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## MASSLESS THIRRING MODEL IN CANONICAL QUANTIZATION SCHEME

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It is shown that the exact solubility of the massless Thirring model in the canonical quantization scheme originates from the intrinsic hidden linearizability of its Heisenberg equations in the method of dynamical mappings. The corresponding role of inequivalent representations of free massless Dirac field and appearance of Schwinger terms are elucidated.

*Keywords:* Heisenberg equations; bosonization; inequivalent representations.

### 1. Introduction

Despite a considerable age the two-dimensional Thirring model [1–3] is still remained as important touchstone for non-perturbative methods of quantum field theory [4–8] revealing new features both in the well-known [9–13] and in newly obtained solutions [15]. At the same time the methods of integration of such two-dimensional models provide a key for understanding some non-linear theories of higher dimension [13]. In particular the Thirring model turns out to be a two-dimensional analog of the well-known Nambu–Jona-Lasinio model [13, 15] and together with the Schwinger model provides an important example of using the well-known bosonization procedure (BP) [7–16].

In the present work the BP for Thirring model is considered as a special case of dynamical mapping (DM) [17, 18], what for Schwinger model was previously done in Greenberg’s works [19]. In the framework of canonical quantization scheme [20] the DM method consists in the construction of Heisenberg field (HF)  $\Psi(x)$  as a solution of Heisenberg equations of motion (HEq) in the form of Haag expansion built on normal products [21] of free “physical” fields  $\psi(x)$ , whose representation space accords with unknown *a priori* physical states of the given field theory [17]. The DM  $\Psi(x) \stackrel{w}{=} \Upsilon[\psi(x)]$ , being generally speaking a weak equality, implies the choice of appropriate initial conditions for the HEq. For example [16, 17], when both sets of fields are complete, irreducible and coincide asymptotically as  $t \rightarrow -\infty$ , the HF will tend in a weak sense to appropriate asymptotic physical field  $\psi_{in}(x)$ :  $\lim_{t \rightarrow -\infty} \Psi(x^1, t) \stackrel{w}{=} \Upsilon[\psi_{in}(x^1, -\infty)]$ . However the (asymptotic) completeness and irreducibility are not true in the presence of bound states [17, 18]. In particular for the

exactly solvable two-dimensional models of Thirring and Schwinger [13, 16] the physical asymptotic states of propagated physical particles have nothing to do with massless free Dirac asymptotic fields (confinement).

As was shown in the works [22–24] it is more convenient generally to make DM onto the “Schrödinger” physical field  $\psi_s(x)$ , associated with the HF at  $t \rightarrow 0$ :  $\lim_{t \rightarrow 0} \Psi(x^1, t) \stackrel{w}{=} \Upsilon[\psi_s(x^1, 0)]$ , which is a generalization [23, 24] of the well-known interaction representation and is closely related to the procedure of canonical quantization [16, 20]. In this representation the time-dependent coefficient functions of DM [22, 23] contain all the information about bound states and scattering, and exactly solvable Federbush model [6] leads to the exactly linearizable HEq [24].

The present paper shows that HEq of the Thirring model admits a similar linearization and that the choice of free massless (pseudo-) scalar fields as the physical ones is a consequence of reducibility of the massless Dirac field [16] in the space of these fields. The problem of Schwinger terms in the currents commutator [4], being closely related to BP [9–16], also finds here a natural solution [24] in fact borrowed from QED [25], where it is also sufficient to define this commutator only for the free fields in corresponding “interaction representation”.

Definition of the model in canonical quantization scheme is given in the next section. Then the linearization procedure with corresponding definition of Heisenberg currents is advocated. The bosonization rules that we need for the free fields only are discussed in Sec. 4 with the appropriate choice of (pseudo-) scalar fields. That all is used in Sec. 5 for direct integration of HEq with chosen initial condition. The final remarks are made in Sec. 6.

