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Research Article

On Ramsey Dynamical Model and Closed-Form Solutions

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ABSTRACT

This study focuses on the analysis of Ramsey dynamical model with current Hamiltonian defining an optimal control problem in a neoclassical growth model by utilizing Lie group theory. Lie point symmetries of coupled nonlinear first-order ordinary differential equations corresponding to first-order conditions of maximum principle are analyzed and then first integrals and corresponding closed-form (analytical) solutions are determined by using Lie point symmetries in conjunction with Prelle-Singer and Jacobi last multiplier methods. Additionally, associated λ-symmetries, adjoint symmetries, Darboux polynomials, and the properties of the model are represented.

1. INTRODUCTION

The concept of the development of an economic growth model is one of the most important research areas in the theory of economics. The main purpose of an economic growth model as an applied dynamical system is to investigate the active forces to determine the growth and economic development. The classical models are based on the fact that economic growth depends on investments and improving productive capacity in an economy. However, in the case of the neoclassical economic growth models, economic growth depends on land, labor, and capital. In the literature, there are well-known economic growth models. For example, in the Solow model [27], agents and the planner follow up a straightforward linear rule for consumption and investments. However, in the case of the Ramsey model, the agents and the planner follow up consumption and investments to maximize their welfare. As early as 1928 a sophisticated model of a society's optimal saving was published by the British mathematician Frank Ramsey [24]. In the Ramsey model, there is a finite number of agents with an infinite time horizon; further, these agents are completely alike and thus it is a representative agent model. The Ramsey framework can be formulated in discrete time as well as in continuous time. In this study, the continuous-time version as did Ramsey's original contribution is considered.

The essence of the optimal control problem is to choose, for the entire time interval, a path $\vec{c}$ that maximizes $\upsilon(k(t), c(t), t)$ subject to the equation of the motion governing $k(t)$. These optimization problems are commonly defined as:

$$\begin{aligned} \max \quad V &= \int_0^T \upsilon(k(t), c(t), t)dt, \\ \dot{k}(t) &= g(k(t), c(t), t), \qquad t \in [0, T], \\ k(T)e^{-R(T)T} &\geq 0. \end{aligned} \tag{1.1}$$

In this formulation, the objective function is the integral over a predetermined interval $[0, T]$ of the payoff function υ. This payoff function depends, at each time instance, on the control variable $c(t)$, which the planner can directly control, the state variable $k(t)$, which the planner cannot directly control but is affected by $c(t)$ via its equation of motion, and the time t. $R(s)$ denotes the average discount rate between time zero and time t. For example, in an optimal savings problem, the control and state variables can be considered as the consumption and the level of assets, respectively. In general, the control problem involves several control and state variables. The constraint function expressing the change in the state variable may depend on the state variable, control variable, and time. The constraints state that: (1) At each moment, the change in state variable depends on the state variable itself and/or control variable and/or time; (2) the initial level of the state variable is given; (3) the discounted value of the state variable at the end of the planning horizon has to be weakly positive. If the planning horizon T is infinite, the last constraint can equivalently be stated as requiring that $k(T) \geq 0$.

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The dynamical optimization problems can be analyzed by utilizing Hamiltonian, which is set in two different ways. The first one can be written with the discount factor ahead and then a multiplier to every constraint is assigned. It is known as the present value Hamiltonian since it includes the discount of the future. Furthermore, the multipliers are called costate variables since it is associated with the constraint of the accumulation of the state variable. The second one called current value Hamiltonian can be set without taking into account the discount factor. In this study, the analysis of the Ramsey model with current value Hamiltonian is carried out by considering the first-order conditions (FOCs).

In the literature, there are some qualitative and quantitative methods to investigate closed-form and numerical solutions for the optimal control models including current value Hamiltonian. Hamiltonian dynamical systems in economics were introduced and its potential usefulness to economic theory was presented in the study [3]. A method for solving a family of four-dimensional nonlinear modified Hamiltonian dynamic systems in closed form was proposed in the studies [9,25,26]. A closed-form solution to the Ramsey model with logistic population growth was investigated in [8]. Furthermore, a closed-form solution to the transitional dynamics of the Lucas-Uzawa model was given in [5]. However, it can be said that there is no general solution procedure for these kinds of optimal control problems.

It is a fact that the theory of Lie groups is one of the most important methods for the investigation of invariant solutions of systems for differential equations [1,2,14,16,17,19–23]. This approach can also be used effectively for the analysis of optimal controls problem of dynamical systems in the economic growth models. In the literature, some economic growth models are analyzed by the partial Hamiltonian and partial Lagrangian approaches [15,16]. The integrability and linearization properties of the coupled ODEs are evaluated based on the Prelle-Singer procedure in the study [4]. Besides, the JLM approach was built up by Jacobi [10] for the application of the multipliers to classical mechanics problems and it has a direct relationship with Lie point symmetries. It is important to mention that all these methods have important properties for the investigation of solutions of partial or ordinary differential equations [18,19]. Here, we deal with FOCs of the Ramsey model representing by the coupled nonlinear ordinary differential equations (ODEs) by using the aforementioned methods, first, by determining its Lie point symmetry properties of the Ramsey model. However, it is a fact that the determination of Lie point symmetries of the model requires some tough mathematical computations because of its nonlinearity and order of the system. In this study, we represent the theory of Lie group as a general analysis and solution procedure for the optimal control problems with economic growth models by using its relations with Prelle-Singer (PS) and Jacobi last multiplier (JLM) methods.

The rest of the paper is arranged as follows: In section 2, we introduce the preliminaries on definitions for current value Hamiltonian, as well as mathematical methods such as Lie point symmetries, Prelle-Singer, Jacobi last multiplier methods, and related mathematical relations. Section 3 deals with the analytical analysis and shows the results for Ramsey's economic growth model with current value Hamiltonian including first integrals and closed-form solutions as well as λ-symmetries and Darboux polynomials. From the mathematical point of view, analytical solutions of the coupled system of nonlinear ordinary differential equations (ODEs) are investigated. The relationships between PS procedure and some analytical methods such as Lie point symmetries, Jacobi last multiplier (JLM), λ-symmetries, and Darboux polynomials [12] are indicated. In section 3, some conclusions and discussions on the analysis of the model are represented.

