3.1. *The GHG distribution is log-concave.*

Proof. Consider the function

$$b(x, \alpha, \beta, \gamma) = \frac{P(X = x + 1)}{P(X = x)} = \frac{\alpha\ H_{n,m}(x + \gamma + 1, \beta)}{H_{n,m}(x + \gamma, \beta)}.$$

Its derivative is given by

$$\frac{d\ b(x, \alpha, \beta, \gamma)}{dx} = \frac{\alpha \left(\sum_{[\frac{n}{m}]}^{k=0}, \frac{\beta^k\ n!\ (x + \gamma + 1)^{n-mk-1}}{k!\ (n - mk - 1)!} \right)}{(H_{n,m}(x + \gamma, \beta))} - \frac{\alpha \left(H_{n,m}, (, x, +, \gamma, +, 1, , , \beta,), \sum_{[\frac{n}{m}]}^{k=0}, \frac{\beta^k\ n!\ (x + \gamma)^{n-mk-1}}{k!\ (n - mk - 1)!} \right)}{(H_{n,m}(x + \gamma, \beta))^2} < 0.$$

Note that $b(x, \alpha, \beta, \gamma)$ is decreasing function in x for $0 < \alpha < 1, \beta > 0$ and $\gamma > 0$ thus, the $G(\alpha, \beta, \gamma)$ is log-concave. The behavior of GHG distribution can be illustrated as in Figure 1. ☒

Corollary 3.2. As a direct consequence of log-concavity, see [11], the following results hold for GHG distribution:

1 It is strongly unimodal.

2 It has all moments.

3 It has an increasing failure rate distribution.

4 It has monotonically decreasing mean residual function.

5 It remains log-concave if truncated.

6 It gives unimodal and log-concave distribution when convoluted with any other discrete distribution.

4. RELIABILITY PROPERTIES

The survival function of GHGD is given by

$$\begin{aligned} S(x) &= P(X > x) = 1 - P(X \leq x) \\ &= \frac{\alpha^{x+1}\ G(\alpha; \beta, x + \gamma + 1)}{G(\alpha; \beta, \gamma)}. \end{aligned} \tag{4.1}$$

In Figure 5, shape and behaviour of survival function plots of GHG distribution with several values of parameters α, β and γ.

Also, the hazard rate function is given by

$$\begin{aligned} h(x) &= \frac{P(X = x)}{S(x)} \\ &= \frac{H_{n,m}(x + \gamma, \beta)}{\alpha\ G(\alpha; \beta, \gamma + x + 1)}. \end{aligned} \tag{4.2}$$

The failure rate is increasing, see (Theorem 3.1) and (Corollary 3.2) and Figure 6.

The mean residual life (MRL) of the GHGD is given by

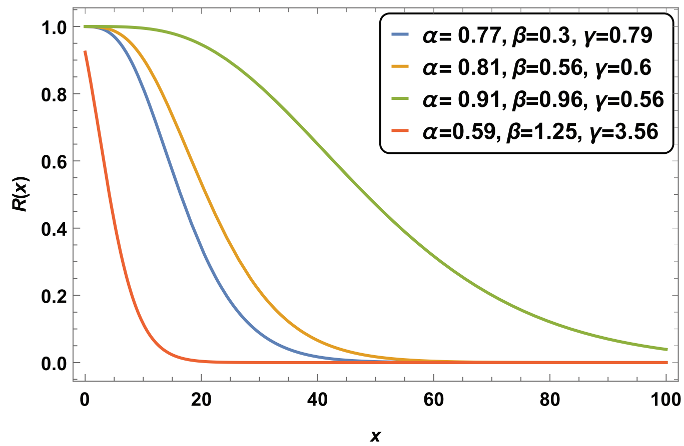


Figure 5 Survival function for GHGD.