## 2. Thirring Model

Following to the canonical quantization procedure [20] we start with the formal Hamiltonian of the Thirring model [1], which in two-dimensional space-time<sup>a</sup> defines a Fermi self-interaction, with fixed (and further unrenormalizable) dimensionless coupling constant  $g$ , for spinor field with spin 1/2 and zero mass:

$$H[\Psi] = H_{0[\Psi]}(x^0) + H_{I[\Psi]}(x^0), \quad (2.1)$$

$$H_{I[\Psi]}(x^0) = \frac{g}{2} \int_{-\infty}^{\infty} dx^1 J_{(\Psi)\mu}(x) J_{(\Psi)}^{\mu}(x), \quad (2.2)$$

$$H_{0[\Psi]}(x^0) = \int_{-\infty}^{\infty} dx^1 \Psi^{\dagger}(x) E(P^1) \Psi(x), \quad E(P^1) = \gamma^5 P^1, \quad (2.3)$$

satisfying the equal-time canonical anticommutation relations:

$$\{\Psi_{\xi}(x), \Psi_{\xi'}^{\dagger}(y)\}|_{x^0=y^0} = \delta_{\xi, \xi'} \delta(x^1 - y^1), \quad (2.4)$$

$$\{\Psi_{\xi}(x), \Psi_{\xi'}(y)\}|_{x^0=y^0} = 0,$$

$$\{\Psi_{\xi}(x), \Psi_{\xi'}^{\#}(y)\}|_{(x-y)^2 < 0} = 0, \quad \text{with: } \Psi_{\xi}^{\#}(y) = \Psi_{\xi}(y), \quad \Psi_{\xi}^{\dagger}(y). \quad (2.5)$$

<sup>a</sup>Here:  $x^{\mu} = (x^0, x^1)$ ;  $x^0 = t$ ;  $\hbar = c = 1$ ;  $\partial_{\mu} = (\partial_0, \partial_1)$ ; for  $g^{\mu\nu}$ :  $g^{00} = -g^{11} = 1$ ; for  $\epsilon^{\mu\nu}$ :  $\epsilon^{01} = -\epsilon^{10} = 1$ ;  $\bar{\Psi}(x) = \Psi^{\dagger}(x)\gamma^0$ ;  $\gamma^0 = \sigma_1$ ,  $\gamma^1 = -i\sigma_2$ ,  $\gamma^5 = \gamma^0\gamma^1 = \sigma_3$ ,  $\gamma^{\mu}\gamma^5 = -\epsilon^{\mu\nu}\gamma_{\nu}$ , where  $\sigma_i$  — Pauli matrices, and  $I$  — unit matrix;  $x^{\xi} = x^0 + \xi x^1$ ,  $2\partial_{\xi} = 2\partial/\partial x^{\xi} = \partial_0 + \xi\partial_1$ ,  $P^1 = -i\partial_1$ ; summation over  $\xi$  is nowhere implied.

Here indices  $\xi, \xi' = \pm$ , as well as for the  $x^\xi$ , enumerate the components of HF by the rule:

$$\Psi(x) = \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \end{pmatrix} = \begin{pmatrix} \Psi_+(x) \\ \Psi_-(x) \end{pmatrix}, \quad (2.6)$$

and the vector current  $J_{(\Psi)}^\mu(x)$ , together with the axial current  $J_{(\Psi)}^{5\mu}(x)$ , for  $\mu, \nu = 0, 1$ , is their yet formal local bilinear functional of the form:

$$J_{(\Psi)}^\mu(x) \mapsto \bar{\Psi}(x)\gamma^\mu\Psi(x), \quad (2.7)$$

$$J_{(\Psi)}^{5\mu}(x) \mapsto \bar{\Psi}(x)\gamma^\mu\gamma^5\Psi(x) = -\epsilon^{\mu\nu}J_{(\Psi)\nu}(x),$$

which due to (2.1)–(2.6) formally appears also in the canonical equations of motion<sup>b</sup> [3–6]:

$$i\partial_0\Psi(x) = [\Psi(x), H[\Psi]] = [E(P^1) + g\gamma^0\gamma_\nu J_{(\Psi)}^\nu(x)]\Psi(x), \quad \text{or:} \quad (2.8)$$

$$2\partial_\xi\Psi_\xi(x) = -igJ_{(\Psi)}^{-\xi}(x)\Psi_\xi(x), \quad \xi = \pm, \quad (2.9)$$

— for each  $\xi$ -component of the field (2.6) that formally are also related to the corresponding current components as:

$$J_{(\Psi)}^\xi(x) = J_{(\Psi)}^0(x) + \xi J_{(\Psi)}^1(x) \mapsto 2\Psi_\xi^\dagger(x)\Psi_\xi(x), \quad \xi = \pm. \quad (2.10)$$

The correct definitions of these formal operator products will be discussed hereinafter.