2. RAMSEY DYNAMICAL MODEL: AN ECONOMICAL GROWTH MODEL INCLUDING AN OPTIMAL CONTROL PROBLEM

In general, an optimal control problem is defined as

$$\max \quad \mathcal{F} = \int_0^T F(t, q^i, c_i) e^{-\rho t} dt \tag{2.1}$$

$$\text{subject to} \quad \dot{q}^i = f^i(t, q^i, c_i), \quad i = 1, \dots, n, \tag{2.2}$$

where $\dot{q}^i = dq^i/dt$, $c_i = (c_1, \dots, c_m)$ is the control vector, $m \leq n$ or $m > n$ with the proper boundary conditions, t is the independent variable representing time and the pair $(q^i, p_i) = (q^1, \dots, q^n, p_1, \dots, p_n)$ are the space coordinates, where $q^1, \dots, q^n$ are called the state variables and $p_1, \dots, p_n$ are called the costate variables. The integrand in the integral (2.1) contains the discount factor $e^{-\rho t}$. Furthermore, the investigation of the optimal control problems in economic growth models can be considered by taking into account Hamiltonian function, namely current value Hamiltonian.

Definition 2.1. *The current value Hamiltonian is defined by*

$$H(t, q^i, c_i) = F(t, q^i, c_i) + p_i f^i(t, q^i, c_i). \tag{2.3}$$

Remark 1. The necessary conditions for optimal control based on the Pontryagin maximum principle for current value Hamiltonian H are

$$\frac{\partial H}{\partial c_i} = 0, \tag{2.4}$$

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \tag{2.5}$$

$$\dot{p}^i = -\frac{\partial H}{\partial q^i} + \Gamma_i, \quad i = 1, \ldots, n, \tag{2.6}$$

where Γ_i generally is taken as a nonzero function of variables t, p_i, and q^i.

The conditions (2.4–2.6) derive optimality with all variables starting out from any given initial position. In this way, the optimal control problems and associated optimal initial values become much simpler for determining the optimality.

In the literature, Ramsey's neoclassical growth model based on a dynamical system[24] is one of the principal models in the case of the theory of dynamic macroeconomics, the investigation of the related optimal control problem is considered by taking into account Hamiltonian. The extended types of this growth model are considered to analyze some important macroeconomic problems. In general, the Ramsey neoclassical growth model as a dynamical system dealing with representative consumer's utility maximization problem is defined as

$$Max \quad \int_0^\infty e^{-rt} c^{1-\sigma} dt, \qquad \sigma \neq 0, 1, \quad r \neq 0 \tag{2.7}$$

subject to the capital accumulation equation

$$\dot{k}(t) = k^\beta(t) - \delta k(t) - c(t), \quad k(0) = k_0, \quad 0 < \beta < 1, \quad \delta \neq 0 \tag{2.8}$$

where $c = c(t)$ is consumption per person, $k(t)$ is capital-labor ratio, β is capital share, δ is depreciation rate, and r is time preferences. The intertemporal elasticity of substitution is given by $1/\sigma$ and k_0 is the capital stock. In the economic growth models with a dynamical system, the maximum principle deals with the system of coupled and nonlinear first-order ordinary differential equations in terms of state and costate variables. In this approach, the main aim is to maximize the current value Hamiltonian function via control variables.

The current value Hamiltonian function for the model (2.7) by using Definition 2.1 from the equation (2.3) can be written as

$$H(t, c, k, p) = c^{1-\sigma} + p(k^\beta - \delta k - c) \tag{2.9}$$

in which $p = p(t)$ is the costate variable. Secondly, the necessary first-order conditions for the optimal control problem are obtained from equations (2.4–2.6) of the form

$$\begin{aligned} p(t) &= (1-\sigma)c(t)^{-\sigma}, \\ \dot{k}(t) &= k(t)^\beta - \delta k(t) - c(t), \\ \dot{p}(t) &= -p(t)(\beta k(t)^{\beta-1} - \delta) + \Gamma, \end{aligned} \tag{2.10}$$

where the "overdot" represents the first-order derivatives concerning time parameter t and it can be checked that $\Gamma = rp(t)$. Besides, we can transform the nonlinear coupled system (2.10) into the following coupled first-order ODEs by eliminating control variable $c(t)$ by using the application of the implicit function theorem based on the fact that the second derivative of the Hamiltonian function with respect to variable $c(t)$ is an invertible map

$$\begin{aligned} \dot{p}(t) &= -p(t)(\beta k(t)^{\beta-1} - \delta) + rp(t), \\ \dot{k}(t) &= k(t)^\beta - \delta k(t) + (-1)^{1+\frac{1}{\sigma}}(\sigma - 1)^{\frac{1}{\sigma}} p(t)^{-\frac{1}{\sigma}}, \qquad \sigma \neq 0, 1. \end{aligned} \tag{2.11}$$

Remark 2. Besides, it is important to mention about some comments on the difference between the Ramsey model and the economic growth model with logarithmic utility function analyzed in the study [22] In general, especially, the definition of Ramsey model includes a parameter σ, which means "the intertemporal elasticity of substitution is given by $1/\sigma$ ($\sigma \neq 0, 1$)" in which the economic growth model with logarithmic utility function has not a similar term. Hence, because of this parameter, the economic growth model with logarithmic utility function and Ramsey dynamical system are quite different not only from mathematical point of view but from economic growth models point of view.

Remark 3. Let us suppose that a system of two coupled first-order ODEs of the form

$$\dot{p} = \phi_1(t, p, q), \qquad \dot{q} = \phi_2(t, p, q), \tag{2.12}$$

in which $\phi_i, \ i = 1, 2$ are analytic functions. The PS method is based on the solution of the following set of equations

$$S_t + \phi_1 S_p + \phi_2 S_q = -\phi_{2_q} + (\phi_{2_q} - \phi_{1_p})S + \phi_{1_q} S^2, \tag{2.13}$$

$$K_t + \phi_1 K_p + \phi_2 K_q = -K(S\phi_{1_q} + \phi_{2_q}), \tag{2.14}$$

$$K_p = SK_q + KS_q. \tag{2.15}$$

where $R(t,p,q)$ and $K(t,p,q)$ are integrating factors of the system and $S(t,p,q)$ is called a null form. Furthermore, the associated first integral is obtained as

$$I = r_1 + r_2 - \int \left[K + \frac{d}{dq}(r_1 + r_2) \right], \tag{2.16}$$

where

$$r_1 = \int (R\phi_1 + K\phi_2)dt \qquad \text{and} \qquad r_2 = -\int (R + \frac{d}{dp}(r_1))dp. \tag{2.17}$$