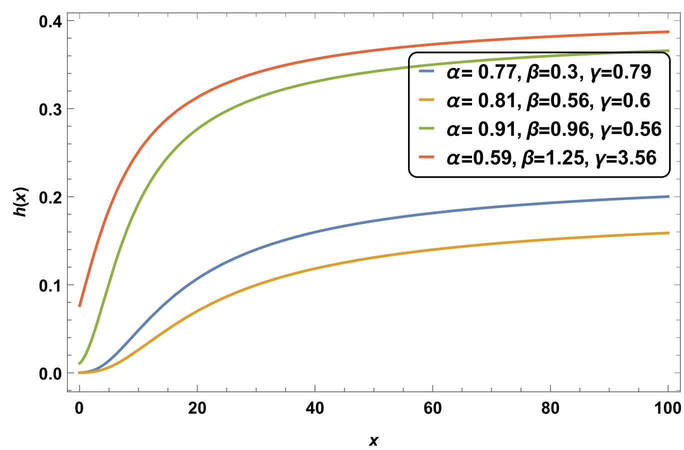


Figure 6 Hazard function for GHGD.

$$\mu(x) = E(X - \mathrm{x}|X \geq \mathrm{x}) = \frac{1}{S(x-1)} \sum_{y=x}^{\infty} S(y-1)$$
$$= \frac{\sum_{y=x}^{\infty} \alpha^y \; G(\alpha; \beta, \gamma + y)}{\alpha^x \; G(\alpha; \beta, \gamma + x)}.$$

The mean time to failure (MTTF) of GHGD is given by

$$\mu = \sum_{x=0}^{\infty} S(x)$$
$$= \sum_{x=0}^{\infty} \frac{\alpha^{x+1} \; G(\alpha; \beta, x + \gamma + 1)}{G(\alpha; \beta, \gamma)}.$$

The reversed hazard rate is given by

$$h^*(x) = \frac{P(X = t)}{P(X \leq t)}$$
$$= \frac{\alpha^x \; H_{n,m}(\gamma + x, \beta)}{G(\alpha; \beta, \gamma) - \alpha^{x+1} \; G(\alpha; \beta, \gamma + x + 1)}.$$

The shape and behavior of reversed hazard rate GHG distribution with several values of parameters α, β and γ, see Figure 7.

Definition 4.1. [13] A discrete distribution of nonnegative random variable is said to be

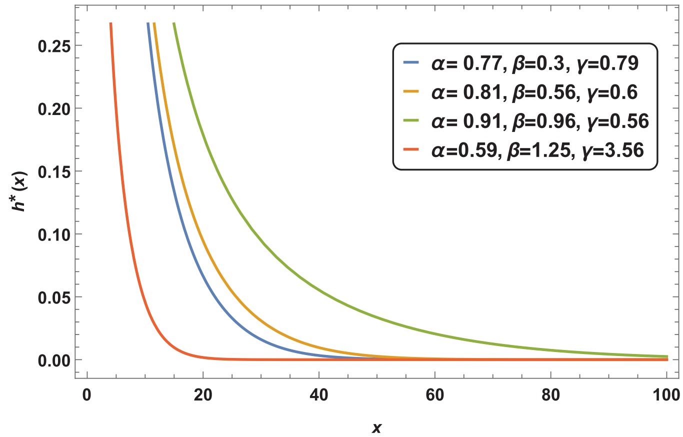


Figure 7 Reversed hazard function for GHGD.

- New better (worse) than used, denoted by NBU(NWU) if

$$S(x+y) \leq (\geq) S(x)S(y).$$

- New better (worse) than used in expectation, denoted by NBUE(NWUE) if

$$\sum_{j=0}^{\infty} S(t+j) \leq (\geq) \sum_{j=0}^{\infty} S(j)S(t).$$

Corollary 4.2. As a result of IFR, see [13], the following results hold:

1 GHGD is IFRA

2 GHGD is NBU

3 GHGD is DMRL

4 GHGD is NBUE.

5. PARAMETER ESTIMATION AND SIMULATION

5.1. Maximum Likelihood Estimators

Let $\mathbf{x} = (x_1, x_2, ..., x_n)$ be a random sample of size n drawn from GHGD. Then, the likelihood function of vector (α, β, γ) is given by

$$\begin{aligned} L(\alpha,\beta,\gamma) &= \prod_{i=1}^{N} \frac{\alpha^{x_i} \ H_{n,m}\left(x_i+\gamma,\beta\right)}{G(\alpha;\beta,\gamma)} \\ &= \left(\frac{1}{G(\alpha;\beta,\gamma)}\right)^{N} \alpha^{\sum_{i=1}^{N} x_i} \prod_{i=1}^{N} H_{n,m}\left(x_i+\gamma,\beta\right). \end{aligned}$$