### 3. Linearization of the Heisenberg Equation

An immediate consequence of the field equations of motion (2.8), (2.9) are the local conservation laws [3–6] for the currents (2.7), (2.10):

$$\partial_\mu J_{(\Psi)}^\mu(x) = 0, \quad \partial_\mu J_{(\Psi)}^{5\mu}(x) = -\epsilon_{\mu\nu}\partial^\mu J_{(\Psi)}^\nu(x) = 0, \quad \text{or: } \partial_\xi J_{(\Psi)}^\xi(x) = 0, \quad \xi = \pm, \quad (3.1)$$

that fully determine their dynamics as a free one [4, 5]. Therefore it is not surprising that by means of the same equations of motion (2.8), (2.9), as well as by means of the anti-commutation relations (2.4) for HF, it is a simple matter to show [24] that:

$$i\partial_0\gamma^0\gamma_\nu J_{(\Psi)}^\nu(x) - [\gamma^0\gamma_\nu J_{(\Psi)}^\nu(x), H_{0[\Psi]}(x^0)] = iI\partial_\mu J_{(\Psi)}^\mu(x) + i\gamma^5\epsilon_{\mu\nu}\partial^\mu J_{(\Psi)}^\nu(x) \equiv 0, \quad (3.2)$$

where the first term on the right-hand side of equality (3.2) comes evidently from the left terms with  $\nu = 0$ , while the second term on the right-hand side comes from the left terms with  $\nu = 1$ . The canonical equation of motion for this operator of “total current” in Eq. (2.8), containing of course its commutator with the full Hamiltonian  $H[\Psi]$  given by Eqs. (2.1)–(2.3), recasts then to the following equation:

$$i\partial_0\gamma^0\gamma_\nu J_{(\Psi)}^\nu(x) - [\gamma^0\gamma_\nu J_{(\Psi)}^\nu(x), H_{0[\Psi]}(x^0)] = [\gamma^0\gamma_\nu J_{(\Psi)}^\nu(x), H_{I[\Psi]}(x^0)] = 0, \quad (3.3)$$

which thus cannot contain a contribution from the commutator with the interaction Hamiltonian  $H_{I[\Psi]}(x^0)$  given by Eq. (2.2). Hence, as well as for the Federbush model [24], a

<sup>b</sup>Contribution to (2.8) due to non-commutativity of  $J_{(\Psi)}^\nu(x)$  and  $\Psi(x)$  is formally proportional to  $\delta(0)\gamma^0\gamma_\nu\gamma^0\gamma^\nu = 0$ .

non-zero contribution of Schwinger terms in HEq (3.3) would be premature, because, due to Eq. (3.2), it leads to violation of the current conservation laws (3.1).

On the one hand, within the framework of canonical quantization procedure [20], the vanishing of expressions (3.2), (3.3) means that temporal evolution of this “total current” is governed by a free Hamiltonian  $H_{0[\chi]}(x^0)$  of the same form (2.3) quadratic on some kind of free massless trial physical Dirac fields  $\chi(x)$ , furnished by the same anti-commutation relations and by the same conservation laws for corresponding currents  $J_{(\chi)}^\nu(x)$ ,  $J_{(\chi)}^{5\nu}(x)$ , defined formally by Eqs. (2.4)–(2.7), (2.10), (3.1) with  $\Psi(x) \mapsto \chi(x)$ :

$$i\partial_0\gamma^0\gamma_\nu J_{(\chi)}^\nu(x) - [\gamma^0\gamma_\nu J_{(\chi)}^\nu(x), H_{0[\chi]}(x^0)] = iI\partial_\mu J_{(\chi)}^\mu(x) + i\gamma^5\epsilon_{\mu\nu}\partial^\mu J_{(\chi)}^\nu(x) = 0. \quad (3.4)$$