Remark 4. Assume that a partial differential equation (PDE) is given by in the form

$$Af = \sum_{i=1}^{n} a_1(x_1,\ldots,x_n)\frac{\partial f}{\partial x_i} = 0, \tag{2.18}$$

and its equivalent Lagrange's system is

$$\frac{dx_1}{a_1} = \frac{dx_2}{a_2} = \ldots = \frac{dx_n}{a_n}. \tag{2.19}$$

The system of first-order differential equation of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{2.20}$$

is given. Jacobi last multiplier of the above system, M, can be determined by the solution of the following partial differential equation

$$\frac{\mathrm{d}}{dt}\log M + \mathrm{div}\mathbf{f} = 0 \tag{2.21}$$

Based on the original formulation of the Jacobi last multiplier method, complete knowledge of the system (2.18) or (2.19) should have been available. Because of the existence of a solution/first integral is consequent upon the existence of symmetry, an alternative formulation in terms of symmetries was presented by Sophus Lie [11]. If $n-1$ symmetries of (2.18)/(2.19) are known

$$\Gamma_i = \sum_{j=1}^{n} \xi_{ij}(x_1,\ldots,x_n)\partial_{x_j}, \qquad i = 1,\ldots,n-1, \tag{2.22}$$

and JLM, that is equal to $M = \Delta^{-1}$, is calculated with the following determinant

$$\Delta = \det \begin{bmatrix} a_1 & \ldots & a_n \\ \xi_{1,1} & & \xi_{1,n} \\ \vdots & & \vdots \\ \xi_{n-1,1} & \ldots & \xi_{n-1,n} \end{bmatrix}, \tag{2.23}$$

in which $\Delta \neq 0$ [17].

Remark 5. Let $\xi(t,p,q)$, $\eta_1(t,p,q)$, and $\eta_2(t,p,q)$ be Lie symmetries of (2.12) and then the characteristic notation Q [19] can be written as

$$Q_1 = \eta_1 - \dot{p}\xi = \eta_1 - \phi_1\xi, \qquad Q_2 = \eta_2 - \dot{q}\xi = \eta_2 - \phi_2\xi. \tag{2.24}$$

Thus, a relation with S null function (2.13) is given by

$$S = \frac{Q_1}{Q_2}. \tag{2.25}$$

If the Lie symmetries of the equation (2.12) are known, and then it is clear that (2.13) is satisfied by using the relation (2.25). Hence, via the equations (2.14) and (2.15), the process of PS method is achieved. Now let us suppose that Λ_1 and Λ_2 are adjoint symmetries of (2.12), then they must satisfy the adjoint equation for the linearized symmetry conditions

$$D[\Lambda_1] = -(\Lambda_1\phi_{1_p} + \Lambda_2\phi_{2_p}), \qquad D[\Lambda_2] = -(\Lambda_1\phi_{1_q} + \Lambda_2\phi_{2_q}). \tag{2.26}$$

The function K an integrating factor satisfying the second equation given by (2.14) can be determined by using the relations between Jacobi last multipliers, Lie point symmetries (2.33), and PS method. However, it is a fact that there are no standard techniques to determine first integrals of ODEs, yet there is several special methods widely discussed in the literature. One of them is called Darboux polynomial method.

There is a simple relation between Darboux polynomials and Jacobi's last multipliers. The determining equation for the Darboux polynomial F [6] of equation (2.12) is defined by

$$D[F] = \alpha F, \tag{2.27}$$

in which $\alpha = (\phi_{2_q} + \phi_{1_p})$ represents the corresponding cofactor. The relation between Jacobi last multiplier and Darboux polynomial [12] is given by

$$F = M^{-1}. \tag{2.28}$$

Remark 6. It is clear that, given a system of first-order differential equations like (2.20) (allowing $\mathbf{f}$ to depend also on t) the determining equations are the solution of the following system of first order PDEs:

$$\frac{\partial \eta_i}{\partial t} + \sum_{i=1}^{N} \frac{\partial \eta_k}{\partial x_k} f_k + f_i \left(\frac{\partial \xi_i}{\partial t} + \sum_{i=1}^{N} \frac{\partial \xi_k}{\partial x_k} f_k \right) = \frac{\partial f_i}{\partial t} + \sum_{i=1}^{N} \frac{\partial f_i}{\partial x_k} \eta_k, \quad i = 1, \ldots, N, \tag{2.29}$$

where $X = \xi \partial_t + \sum_{k=1}^{N} \eta_k \partial_{x_k}$ is the infinitesimal generator of the Lie symmetry. For every $N \geq 1$ the solution of this system will depend on arbitrary functions, so that a system of first order differential equations admits infinite-dimensional Lie algebra.

Proposition 2.1. *The full symmetry group of (2.11) is always infinite-dimensional. However, the symmetry $X = \frac{\partial}{\partial t}$ presents one-dimensional Lie subalgebra if there are no any constraints on the parameters β, σ, δ, and r. On the other hand, the Lie subalgebra L_4 arises only for the given combination of parameters such as $\sigma = -1$, $\beta = \frac{1}{2}$, $\delta = -\frac{2r}{3}$.*

Proof. In order to analyze the solutions of the system (2.11), firstly Lie point symmetries of the coupled-system are examined. In order to determine the Lie point symmetries of the system (2.11), let us consider the vector field $X = \xi(t,p,k)\frac{\partial}{\partial t} + \eta_1(t,p,k)\frac{\partial}{\partial p} + \eta_2(t,p,k)\frac{\partial}{\partial k}$. The first prolongation of this vector field written according to the prolongation formula [19] is applied separately to the equations in the system (2.11). Thus, the overdetermined systems of PDEs, called *determining equations*, are obtained in terms of infinitesimal functions of the infinitesimal generator as follow

$$\begin{aligned}
&(-1)^{1+\frac{1}{\sigma}} k p^{\frac{1}{\sigma}} (\sigma - 1)^{\frac{1}{\sigma}} \eta_1 + p\sigma \big(- p^{2/\sigma} (k^\beta \beta - k\delta)\eta_2 - k\big(p^{\frac{1}{\sigma}} (k\delta - k^\beta) + (-1)^{\frac{1}{\sigma}} (\sigma - 1)^{\frac{1}{\sigma}}\big)^2 \xi_k \\
&\quad + p^{\frac{1}{\sigma}} \big(p(-k^\beta \beta + k(r+\delta))\big(p^{\frac{1}{\sigma}} (k\delta - k^\beta) + (-1)^{\frac{1}{\sigma}} (\sigma - 1)^{\frac{1}{\sigma}}\big)\xi_p + (-1)^{\frac{1}{\sigma}} k(\sigma - 1)^{\frac{1}{\sigma}} (\xi_t - \eta_{2k}) \\
&\quad + p^{\frac{1}{\sigma}} (k(k\delta - k^\beta)\xi_t + k(k^\beta - k\delta)\eta_{2k} - k^\beta p\beta \eta_{2p} + k(p(r+\delta)\eta_{2p} + \eta_{2t}))\big)\big) = 0,
\end{aligned} \tag{2.30}$$