The log-likelihood function can be written as

$$\begin{aligned} \log L = &-N \log\left(\sum_{\ell=0}^{\infty} \alpha^{\ell} \sum_{k=0}^{[\frac{n}{m}]} \frac{\beta^k \ \ n!}{k! \ \ (n-mk)!} \ (\ell+\gamma)^{n-mk}\right) \\ &+ \sum_{i=1}^{N} x_i \log\alpha + \sum_{i=1}^{N} \log\left(\sum_{k=0}^{[\frac{n}{m}]} \frac{\beta^k \ \ n!}{k! \ \ (n-mk)!} \ (x_i+\gamma)^{n-mk}\right). \end{aligned} \tag{5.1}$$

Computing the first partial derivatives of (5.1) with respect to $\alpha,\ \beta$ and γ, we get

$$\begin{aligned}\frac{\partial}{\partial\alpha}\log L &= \frac{-N}{G(\alpha;\beta,\gamma)}\sum_{\ell=0}^{\infty}\ell\ \alpha^{\ell-1}\ H_{n,m}\left(\ell+\gamma,\beta\right)+\frac{\sum_{i=1}^{N}x_i}{\alpha}\\ &= \frac{\sum_{i=1}^{N}x_i}{\alpha}-\frac{N}{\alpha}E(X).\end{aligned} \tag{5.2}$$

$$\begin{aligned}\frac{\partial}{\partial\beta}\log L &= \frac{-N}{G(\alpha;\beta,\gamma)}\sum_{\ell=0}^{\infty}\alpha^{\ell}\sum_{k=0}^{[\frac{n}{m}]}\frac{k\ \beta^{k-1}\ n!}{k!\ (n-mk)!}\left(\ell+\gamma\right)^{n-mk}\\ &+\sum_{i=1}^{N}\frac{1}{H_{n,m}\left(x_i+\gamma,\beta\right)}\sum_{k=0}^{[\frac{n}{m}]}\frac{k\ \beta^{k-1}\ n!}{k!\ (n-mk)!}\left(x_i+\gamma\right)^{n-mk}.\end{aligned} \tag{5.3}$$

$$\begin{aligned}\frac{\partial}{\partial\gamma}\log L &= \frac{-N}{G(\alpha;\beta,\gamma)}\sum_{\ell=0}^{\infty}\alpha^{\ell}\sum_{k=0}^{[\frac{n}{m}]}\frac{\beta^{k}\ n!}{k!\ (n-mk)!}\left(n-mk\right)\left(\ell+\gamma\right)^{n-mk-1}\\ &+\sum_{i=1}^{N}\frac{1}{H_{n,m}\left(x_i+\gamma,\beta\right)}\sum_{k=0}^{[\frac{n}{m}]}\frac{\beta^{k}\ n!\ (n-mk)}{k!\ (n-mk)!}\left(x_i+\gamma\right)^{n-mk-1}.\end{aligned} \tag{5.4}$$

Equating the Equations (5.2–5.4) to zero and solving them with the help of R software, the MLES can be obtained. We notice that, these equations cannot solve analytically, there is an alternative procedure like Newton-Raphson is required to solve them numerically.

5.2. Simulation

In this section, we evaluate MLE performance to sample n. Evaluation based on simulation study described in the following steps:

1 Generate 1000 samples with $n = 50,\ 100,\ 500,\ 800$ and 1000 from GHGD.

2 Calculate MLES for 1000 sampls.

3 Calculating absolute bias, standard errors and mean square errors (MSE).

Table 1 Result from the simulated data.

n	Parameter	MLE	Standard Error	Abs. Bias	MSE
50	$\alpha = 0.2$	0.2187	0.011	0.0187	0.0005
	$\beta = 0.05$	0.0809	0.057	0.0309	0.0038
	$\gamma = 0.3$	0.2462	0.1434	0.0538	0.0212
100	$\alpha = 0.2$	0.2118	0.0137	0.0118	0.0003
	$\beta = 0.05$	0.0573	0.018	0.0073	0.0003
	$\gamma = 0.3$	0.3169	0.123	0.0169	0.0124
500	$\alpha = 0.2$	0.2086	0.0063	0.0086	0.0001
	$\beta = 0.05$	0.0447	0.0069	0.0053	0.00007
	$\gamma = 0.3$	0.2947	0.0414	0.0053	0.0017
800	$\alpha = 0.2$	0.2079	0.0059	0.0079	0.00009
	$\beta = 0.05$	0.0455	0.004	0.0045	0.00004
	$\gamma = 0.3$	0.3051	0.0408	0.0051	0.0016
1000	$\alpha = 0.2$	0.2065	0.0056	0.0065	0.00007
	$\beta = 0.05$	0.0456	0.0045	0.0044	0.000039
	$\gamma = 0.3$	0.3003	0.0348	0.0003	0.00121

The results obtained in Table 1.