On the other hand, the Heisenberg current operators appearing in (3.2), (3.3) acquire precise operator meaning — with non-vanishing Schwinger term — only after the choice of the representation space [4, 20, 26] for anticommutation relations (2.4), (2.5) and subsequent reduction in this representation to the normal-ordered form by means of renormalization, for example, via point-splitting and subtraction of the vacuum expectation value [16]:

$$J_{(\Psi)}^0(x) \mapsto \lim_{\tilde{\varepsilon} \rightarrow 0} \hat{J}_{(\Psi)}^0(x; \tilde{\varepsilon}) = \hat{J}_{(\Psi)}^0(x), \quad J_{(\Psi)}^1(x) \mapsto \lim_{\varepsilon \rightarrow 0} \hat{J}_{(\Psi)}^1(x; \varepsilon) = \hat{J}_{(\Psi)}^1(x), \quad (3.5)$$

$$\text{where at first: } \tilde{\varepsilon}^0 = \varepsilon^1 \rightarrow 0, \quad \text{with fixed: } \tilde{\varepsilon}^1 = \varepsilon^0, \quad \varepsilon^2 = -\tilde{\varepsilon}^2 > 0, \quad (3.6)$$

$$\text{for: } \hat{J}_{(\Psi)}^\nu(x; a) = Z_{(\Psi)}^{-1}(a)[\bar{\Psi}(x+a)\gamma^\nu\Psi(x) - \langle 0|\bar{\Psi}(x+a)\gamma^\nu\Psi(x)|0\rangle], \quad (3.7)$$

and accordingly for the components (2.10). The renormalization “constant”  $Z_{(\Psi)}(a)$  is defined below in (5.16). The definition of renormalized current (3.5)–(3.7) used here corresponds to the well-known Schwinger prescription [25] specified in the work [11] and, unlike Johnson definition [2], directly depends on the representation choice via the vacuum expectation value [16] in Eq. (3.7) like the very meaning of Schwinger term [4, 13]. One can show [11] that for the massless case these different current definitions lead to coincident expressions only for the free Dirac fields (cf. Eqs. (4.1) and (5.12)).

The comments given above jointly with the foregoing arguments deduced from Eq. (3.1)–(3.4) allow to identify in HEq (2.8), at least in a weak sense, the Heisenberg operator of “total current”, defined by Eqs. (2.7) and (3.2), with that operator, defined by Eqs. (2.7), (3.4) for the free massless trial physical Dirac fields  $\chi(x)$  and renormalized in the sense of normal form (3.5)–(3.7) up to an unknown yet constant  $\beta$ :

$$\gamma^0\gamma_\nu J_{(\Psi)}^\nu(x) \xrightarrow{w} \frac{\beta}{2\sqrt{\pi}}\gamma^0\gamma_\nu \hat{J}_{(\chi)}^\nu(x), \quad (3.8)$$

$$\hat{J}_{(\chi)}^\nu(x) = \lim_{\varepsilon, (\tilde{\varepsilon}) \rightarrow 0} \hat{J}_{(\chi)}^\nu(x; \varepsilon(\tilde{\varepsilon})) \equiv: J_{(\chi)}^\nu(x) : . \quad (3.9)$$

Here for  $Z_{(\chi)}(a) = 1$  the symbol  $: \dots :$  means the usual normal form [21] with respect to free field  $\chi(x)$ . This identification leads to linearization of both Eqs. (2.8) and (2.9) in the representation of these trial fields  $\chi(x)$ . Of course, the Eq. (2.8) is linearized with respect to  $x^0$ , while the Eq. (2.9) — with respect to  $x^\xi$ . However, the latter equation is the preference of two-dimensional world with initial condition being far from evidence. Whereas the former equation admits the above-mentioned in the Introduction physically reasonable

initial condition at  $x^0 = 0$ . Unlike [4, 16, 24], this initial condition does not fix here the constant  $\beta$ , which will be defined dynamically in subsequent sections.