and

$$\begin{aligned}
&- k p^{\frac{1}{\sigma}} (-k^\beta \beta + k(r+\delta))\eta_1 + k^\beta p^{1+\frac{1}{\sigma}} (\beta - 1)\beta \eta_2 - k^{2+\beta} p^{1+\frac{1}{\sigma}} r \xi_k + k^{1+2\beta} p^{1+\frac{1}{\sigma}} \beta \xi_k \\
&- k^{2+\beta} p^{1+\frac{1}{\sigma}} \delta \xi_k + k^3 p^{1+\frac{1}{\sigma}} r\delta \xi_k - k^{2+\beta} p^{1+\frac{1}{\sigma}} \beta\delta \xi_k + k^3 p^{1+\frac{1}{\sigma}} \delta^2 \xi_k - k^2 p^{2+\frac{1}{\sigma}} r^2 \xi_p + 2k^{1+\beta} p^{2+\frac{1}{\sigma}} r\beta \xi_p \\
&- k^{2\beta} p^{2+\frac{1}{\sigma}} \beta^2 \xi_p - 2k^2 p^{2+\frac{1}{\sigma}} r\delta \xi_p + 2k^{1+\beta} p^{2+\frac{1}{\sigma}} \beta\delta \xi_p - k^2 p^{2+\frac{1}{\sigma}} \delta^2 \xi_p \\
&- k^2 p^{1+\frac{1}{\sigma}} r \xi_t + k^{1+\beta} p^{1+\frac{1}{\sigma}} \beta \xi_t - k^2 p^{1+\frac{1}{\sigma}} \delta \xi_t + k^{2+\beta} p^{\frac{1}{\sigma}} \eta_{1k} - k^3 p^{\frac{1}{\sigma}} \eta_{1k} \\
&+ (-1)^{\frac{1}{\sigma}} k(\sigma - 1)^{\frac{1}{\sigma}} \big(p(-k^\beta \beta + k(r+\delta))\xi_{1k} - k\eta_{1k}\big) + k^2 p^{1+\frac{1}{\sigma}} r \eta_{1p} - k^{1+\beta} p^{1+\frac{1}{\sigma}} \beta \eta_{1p} \\
&+ k^2 p^{1+\frac{1}{\sigma}} \delta \eta_{1p} + k^2 p^{\frac{1}{\sigma}} \eta_{1t}.
\end{aligned} \tag{2.31}$$

Based on the above Remark 6, the solution of this system of first-order differential equations as determining equations depends on an arbitrary function and then it admits an infinite-dimensional Lie algebra. On the other hand, in order to obtain a finite-dimensional subalgabra, from the direct integration of the determining equations (2.30) and (2.31), the following algebraic equations including the model parameters, namely β, δ, σ, r and the group parameters, namely s_1, s_2, and s_3

$$\begin{aligned}
&s_1(-2+\beta)\beta\delta^2(-r^2(-1+\beta) - r(2 - 2\beta + \beta^2)\delta + (1 - \beta + \beta^2)\,\delta^2) = 0, \\
&s_2(-1)((1/\beta))r^3(1/(-1+\beta))^{(-4+1/\beta)}(-1+2\beta) = 0, \\
&s_3(-1+\beta)\delta(r + \delta - \beta\delta\sigma) = 0, \qquad \delta \neq 0, \quad \beta \neq 1,
\end{aligned} \tag{2.32}$$

are obtained. If there are no any constraints on the parameters and then we have $s_1 = 0$, $s_2 = 0$, and $s_3 = 0$ in (2.32) and then it can be shown that the analytical integration of the associated determining equations yields a finite subalgebra L_1 with basis operator of the form $X = \frac{\partial}{\partial t}$.

In addition, the solution of (2.32) gives $\sigma = -1, \beta = \frac{1}{2}, \delta = -\frac{2r}{3}$ for $s_1 \neq 0, s_2 \neq 0, s_3 \neq 0$ (for a nontrivial symmetry group). Consequently, under these constraints, the solution of determining equations (2.30) and (2.31) yields a four-dimensional infinite subalgabra given by

$$\begin{aligned} X_1 &= -\frac{3}{2r}\frac{\partial}{\partial t}, \quad X_2 = e^{\frac{rt}{3}}\frac{\partial}{\partial t} + \frac{r}{3}e^{\frac{rt}{3}}p\frac{\partial}{\partial p} + \frac{2r}{3}e^{\frac{rt}{3}}k\frac{\partial}{\partial k}, \\ X_3 &= \frac{e^{\frac{rt}{3}}(3+2r\sqrt{k})}{3\sqrt{k}}\frac{\partial}{\partial p} + e^{\frac{rt}{3}}\frac{\partial}{\partial k}, \quad X_4 = \frac{e^{\frac{2rt}{3}}}{\sqrt{k}}\frac{\partial}{\partial p} + e^{\frac{2rt}{3}}\frac{\partial}{\partial k}. \end{aligned} \tag{2.33}$$

□

Remark 7. It is clear that the use of Lie point symmetry $X = \frac{\partial}{\partial t}$ with Prelle-Singer procedure and Jacobi last multipliers method does not yield functions S and K. Therefore, Lie point symmetry $X = \frac{\partial}{\partial t}$ does not give rise to an invariant by using Prelle-Singer procedure for the system (2.11) without any constraints on the model parameters.

Remark 8. It is important to mention that the relation between Jacobi last multiplier and Lie symmetries enable to us to determine first integrals. Since this case has four Lie symmetries, then six Jacobi last multipliers of this system can be obtained. Hence, by using the ratio of two multipliers, thirty first integrals of the system are achieved, it is clear that some of these first integrals should be equivalent. Via the first integrals for this case, K and S functions can be found based on Remark 4. As a result, six different S null forms by using first integrals that obtained by JLM method are determined. Four of them overlap the null forms $S_1, \ldots, S_4$ that are derived by Lie symmetries and other two of them S_5 and S_6, which cannot be obtained directly from Lie symmetries are given at the Table 1.