It can be seen that

1 The bias values decrease as $n \to \infty$.

2 MSEs decrease as $n \to \infty$. This shows the consistency of the estimators.

3 The MLE method performs well for the parameters.

6. DATA ANALYSIS

In this section, we explain the empirical importance of GHGD using real data applications. The fitted model is compared using χ^2 statistic, Akaike information criterion (AIC), Bayesian information criterion (BIC) and correct Akaike information criterion (AICc).

6.1. Data Set 1

This data represents counts of cysts in embryonic mouse kidneys which subjected to steroids, taken from McElduff *et al.* [15] and [16]. We compare the fits of GHGD with HD, zero-inflated Poisson distribution (ZIPD), negative binomial distribution (NBD), zero-inflated negative binomial distribution (ZINBD), zero-inflated generalized Poisson distribution (ZIGPD) and zero-inflated Hermite distribution (ZIHD). The MLES and goodness of fit are presented in Table 2.

From the plots of the log-likelihood function of α, β and γ in Figure 8a–8c, we observe that the likehood equations have a unique solution.

6.2. Data Set 2

This data represents the distribution of mistakes in copying groups of random digits, see [17]. We compare the fits of GHGD with hyper-Poisson distribution (HPD), zero- inflated Poisson distribution (ZIPD), zero-inflated Conway–Maxwell–Poisson distribution (ZICMPD),ZINBD, ZIGPD and zero-inflated hyper-Poisson distribution (ZIHPD). The MLES and goodness of fit are presented in Table 3.

Table 2 Distribution of the counts of cysts from 111 steroid-treated kidneys [15] and the expected frequencies computed using HD, ZIPD, NBD, ZIGPD, ZINBD, ZIHD and GHGD.

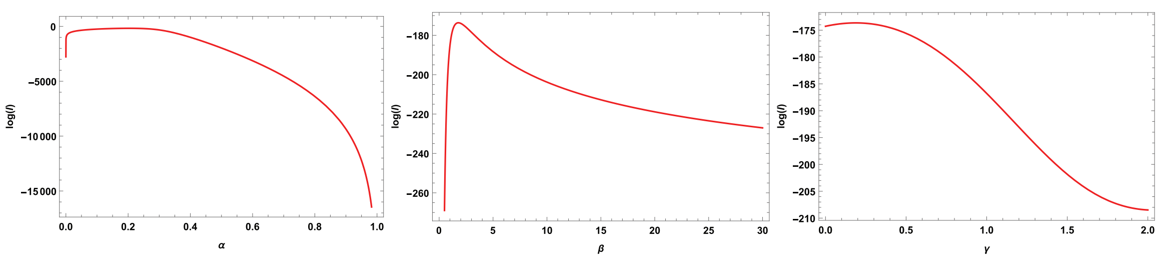
Count	Observed Frequency	HD	ZIPD	NBD	ZIGPD	ZINBD	ZIGH	GHGD
0	65	17.938	60.87	20.87	85.113	63.7	65.02	64.86
1	14	24	3.5	24.6	8.64	5.5	11	14.16
2	10	24	7.5	20.973	6.307	6.02	9	7
3	6	19	9.5	15.57	5.9	6.2	8	6.52
4	4	12	9.6	10.71	4.99	5	5.5	5.36
5	2	6.77	7.8	7.02	0.05	6	2.5	3.9
6	2	3.3	5.44	4.449	4.53×10^{-7}	4.2	2.52	2.77
7	2	1.48	3	2.74	2.03×10^{-12}	3.17	2.05	1.97
8	1	0.612	1.5	1.7	1.20×10^{-19}	2.27	0.9	1.4
9	1	0.5	1.02	1.25	6.76×10^{-29}	1.5	0.61	0.98
10	1	0.5	0.27	0.58	2.642×10^{-40}	3.04	0.6	0.68
11	2	0.7	0.7	0.34	5.57×10^{-54}	2.5	1.7	0.48
12	1	0.2	0.5	0.198	5.05×10^{-70}	1.9	1.6	0.92
Total	111	111	111	111	111	111	111	111
df		4	5	4	1	4	3	2
Estimates of the parameter		$\lambda = 1.5$	$\lambda = 4.0$	$\lambda = 2.25$	$\lambda = 1.05$	$\lambda = 3.85$	$\lambda = 1.15$	$\alpha = 0.203$
		$\theta = 0.4$	$\omega = 0.54$	$\theta = 2.48$	$\omega = 0.56$	$\omega = 0.56$	$\omega = 0.53$	$\beta = 1.79$
					$\theta = 1.35$	$\theta = 4.33$	$\theta = 1.01$	$\gamma = 0.182$
χ^2 value		154.39	27.66	117.43	34.07	22.62	2.32	1.77
P value		0.0001	0.0001	0.0001	0.0001	0.0001	0.0914	0.8804
AIC		476.238	383.634	450.82	3238.76	371.02	368.4	353.287
BIC		477.368	384.7	451.95	3240.45	372.71	370.24	361.415
AICc		473.83	384.83	448.42	3236.09	368.35	365.22	353.511