#### 4. Bosonization and Scalar Fields

As was shown in [24] such kind of linearization of HEq for the Federbush model directly leads to its solution in the form of DM  $\Psi(x) = \Upsilon[\psi_1(x), \psi_2(x)]$  onto the free massive Dirac fields  $\psi_{1,2}(x)$  with different non-zero masses  $m_{1,2}$ . Unlike the massive one, the components  $\chi_\xi(x)$  of two-dimensional free massless field become completely decoupled,  $\partial_\xi \chi_\xi(x) = 0$ . As a consequence, this field turns out to be essentially non-uniquely defined or reducible and equipped by many inequivalent representations both in the spaces of a free massless (pseudo-) scalar field [16] ( $\phi(x)$ ),  $\varphi(x)$  and massive scalar field [12]  $\phi_m(x)$ . Because the DM is physically meaningful only onto the complete, irreducible sets of fields:  $\Psi(x) = \Upsilon[\varphi(x), \phi(x)]$ , or  $\Psi(x) = \Upsilon[\phi_m(x)]$ , or  $\Psi(x) = \Upsilon[\psi_M(x)]$ , — for the phase with spontaneously broken chiral symmetry [13, 15], further we consider here only the first possibility.

The corresponding BP allows to operate with functionals of boson fields instead of fermion operators and forms a powerful tool for obtaining non-perturbative solutions in various two-dimensional models [9, 13, 16, 24]. Its use also simplifies integration of the linearized HEq (2.8).

Being a formal consequence of the current conservation conditions (3.1) only, the bosonization rules have, generally speaking, the sense of weak equalities only for the current operator in the normal-ordered form (3.5)–(3.7), that already implies a choice of certain representations of (anti-) commutation relations (2.4) and (4.5) below. However, for the free massless fields  $\chi(x)$ ,  $\varphi(x)$ ,  $\phi(x)$ , this choice is carried out automatically. This, due to the linearization condition (3.8) and (3.9), becomes enough for our purposes, since for the free fields these relationships appear as operator equalities [16]:

$$\widehat{J}_{(\chi)}^\mu(x) = \frac{1}{\sqrt{\pi}} \partial^\mu \varphi(x) = -\frac{1}{\sqrt{\pi}} \epsilon^{\mu\nu} \partial_\nu \phi(x), \quad \widehat{J}_{(\chi)}^{-\xi}(x) = \frac{2}{\sqrt{\pi}} \partial_\xi \varphi^\xi(x^\xi). \quad (4.1)$$

Here, unlike [9], the free massless scalar field  $\varphi(x)$ ,  $\partial_\mu \partial^\mu \varphi(x) = 0$ , and pseudoscalar field  $\phi(x)$ ,  $\partial_\mu \partial^\mu \phi(x) = 0$ , are mutually dual and coupled by symmetric integral relations:

$$\left. \begin{array}{l} \phi(x) \\ \varphi(x) \end{array} \right\} = -\frac{1}{2} \int_{-\infty}^{\infty} dy^1 \varepsilon(x^1 - y^1) \partial_0 \left\{ \begin{array}{l} \varphi(y^1, x^0) \\ \phi(y^1, x^0) \end{array} \right\}, \quad (4.2)$$

where the step function  $\varepsilon(x^1) = 1$ , for  $x^1 > 0$ ,  $\varepsilon(x^1) = -1$ , for  $x^1 < 0$ ,  $\varepsilon(0) = 0$ , and the corresponding charges for these fields have the form similar to [13, 16]:

$$\left. \begin{array}{l} O \\ \overline{O} \end{array} \right\} = \int_{-\infty}^{\infty} dy^1 \partial_0 \left\{ \begin{array}{l} \varphi(y^1, x^0) \\ \phi(y^1, x^0) \end{array} \right\} = \left\{ \begin{array}{l} \phi(-\infty, x^0) - \phi(\infty, x^0) \\ \varphi(-\infty, x^0) - \varphi(\infty, x^0) \end{array} \right\}. \quad (4.3)$$

Right and left fields  $\varphi^\xi(x^\xi)$  and their charges  $\mathcal{Q}^\xi$  are defined by linear combinations [16]:

$$\varphi^\xi(x^\xi) = \frac{1}{2} [\varphi(x) - \xi \phi(x)], \quad \mathcal{Q}^\xi = \frac{1}{2} [O - \xi \overline{O}] = \pm 2\varphi^\xi(x^0 \pm \infty), \quad (4.4)$$

for  $\xi = \pm$ . All commutation relations [9, 14, 16] for the fields  $\varphi(x)$ ,  $\phi(x)$ ,  $\varphi^\xi(x^\xi)$ , and  $\mathcal{Q}^\xi$ :