It is clear that due to the presence of symmetries (2.33), such an invariant can be derived in a purely algebraic way from JLM. For instance, from JLMs $M_{13} = \frac{6re^{-rt/3}}{6pr-(3+2r\sqrt{k})^2}$ and $M_{24} = e^{-rt}$ corresponding symmetries X_1, X_3 and X_2, X_4 respectively, the first integral

$$I = \frac{6re^{2rt/3}}{6pr - (3+2r\sqrt{k})^2}, \tag{2.34}$$

Table 1 Prelle-Singer Procedure of the system (2.11) for $\sigma = -1, \beta = \frac{1}{2}, \delta = -\frac{2r}{3}$

Generators	Null Forms	Integration Factors	λ-symmetries and First Integrals
X_1	$S_1(t,p,k) = \frac{\sqrt{k}(3p-6\sqrt{k}-4kr)}{p(2r\sqrt{k}-3)}$	$K_1(t,p,k) = \Lambda_2 = -\frac{(4kr^2-9+pr(3-2r\sqrt{k}))}{3p(p-4\sqrt{k})^2 r\sqrt{k}}$	$\lambda_1(t,p,k) = \frac{r}{3} + \frac{3(p-4\sqrt{k})}{12k-8rk^{3/2}}$
		$R_1(t,p,k) = \Lambda_1 = \frac{4(pr-3-2r\sqrt{k})(3p-6\sqrt{k}-4kr)}{3rp^2(p-4\sqrt{k})^2}$	$V_1 = \frac{\partial}{\partial p} - \left(\frac{\sqrt{k}(3p-6\sqrt{k}-4kr)}{p(2r\sqrt{k}-3)}\right)\frac{\partial}{\partial k}$
			$I_1(t,p,k) = -\frac{2((3+2\sqrt{k})^2-6pr)}{3pr(p-4\sqrt{k})}$
X_2	$S_2(t,p,k) = \frac{2k}{p} - \sqrt{k}$	$K_2(t,p,k) = \Lambda_2 = -\frac{8e^{2rt/3}}{p(p-4\sqrt{k})^2\sqrt{k}}$	$\lambda_2(t,p,k) = \frac{p}{4k} - \frac{1}{\sqrt{k}} + \frac{r}{3}$
		$R_2(t,p,k) = \Lambda_1 = \frac{8e^{2rt/3}(p-2\sqrt{k})}{p^2(p-4\sqrt{k})^2}$	$V_2 = \frac{\partial}{\partial p} - \left(\frac{2k}{p} - \sqrt{k}\right)\frac{\partial}{\partial k}$
			$I_2(t,p,k) = \frac{4e^{2rt/3}}{p^2-4p\sqrt{k}}$
X_3	$S_3(t,p,k) = \frac{3\sqrt{k}}{3+2r\sqrt{k}}$	$K_3(t,p,k) = \Lambda_2 = -\frac{12r^2e^{2rt/3}(3+2r\sqrt{k})}{\sqrt{k}((3+2r\sqrt{k})^2-6pr)^2}$	$\lambda_3(t,p,k) = \frac{1}{12}\left(-\frac{6}{\sqrt{k}} + 4r + \frac{9p}{3k+2rk^{3/2}}\right)$
		$R_3(t,p,k) = \Lambda_1 = \frac{36e^{2rt/3}r^2}{(3+2r\sqrt{k})^2-6pr)^2}$	$V_3 = \frac{\partial}{\partial p} - \left(\frac{3\sqrt{k}}{3+2r\sqrt{k}}\right)\frac{\partial}{\partial k}$
			$I_3(t,p,k) = \frac{6re^{2rt/3}}{6pr-(3+2r\sqrt{k})^2}$
X_4	$S_4(t,p,k) = -\sqrt{k}$	$K_4(t,p,k) = \Lambda_2 = -\frac{r^2e^{2rt/3}}{\sqrt{k}(3-pr+2r\sqrt{k})^2}$	$\lambda_4(t,p,k) = \frac{3p-6\sqrt{k}+4kr}{12k}$
		$R_4(t,p,k) = \Lambda_1 = \frac{r^2e^{2rt/3}}{(3-pr+2r\sqrt{k})^2}$	$V_4 = \frac{\partial}{\partial p} + \sqrt{k}\frac{\partial}{\partial k}$
			$I_4(t,p,k) = \frac{2re^{2rt/3}}{-3+pr+2r\sqrt{k}}$
	$S_5(t,p,k) = -\frac{\sqrt{k}(-6p+12\sqrt{k}+(p^2-4p\sqrt{k}+8k)r)}{p(pr-6)}$	$K_5(t,p,k) = \Lambda_2 = -\frac{2e^{\frac{rt}{3}}(pr-6)}{pr(p-4\sqrt{k})^2\sqrt{k}}$	$\lambda_5(t,p,k) = \frac{72\sqrt{k}+p(-18+3r(p-6\sqrt{k})+4r^2k)}{12k(pr-6)}$
		$R_5(t,p,k) = \Lambda_1 = \frac{2e^{\frac{rt}{3}}(-6p+12\sqrt{k}+r(p^2-4p\sqrt{k}+8k))}{p^2r(p-4\sqrt{k})^2}$	$V_5 = \frac{\partial}{\partial p} + \left(\frac{\sqrt{k}(-6p+12\sqrt{k}+(p^2-4p\sqrt{k}+8k)r)}{p(pr-6)}\right)\frac{\partial}{\partial k}$
		$I_5(t,p,k) = \frac{2e^{\frac{rt}{3}}(pr-3-2r\sqrt{k})}{pr(p-4\sqrt{k})}$	
	$S_6(t,p,k) = \frac{\sqrt{k}(4kr^2-9)}{4kr^2+9-4\sqrt{k}r(pr-3)}$,	$K_6(t,p,k) = \Lambda_2 = \frac{e^{\frac{rt}{3}}4kr^2+9-4\sqrt{k}r(pr-3)}{\sqrt{k}((2r\sqrt{k}+3)^2-6pr)^2}$	$\lambda_6(t,p,k) = \frac{1}{6}(2r - \frac{3}{\sqrt{k}}) + \frac{p(9-4kr^2)}{4k(9+4kr^2-4r\sqrt{k}(pr-3))}$
		$R_6(t,p,k) = \Lambda_1 = \frac{3e^{\frac{rt}{3}}r(4kr^2-9)}{((2r\sqrt{k}+3)^2-6pr)^2}$	$V_6(t,p,k) = \frac{\partial}{\partial p} - \left(\frac{\sqrt{k}(4kr^2-9)}{4kr^2+9-4\sqrt{k}r(pr-3)}\right)\frac{\partial}{\partial k}$
			$I_6(t,p,k) = \frac{3e^{\frac{rt}{3}}(2r\sqrt{k}+3-pr)}{(2r\sqrt{k}+3)^2-6pr}$