From the plots of the log-likelihood function of α, β and γ in Figure 9a–9c, we observe that the likehood equations have a unique solution.

6.3. Data Set 3

This data represents counts of Collenbola microarthropods in 200 samples of forest soil, see [18,19]. We compare the fits of GHGD with (HPD), (ZIPD), (ZICMPD),(ZINBD), (ZIGPD) and (ZIHPD). The MLES and goodness of fit are presented in Table 4.

From the plots of the log-likelihood function of α, β and γ in Figure 10a–10c, we observe that the likelihood equations have a unique solution.



(a) The profit of the log-likehood function of α.

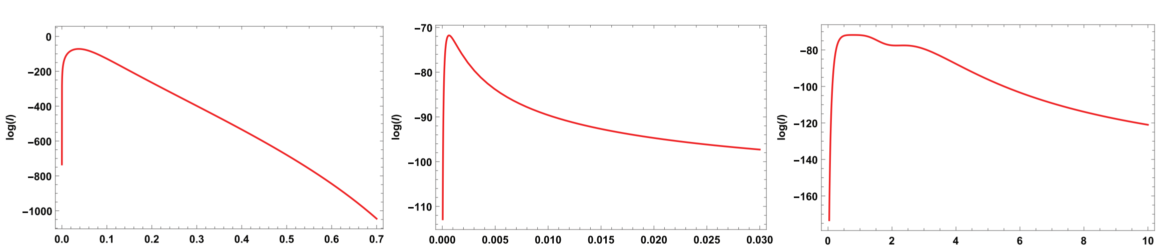
(b) The profit of the log-likehood function of β.

(c) The profit of the log-likehood function of γ.

Figure 8 The profiles of the log-likelihood function of α, β and γ.

Table 3 Distribution of mistakes in copying groups of random digits [17] and the expected frequencies computed using HPd, ZIPD, ZIHPD, ZICMPD, ZINBD, ZIGPD and GHGD distribution.

Count	Observed Frequency	HPD	ZIPD	ZIHPD	ZICMPD	ZINBD	ZIGPD	GHGD
0	35	24.41	41.1937	36.84	40.6	43.69	41.999	34.67
1	11	21.09	9.039	7.5	5.4	8.74	7.98	10.5
2	8	9.69	6.24	8.5	5.7	5.5	7.981	8.5
3	4	3.07	3.018	5.01	5.1	1.51	1.98	4.59
4	2	0.74	0.05093	2.05	3.2	0.56	0.06	1.74
Total	60	60	60	60	60	60	60	60
df		1	1	1	1	1	1	1
Estimates of the parameter		$\lambda = 1.23$	$\lambda = 1.45$	$\lambda = 0.63$	$\lambda = 2.3$	$\lambda = 0.54$	$\lambda = 2.0$	$\alpha = 0.0382$
		$\theta = 1.02$	$\omega = 0.579$	$\omega = 0.601$	$\omega = 0.7$	$\omega = 0.2$	$\omega = 0.55$	$\beta = 0.0006$
				$\theta = 1.23$	$\theta = 0.7$	$\theta = 0.5$	$\theta = 0.00000035$	$\gamma = 0.845$
χ^2 value		11.168	6.53	1.968	8.09	11.25	67.07	0.074
P value		0.0008	0.0106	0.1607	0.051	0.0008	0.0001	0.9948
AIC		224.34	181.87	169.233	206.22	206.244	301.45	149.57
BIC		221.74	179.27	165.332	205.13	205.072	300.82	155.853
AICc		223.746	181.272	170.033	224.30	230.24	325.45	149.998



(a) The profit of the log-likehood function of α.