$$[\varphi(x), \partial_0 \varphi(y)]|_{x^0=y^0} = [\phi(x), \partial_0 \phi(y)]|_{x^0=y^0} = i\delta(x^1 - y^1), \quad (4.5)$$

$$[\varphi(x), \varphi(y)] = [\phi(x), \phi(y)] = -\frac{i}{2}\varepsilon(x^0 - y^0)\theta((x - y)^2), \quad (4.6)$$

$$[\varphi^\xi(s), \varphi^{\xi'}(\tau)] = -\frac{i}{4}\varepsilon(s - \tau)\delta_{\xi, \xi'}, \quad [\varphi^\xi(s), \mathcal{Q}^{\xi'}] = \frac{i}{2}\delta_{\xi, \xi'}, \quad (4.7)$$

are reproduced by commutators of their frequency parts and corresponding charges [5, 9, 13]:

$$[\varphi^{\xi(\pm)}(s), \varphi^{\xi'(\mp)}(\tau)] = \mp \frac{1}{4\pi} \ln(i\kappa\{\pm(s - \tau) - i0\})\delta_{\xi, \xi'}, \quad (4.8)$$

$$[\varphi^{\xi(\pm)}(s), \mathcal{Q}^{\xi'(\mp)}] = \frac{i}{4}\delta_{\xi, \xi'}, \quad [\mathcal{Q}^{\xi(\pm)}, \mathcal{Q}^{\xi'(\mp)}] = \pm \frac{1}{4}\delta_{\xi, \xi'}, \quad (4.9)$$

defined here by the creation/annihilation operators  $c^\dagger(k^1), c(k^1)$  of the pseudoscalar field  $\phi(x)$ :  $\mathcal{P}c(k^1)\mathcal{P}^{-1} = -c(-k^1)$ , with  $[c(k^1), c^\dagger(q^1)] = 4\pi k^0 \delta(k^1 - q^1)$ , and  $k^0 \equiv |k^1|$ , as:

$$\varphi^{\xi(+)}(s) = -\frac{\xi}{2\pi} \int_{-\infty}^{\infty} \frac{dk^1}{2k^0} \theta(-\xi k^1) c(k^1) e^{-ik^0 s}, \quad \varphi^{\xi(-)}(s) = [\varphi^{\xi(+)}(s)]^\dagger, \quad (4.10)$$

$$\mathcal{Q}^{\xi(+)} = \lim_{L \rightarrow \infty} \frac{iL\xi}{4\sqrt{\pi}} \int_{-\infty}^{\infty} dk^1 \theta(-\xi k^1) c(k^1) e^{-ik^0 x^0} e^{-(k^1 L/2)^2}, \quad \mathcal{Q}^{\xi(-)} = [\mathcal{Q}^{\xi(+)}]^\dagger. \quad (4.11)$$

According to [14], the invariance under the parity transformation  $\mathcal{P}\{\dots\}\mathcal{P}^{-1}$  for generating functional of a free massless pseudoscalar field, unlike the scalar field theory, leads to its well definiteness and the gauge invariance also under field's shift by arbitrary constant. According to [16], in such a well-defined space of bosonic fields (4.2)–(4.11) one can construct the variety of different inequivalent representations of solutions of the Dirac equation for massless free trial field,  $\partial_\xi \chi_\xi(x) = 0$ , in the form of local normal-ordered exponentials of left and right boson fields  $\varphi^\xi(x^\xi)$  and their charges  $\mathcal{Q}^\xi$  (4.4), (4.7). Let us choose the most simple of them [16], which leads to the bosonization relations (4.1) for the currents (3.5)–(3.7) of trial fields  $\chi(x)$  with  $Z_{(\chi)}(a) = 1$ :

$$\begin{aligned} \chi_\xi(x) &= \chi_\xi(x^{-\xi}) = \mathcal{N}_\varphi \left\{ \exp \left( -i\sqrt{\pi} \left[ 2\varphi^{-\xi}(x^{-\xi}) + \frac{\xi}{2}\mathcal{Q}^\xi \right] \right) \right\} u_\xi, \\ u_\xi &= \sqrt{\frac{\kappa}{2\pi}} e^{-\pi/32} e^{-i\pi\xi/4}. \end{aligned} \quad (4.12)$$

The infrared regularization parameter  $\kappa$  from (4.8) can subsequently tend to zero [16] or remain to be fixed,  $\kappa \mapsto M$ , [13], depending on the phase of the model under consideration.