is determined and the analytical solution for the coupled system (2.11) using the invariant (2.34)

$$p(t) = \frac{3}{r} + \frac{2}{3}c_2 re^{rt/3} + \sqrt{4c_2 e^{rt/3} + \frac{9c_1 - 6re^{2rt/3}}{c_1 r^2}},$$
$$k(t) = c_1 e^{rt/3} + \frac{9c_1 - 6re^{2rt/3}}{4c_1 r^2}, \tag{2.35}$$

can be obtained, where c_1 and c_2 are arbitrary constants. Furthermore, Darboux polynomials of the system for this case (2.11) are

$$F_1(t,p,k) = \frac{(p^2 - 4p\sqrt{k})e^{rt/3}}{4}, \quad F_2(t,p,k) = \frac{e^{rt/3}(6pr - (3+2r\sqrt{k})^2)}{6r},$$
$$F_3(t,p,k) = \frac{e^{2rt/3}(-3 + pr - 2r\sqrt{k})}{2r}, \quad F_4(t,p,k) = \frac{e^{2rt/3}(3 - pr + 2r\sqrt{k})}{3}, \tag{2.36}$$
$$F_5(t,p,k) = e^{rt}, \quad F_6(t,p,k) = \frac{2re^{rt}}{3},$$

which satisfy the condition (2.27).

Proposition 2.2. *From the analytical integration of the determining equations (2.30) and (2.31), the following finite subalgebra corresponding to the model without constraints*

$$\tilde{X}_1 = \frac{\partial}{\partial t}. \tag{2.37}$$

can be found and for the constraint $\sigma = \frac{r+\delta}{\beta\delta}$, the system (2.11) admits two-dimensional finite subalgebra L_2.

Proof. First of all, similarly, from the direct integration of the determining equations (2.30) and (2.31), we have the following relation for the model parameters

$$s(\beta - 1)\delta(r + \delta - \beta\delta\sigma) = 0, \tag{2.38}$$

where s is a group parameter. It can be shown that $s = 0$ gives the trivial symmetry group as given in (2.37) and $s \neq 0$ yields a nontrivial symmetry group and for the conditions $\delta \neq 0$, and $\beta \neq 1$, thus we have a constraint $\sigma = \frac{r+\delta}{\beta\delta}$ from (2.38). By similar calculations in the proof of the Proposition 2.1 for the constraint $\sigma = \frac{r+\delta}{\beta\delta}$, the analytical integration of the determining equations (2.30) and (2.31) yields Lie subalgebra with basis operators of the form

$$\tilde{X}_2 = \frac{e^{t\delta(\beta-1)}}{\delta(\beta-1)}\frac{\partial}{\partial t} + \frac{e^{t\delta(\beta-1)}p(r+\delta)}{\delta(\beta-1)}\frac{\partial}{\partial p} + \frac{e^{t\delta(\beta-1)}k}{(\beta-1)}\frac{\partial}{\partial k}. \tag{2.39}$$

□

Remark 9. For the condition $\sigma = \frac{r+\delta}{\beta\delta}$, Lie point symmetry $\tilde{X}_1$ does not produce an analytical solution for the system (2.11) by using PS aproach.

For the condition $\sigma = \frac{r+\delta}{\beta\delta}$ and Lie point symmetry $\tilde{X}_1 = \frac{\partial}{\partial t}$, the null function

$$\tilde{S}_1(t,p,k) = \frac{(-1)^{\frac{1}{\sigma}}p^{-\frac{1}{\sigma}}(\sigma-1)^{\frac{1}{\sigma}} + k\delta - k^\beta}{p(r - \beta k^{\beta-1} + \delta)}, \tag{2.40}$$

is determined, which clearly satisfies the equation in (2.13). The determining equation (2.14) requires some demanding calculations for obtaining the solution of the function K. For this purpose, let us consider the ansatz of the form

$$\tilde{K}(t,p,k) = \frac{S_d}{(A(t,p) + B(t,p)k)^z}, \tag{2.41}$$

where S_d is the denominator of $\tilde{S}_1$, and $A(t,p)$, $B(t,p)$ are functions of t and p, and z is a constant. Since the function K is in a rational form, the differentiation or integration the form of the denominator remains the same. On the other hand, the power of the denominator decreases or increases by a unit order from that of the initial one [4]. Using (2.41) for function K by considering the expressions

$$\phi_1(t,p,k) = -p(\beta k^{\beta-1} - \delta) + pr, \phi_2(t,p,k) = k^\beta - \delta k + (-1)^{1+\frac{1}{\sigma}}(\sigma-1)^{\frac{1}{\sigma}}p^{-\frac{1}{\sigma}}, \tag{2.42}$$

it can be checked that the equation (2.13) is satisfied and the solution of the equation (2.14) in terms of the function K using the ansatz form (2.41) is

$$\tilde{K}(t,p,k) = \alpha_1^{-z}e^{-rt}p(r - \beta k^{\beta-1} + \delta), \tag{2.43}$$

where α_1 is a constant. In addition, the compatibility condition (2.15) gives $\alpha_1^{-z}e^{-rt}r = 0$. However, for the time preference parameter r, the condition $r \neq 0$ means $\alpha_1 = 0$ $(z < 0)$, that is $K = 0$. Thus, we conclude that the symmetry $\tilde{X}_1$ does not produce a nontrivial solution for the system (2.11). Furthermore, λ-function and λ-symmetry are determined as

$$\begin{aligned} \tilde{\lambda}_1(t,p,k) &= r - \beta k^{\beta-1} + \delta + \frac{k^{\beta-2}(\beta-1)\beta((-1)^{\frac{1}{\sigma}}p^{-\frac{1}{\sigma}}(\sigma-1)^{\frac{1}{\sigma}} + k\delta - k^{\beta})}{r - \beta k^{\beta-1} + \delta}, \\ \tilde{V}_1 &= \frac{\partial}{\partial p} - \left(\frac{(-1)^{\frac{1}{\sigma}}p^{-\frac{1}{\sigma}}(\sigma-1)^{\frac{1}{\sigma}} + k\delta - k^{\beta}}{p(r - \beta k^{\beta-1} + \delta)}\right)\frac{\partial}{\partial k}. \end{aligned} \tag{2.44}$$

Remark 10. For the condition $\sigma = \frac{r+\delta}{\beta\delta}$, Lie point symmetry $\tilde{X}_2$ produces a nontrivial analytical solution for system (2.11) via PS approach.