(b) The profit of the log-likehood function of β.

(c) The profit of the log-likehood function of γ.

Figure 9 The profiles of the log-likelihood function of α, β and γ.

6.4. Graphical Assesment of Goodness of Fit

Plotting both the empirical probability generating function (EPGF) and log *pgf's* on the same graph allows us to compare the fit of a number of discrete distributions using only one plot, see [20].

The log of the EPGF of data set 1 is plotted in Figure 11. The EPGF is shown as black line, whilst a series of distributions fitted to data. The GHGD *pgf* shown by the red line, indicates that the GHGD is a good fit to the data.

The log of the EPGF of data set 2 is plotted in Figure 12. The EPGF is shown as black line, whilst a series of distributions fitted to data. The GHGD *pgf* shown by the red line, indicates that the GHGD is a good fit to the data.

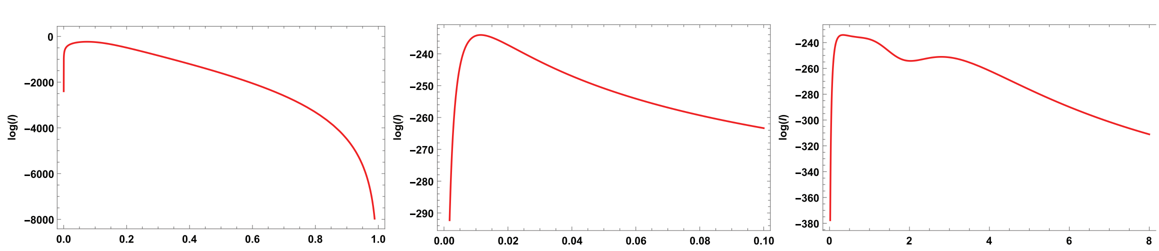
The log of the EPGF of data set 3 is plotted in Figure 13. The EPGF is shown as black line, whilst a series of distributions fitted to data. The GHGD *pgf* shown by the red line, indicates that the GHGD is a good fit to the data.

7. CONCLUSION

A new three parameters discrete distribution is proposed and its important monotonic and reliability concepts are introduced. The model proposed parameters are estimated by Maximum likelihood and the simulation study is performed to establish the accuracy of the maximum likelihood estimators. Applications of the new model in the analysis of three real-life data are presented. We show by three applications of the real data that the proposed distribution can yield better fits than some other distributions.

Table 4 Distribution of the counts of Collenbola microarthropods in 200 samples of fort soil [19] and the expected frequencies computed using HPD, ZIPD, ZIHPD, ZICMPD, ZINBD, ZIGPD and GHGD distribution.

Count	Observed Frequency	HPD	ZIPD	ZIHPD	ZICMPD	ZINBD	ZIGPD	GHGD
0	122	135.133	134.46	118.5	129.6	133.79	157.33	120.09
1	40	54	28.7	36.56	40	41.2	25.75	39.85
2	14	7.31	21.1	23.24	24	17.2	9.5	18.52
3	16	1.58	11.05	14.25	5.2	5.61	5.5	13.07
4	4	1.5	3.64	5.5	0.5	1.72	1.35	5.97
5	2	0.74	1.05	1.95	0.8	0.48	0.57	2.5
Total	200	200	200	200	200	200	200	200
df		1	2	2	1	1	1	2
Estimates of the parameter		$\lambda = 2.5$	$\lambda = 1.45$	$\lambda = 0.25$	$\lambda = 3.95$	$\lambda = 4.76$	$\lambda = 0.73$	$\alpha = 0.075$
		$\theta = 0.2$	$\omega = 0.578$	$\omega = 0.55$	$\omega = 0.60$	$\omega = 0.37$	$\omega = 0.65$	$\beta = 0.0011$
				$\theta = 1.02$	$\theta = 2.74$	$\theta = 0.81$	$\theta =$ 0.000127	$\gamma = 0.331$
χ^2 value		158.27	12.6	4.36	90.07	36.37	57.60	1.817
P value		0.0001	0.0018	0.1130	0.0001	0.0001	0.0001	0.7694
AIC		1228.03	621.6	582.99	744.7	660.2	1298.1	474.151
BIC		1227.6	621.18	582.37	744.15	659.5	1297.5	484.046
AICc		1229.3	625.6	591.9	750.77	669.2	1307.1	474.273



(a) The profit of the log-likehood function of α.