## 5. Integration of the Heisenberg Equation

For the chosen representation (4.1)–(4.8) the operator product in the linearized by means of (3.8) and (3.9) HEq (2.8) or (2.9) is naturally redefined into the normal-ordered form

[16] with respect to the fields  $\varphi^\xi(x^\xi)$ :

$$\partial_0 \Psi_\xi(x) = \left( -\xi \partial_1 - i \frac{\beta g}{2\sqrt{\pi}} \widehat{J}_{(\chi)}^{-\xi(-)}(x) \right) \Psi_\xi(x) - \Psi_\xi(x) \left( i \frac{\beta g}{2\sqrt{\pi}} \widehat{J}_{(\chi)}^{-\xi(+)}(x) \right). \quad (5.1)$$

The famous expression for the derivative of function  $F(x^1)$  in terms of the operator  $P^1$ :  $-i\partial_1 F(x^1) = [P^1, F(x^1)]$ , and its finite-shift equivalent:  $e^{iaP^1} F(x^1) e^{-iaP^1} = F(x^1 + a)$ , allows to transcribe Eq. (5.1) for  $x^0 = t$ ,  $\Psi_\xi(x) \leftrightarrow Y(t)$ , as follows:

$$\frac{d}{dt} Y(t) = A(t)Y(t) - Y(t)B(t), \quad (5.2)$$

and to obtain then its formal solution in the form of time-ordered exponentials:

$$Y(t) = T_A \left\{ \exp \left( \int_0^t d\tau A(\tau) \right) \right\} Y(0) \left[ T_B \left\{ \exp \left( \int_0^t d\tau B(\tau) \right) \right\} \right]^{-1}, \quad (5.3)$$

that are immediately replaced here by the usual ones, recasting the solution already into the normal form:

$$\Psi_\xi(x) = e^{C^{\xi(-)}(x)} \Psi_\xi(x^1 - \xi x^0, 0) e^{C^{\xi(+)}(x)}, \quad (5.4)$$

where operator bosonization (4.1) of the vector current of trial field  $\chi(x)$  (4.12) gives:

$$\begin{aligned} C^{\xi(\pm)}(x) &= -i \frac{\beta g}{2\sqrt{\pi}} \int_0^{x^0} dy^0 \widehat{J}_{(\chi)}^{-\xi(\pm)}(x^1 + \xi y^0 - \xi x^0, y^0) \\ &= -i \frac{\beta g}{2\pi} [\varphi^{(\pm)}(x^1, x^0) - \varphi^{(\pm)}(x^1 - \xi x^0, 0)] \\ &= -i \frac{\beta g}{2\pi} [\varphi^{\xi(\pm)}(x^\xi) - \varphi^{\xi(\pm)}(-x^{-\xi})]. \end{aligned} \quad (5.5)$$

Remarkably, that the completely unknown ‘‘initial’’ HF  $\Psi_\xi(x^1 - \xi x^0, 0) = \lambda_\xi(x^{-\xi})$  appears here also as a solution of free massless Dirac equation,  $\partial_\xi \lambda_\xi(x^{-\xi}) = 0$ , but certainly unitarily inequivalent to the free field  $\chi(x)$  (4.12). The expressions (5.4) and (5.5) suggest to choose it also in the normal-ordered form with respect to the field  $\varphi$ , using appropriate ‘‘bosonic canonical transformation’’ of this field with parameters  $\bar{\alpha} = 2\sqrt{\pi} \cosh \eta$ ,  $\bar{\beta} = 2\sqrt{\pi} \sinh \eta$ , obeying  $\bar{\alpha}^2 - \bar{\beta}^2 = 4\pi$ , which is generated by the operator  $F_\eta$  (for  $y^0 = x^0$ ) in the form:

$$U_\eta^{-1} \varphi(x) U_\eta = \omega(x) \equiv \omega^\xi(x^\xi) + \omega^{-\xi}(x^{-\xi}) = \frac{1}{2\sqrt{\pi}} [\bar{\alpha} \varphi(x^1, x^0) + \bar{\beta} \varphi(x^1, -x^0)], \quad (5.6)$$