To show this, let us consider the Lie point symmetry $\tilde{X}_2 = \frac{e^{t\delta(\beta-1)}}{\delta(\beta-1)}\frac{\partial}{\partial t} + \frac{e^{t\delta(\beta-1)}p(r+\delta)}{\delta(\beta-1)}\frac{\partial}{\partial p} + \frac{e^{t\delta(\beta-1)}k}{1-\beta}\frac{\partial}{\partial k}$ with the condition $\sigma = \frac{r+\delta}{\beta\delta}$ and investigate two different cases as below.

Case I: Based on the applications of the equations (2.25) for Lie point symmetry $\tilde{X}_2$, then the related null form

$$\tilde{S}_2(t,p,k) = \frac{k(1 + (-1)^{1+\frac{1}{\sigma}}p^{-\frac{1}{\sigma}}k^{-\beta}(\sigma-1)^{\frac{1}{\sigma}})}{\beta p}, \tag{2.45}$$

is obtained, indeed a particular solution of determining equation (2.13), and then the function K in the form (2.41) associated with the function $\tilde{S}_2$ can be determined as

$$\tilde{K}_{21}(t,p,k) = \Lambda_2 = e^{-t(r+\delta-\beta\delta)}pk^{\beta-1}. \tag{2.46}$$

The integrating factor R is

$$\tilde{R}_{21}(t,p,k) = \Lambda_1 = \frac{e^{-t(r+\delta-\beta\delta)}k^{\beta}}{\beta}(1 - (-1)^{\frac{1}{\sigma}}p^{-\frac{1}{\sigma}}k^{-\beta}(\sigma-1)^{\frac{1}{\sigma}}), \tag{2.47}$$

where Λ_1 and Λ_2 are the adjoint symmetries of (2.11) and it is clear that they satisfy the adjoint invariance condition $\Lambda_{1k} = \Lambda_{2p}$. Additionally, the first integral of the system (2.11)

$$\tilde{I}_{21}(t,p,k) = -\frac{e^{-t(r+\delta-\beta\delta)}p^{1-\frac{1}{\sigma}}(p^{\frac{1}{\sigma}}k^{\beta}(r+\delta-\beta\delta) - (-1)^{\frac{1}{\sigma}}(r+\delta)(\sigma-1)^{\frac{1}{\sigma}})}{\beta(r+\delta-\beta\delta)}, \tag{2.48}$$

is obtained. As a result, by using first integral above, an analytical solution to the Ramsey neoclassical economic growth model (2.7) corresponding to the condition $\sigma = \frac{r+\delta}{\beta\delta}$ can be determined as

$$\begin{aligned} \tilde{p}(t) &= \left(c_1 e^{\frac{t(\beta-1)(r+\delta)}{\beta\sigma}} + (-1)^{\frac{\beta-1}{\beta\sigma}}\beta^{\frac{1+\beta(\sigma-1)}{\beta\sigma}}\delta^{\frac{1-\beta}{\beta\sigma}}(r+\delta)^{-1/\beta}(r+\delta-\beta\delta)^{-1+\frac{\beta+\sigma-1}{\beta\sigma}}\right)^{\frac{\beta\sigma}{\beta-1}}, \\ \tilde{k}(t) &= (-1)^{\frac{1}{\beta\sigma}}\beta^{-\frac{1}{\beta\sigma}}\delta^{-\frac{1}{\beta\sigma}}(r+\delta)^{1/\beta}(r+\delta-\beta\delta)^{-\frac{1}{\beta}+\frac{1}{\beta\delta}} \\ &\quad \times \left(c_1 e^{\frac{t(\beta-1)(r+\delta)}{\beta\sigma}} + (-1)^{\frac{\beta-1}{\beta\sigma}}\beta^{\frac{1+\beta(\sigma-1)}{\beta\sigma}}\delta^{\frac{1-\beta}{\beta\sigma}}(r+\delta)^{-1/\beta}(r+\delta-\beta\delta)^{-1+\frac{\beta+\sigma-1}{\beta\sigma}}\right)^{\frac{1}{1-\beta}}, \\ \tilde{c}(t) &= (-1)^{\frac{1}{\sigma}}(\sigma-1)^{\frac{1}{\sigma}}\left(c_1 e^{t(\beta-1)\delta} + (-1)^{\frac{(\beta-1)\delta}{r+\delta}}\beta^{\frac{r+2\delta-\beta\delta}{r+\delta}}\delta^{\frac{\delta-\beta\delta}{r+\delta}}\right. \\ &\quad \left. \times (r+\delta)^{-1/\beta}(r+\delta-\beta\delta)^{\frac{(1-\beta)(r+\delta-\beta\delta)}{\beta(r+\delta)}}\right)^{\frac{\beta}{1-\beta}}, \end{aligned} \tag{2.49}$$

where c_1 is an arbitrary constant.

Case II: Now let us consider the other solution in the following form for the function K

$$\tilde{K}_{22}(t,p,k) = \Lambda_2 = \frac{e^{t(1-\sigma)}k^{\beta-1}(\sigma-1)(-1)p(\sigma-1)^{-1}}{\delta}, \tag{2.50}$$

which is the first integrating factor for the system. The second integrating factor R of (2.11) can be found as

$$\tilde{R}_{22}(t,p,k) = \Lambda_1 = 1 - \frac{e^{t(1-\sigma)}k^{\beta}(\sigma-1)(-1)p(\sigma-1)^{-1}(1 + (-1)^{1+\frac{1}{\sigma}}p^{-\frac{1}{\sigma}}k^{-\beta}(\sigma-1)^{\frac{1}{\sigma}})}{\delta\beta p}, \tag{2.51}$$

where Λ_1 and Λ_2 functions are the adjoint symmetries of (2.11) and they clearly satisfy the adjoint invariance condition $\Lambda_{1k} = \Lambda_{2p}$. Thus, the first integral of system (2.11) is

$$\tilde{I}_{22}(t,p,k) = \frac{e^{t(1-\sigma)}p^{-\frac{1}{\sigma}}(-1)p(\sigma-1)^{-1}(p^{\frac{1}{\sigma}}k^{\beta}(\sigma-1) + (-\sigma)^{1+\frac{1}{\sigma}}(r+\delta)(\sigma-1)^{\frac{1}{\sigma}})}{\beta\delta}, \tag{2.52}$$