(b) The profit of the log-likehood function of β.

(c) The profit of the log-likehood function of γ.

Figure 10 The profiles of the log-likelihood function of α, β and γ.

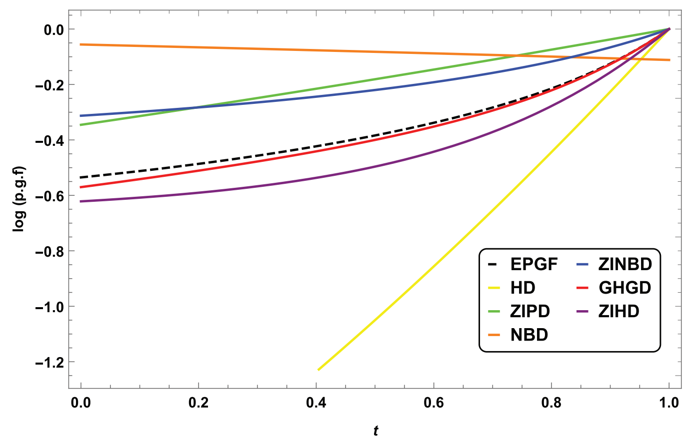


Figure 11 EPGF plot of counts of cysts from 111 steroid-treated kidneys with fitted it log pgf's for the Hermite distribution, zero-inated Poisson distribution, negative binomial distribution, zero-inated negative binomial distribution and generalized Hermite–Genocchi distribution.

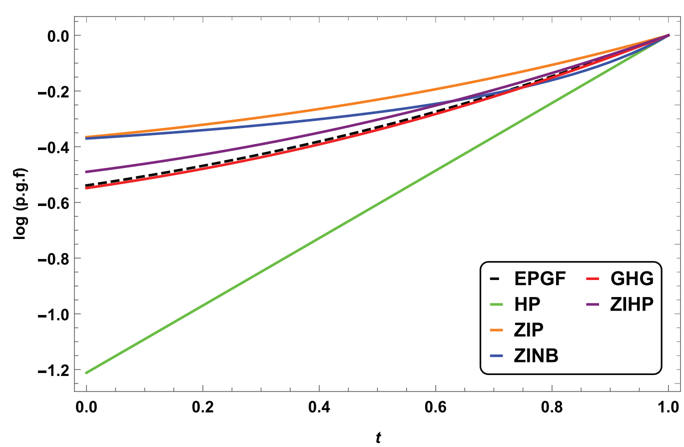


Figure 12 EPGF plot of the distribution of mistakes in copying groups of random digits with fitted it log pgf's for the hyper-Poisson distribution, zero-inflated Poisson distribution, zero-inflated negative binomial distribution, zero-inflated hyper-Poisson distribution and generalized Hermite–Genocchi distribution.

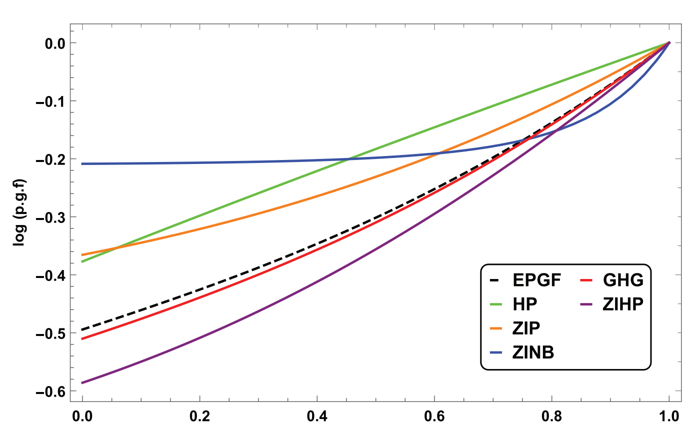


Figure 13 EPGF plot of counts of Collenbola microarthropods of forest soil with fitted it log pgf's for the hyper-Poisson distribution, zero-inflated Poisson distribution, zero-inflated negative binomial distribution, zero-inflated hyper-Poisson distribution and generalized Hermite–Genocchi distribution.

CONFLICTS OF INTEREST

The authors declare they have no conflicts of interest.

AUTHORS' CONTRIBUTIONS

All authors have read and agreed to the published version of the manuscript.

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