Another form of an analytical solution of the model (2.7) for the same condition $\sigma = \frac{r+\delta}{\beta\delta}$ can be obtained by using the first integral (2.52) I_{22} as follows

$$\begin{aligned}
\tilde{p}(t) &= \left(c_1 e^{\frac{t(\beta-1)(r+\delta)}{\beta\delta}} + (-1)^{\frac{\beta-1}{\beta\sigma}} \beta (r+\delta)^{-1/\beta} (\sigma-1)^{\frac{(\beta-1)(1-\sigma)}{\beta\delta}}\right)^{\frac{\beta\delta}{\beta-1}}, \\
\tilde{k}(t) &= (-1)^{\frac{1}{\beta\sigma}} (r+\delta)^{\frac{1}{\beta}} \left(c_1 e^{\frac{t(\beta-1)(r+\delta)}{\beta\delta}} + (-1)^{\frac{\beta-1}{\beta\sigma}} \beta (r+\delta)^{-1/\beta} (\sigma-1)^{\frac{(\beta-1)(\frac{1}{\sigma}-1)}{\beta}}\right)^{\frac{1}{1-\beta}} (\sigma-1)^{\frac{1-\sigma}{\beta\sigma}}, \\
\tilde{c}(t) &= (-1)^{\frac{1}{\sigma}} \left(c_1 e^{\frac{t(\beta-1)(r+\delta)}{\beta\delta}} + (-1)^{\frac{\beta-1}{\beta\sigma}} \beta (r+\delta)^{-1/\beta} (\sigma-1)^{\frac{(1-\beta)(\sigma-1)}{\beta\sigma}}\right)^{\frac{\beta}{1-\beta}} (\sigma-1)^{\frac{1}{\sigma}},
\end{aligned} \tag{2.53}$$

where c_1 is an arbitrary constant. For the system (2.11), its λ-function and λ-symmetry are determined as below

$$\begin{aligned}
\tilde{\lambda}_2(t,p,k) &= r + \delta + \frac{(-1)^{1+\frac{1}{\sigma}} p^{-\frac{1}{\sigma}} (\beta-1)(\sigma-1)^{\frac{1}{\sigma}} - k^{\beta}}{k}, \\
\tilde{V}_2 &= \frac{\partial}{\partial p} - \left(\frac{k(1 + (-1)^{1+\frac{1}{\sigma}} p^{-\frac{1}{\sigma}} k^{-\beta} (\sigma-1)^{\frac{1}{\sigma}})}{\beta p}\right) \frac{\partial}{\partial k}.
\end{aligned} \tag{2.54}$$

In addition, Darboux polynomial for this case is

$$\tilde{F}_1(t,p,k) = \frac{p(k^{\beta}(r+\delta-\beta\delta) + (-1)^{1+\frac{1}{\sigma}} p^{-\frac{1}{\sigma}} (r+\delta)(\sigma-1)^{\frac{1}{\sigma}})}{e^{t\delta(\beta-1)}(\beta-1)\delta}, \tag{2.55}$$

which satisfies the condition (2.27).

3. CONCLUSION AND DISCUSSION

This study introduces an analytical approach for analyzing Ramsey's dynamical model defining a neoclassical growth model in the theory of economics via the theory of Lie groups. In the literature, there are different qualitative and quantitative methods to deal with the optimal control problems related to dynamic economic growth models in terms of Hamiltonians. However, the crucial problem for nonlinear economic growth models is the fact that there is no general analytical solution procedure for the analytical investigation for the related problems. On the other hand, with this study, we represent a unified approach including Lie point symmetry, Jacobi last multiplier (JLM), and Prelle-Singer (PS) methods for Ramsey economic growth model with current value Hamiltonian.

Firstly, we deal with the associated coupled-nonlinear first-order ordinary differential equations (ODEs) corresponding to first-order conditions of maximum principle for current value Hamiltonian form. From the Lie group point of view, it is a fact that the symmetry group analysis of first-order coupled nonlinear ODEs requires troublesome calculations since it is not possible to determine the Lie point symmetries by using the software programs known in the literature. From the theory of economics point of view, we consider the Ramsey economic growth model in terms of the current value Hamiltonian. The problem is that, in general, the Lie symmetries of the system (2.11) cannot be determined in closed form. Indeed, solving such problem is equivalent to the original problem. However, also in the particular cases the full symmetry group of (2.11) is infinite-dimensional. It is pointed out that Lie point symmetries for the Ramsey growth model do not present enough information in order to determine the first integrals and the exact closed-forms solution since the model has a restricted number of point symmetries. For this purpose, the relation between Lie point symmetries and JLM is considered. We also show that the JLM approach can be used as an effective method for analysis of the first-order nonlinear coupled system by considering its Lie point symmetries together.

For the case of current value Hamiltonian, we show that the system has a finite subalgebra such as $X = \frac{\partial}{\partial t}$ and prove that there is no analytical solution for the Ramsey model by using the PS method if there is no any restriction on the model parameters, which corresponds to the general Ramsey model. Additionally, based on the parametric restrictions $\sigma = -1$, $\beta = \frac{1}{2}$, $\delta = -\frac{2r}{3}$, it is shown that the system admits four-dimensional Lie algebra L_4 and six different Jacobi last multipliers. The associated closed-form solutions for each subcase are found. As the other case, for the condition $\sigma = \frac{r+\delta}{\beta\delta}$, we present that the system admits two-dimensional finite Lie subalgebra L_2 and only symmetry $\tilde{X}_2$ gives a nontrivial closed-form solution by using PS method. In this case, two different analytical solutions to the Ramsey dynamical system are represented.

We also take into account the relation between Lie point symmetries and extended PS method to determine integrating factors and first integrals since the JLM approach does not always yield a first integral thus an analytical solution due to the symmetry restriction problem in which the first-order coupled and highly nonlinear ODEs may have. For the determination of the function K called the integrating factor, it is clear that the determination of the solution for the related system of equations is not straightforward and then we consider an ansatz form for the function K. However, it is shown that the compatibility equation (2.15) is not always satisfied by functions K and S, null function, obtained from (2.13) and (2.14), which then gives a trivial solution for function K. For the other cases, the associated first integrals and nontrivial closed-form solutions are obtained and they are represented for the different cases. Additionally, the related λ-symmetries, adjoint symmetries, Darboux polynomials, their properties, and relations for all cases are determined.

This study can be seen as an application of the theory of Lie groups to the Ramsey dynamical system. Based on the theoretical Lie group analysis, the novel closed-form solutions to the Ramsey economic growth model are reported for the first time in the literature.

4. CONFLICTS OF INTEREST

The authors declare they have no conflicts of interest.

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