$$U_\eta^{-1} \varphi^\xi(x^\xi) U_\eta = \omega^\xi(x^\xi) = \frac{1}{2\sqrt{\pi}} [\bar{\alpha} \varphi^\xi(x^\xi) + \bar{\beta} \varphi^{-\xi}(-x^\xi)], \quad U_\eta = \exp F_\eta, \quad (5.7)$$

$$U_\eta^{-1} \mathcal{Q}^\xi U_\eta = \mathcal{W}^\xi = \frac{1}{2\sqrt{\pi}} [\bar{\alpha} \mathcal{Q}^\xi - \bar{\beta} \mathcal{Q}^{-\xi}], \quad \text{with: } [\varphi^{\xi(\pm)}(s), F_\eta] = \eta \varphi^{-\xi(\mp)}(-s), \quad (5.8)$$

$$\text{for: } F_\eta = 2i\eta \int_{-\infty}^{\infty} dy^1 \varphi^\xi(y^\xi) \partial_0 \varphi^{-\xi}(-y^\xi) = 2i\eta \int_{-\infty}^{\infty} dy^1 \omega^\xi(y^\xi) \partial_0 \omega^{-\xi}(-y^\xi), \quad (5.9)$$



— does not depend at all on  $\xi$  and  $y^0$ , and ( $\Lambda$  is ultraviolet cut-off [13]):

$$U_\eta^{-1} \chi_\xi(x^{-\xi}) U_\eta = \lambda_\xi(x^{-\xi}) = \mathcal{N}_\varphi \left\{ \exp \left( -i\sqrt{\pi} \left[ 2\omega^{-\xi}(x^{-\xi}) + \frac{\xi}{2} \mathcal{W}^\xi \right] \right) \right\} v_\xi, \quad (5.10)$$

$$v_\xi = \left( \frac{k}{\Lambda} \right)^{\bar{\beta}^2/4\pi} e^{-\bar{\beta}^2/64} u_\xi = \left( \frac{k}{\Lambda} \right)^{\bar{\beta}^2/4\pi} \sqrt{\frac{\kappa}{2\pi}} e^{-\pi/32} e^{-\bar{\beta}^2/64} e^{-i\pi\xi/4}. \quad (5.11)$$

For the corresponding current  $\hat{J}_{(\lambda)}^\mu(x)$ , defined by Eqs. (3.5)–(3.7), or by the Johnson definition [2, 3, 6], but nevertheless with the same renormalization constant  $Z_{(\lambda)}(a)$ , one finds the previous bosonization rules (4.1) onto the new scalar fields  $\omega(x)$ ,  $\omega^\xi(x^\xi)$ , and  $\mathcal{W}^\xi$ , (5.6)–(5.8), obeying obviously the same commutation relations (4.5)–(4.7):

$$\hat{J}_{(\lambda)}^\mu(x) = \frac{1}{\sqrt{\pi}} \partial^\mu \omega(x), \quad \text{for: } Z_{(\lambda)}(a) = (\Lambda^2 |a^2|)^{-\bar{\beta}^2/4\pi}. \quad (5.12)$$

Substituting the normal form (5.10) into the solution (5.4), we immediately obtain the normal exponential of the DM for Thirring field in the form, analogous to [16]:

$$\Psi_\xi(x) = \mathcal{N}_\varphi \left\{ \exp \left( -i\bar{\alpha}\varphi^{-\xi}(x^{-\xi}) - i\bar{\beta}\varphi^\xi(x^\xi) - i\bar{\alpha}\frac{\xi}{4}\mathcal{Q}^\xi + i\bar{\beta}\frac{\xi}{4}\mathcal{Q}^{-\xi} \right) \right\} v_\xi, \quad (5.13)$$

by imposing the conditions onto the parameters that are necessary to have correct Lorentz-transformation properties corresponding to the spin 1/2, and correct canonical anticommutation relations (2.4) and (2.5), respectively:

$$\bar{\alpha}^2 - \bar{\beta}^2 = 4\pi, \quad \bar{\beta} - \frac{\beta g}{2\pi} = 0. \quad (5.14)$$

Straightforward calculation of the vector current operators (3.5)–(3.7) for the solution (5.13) by means of Eqs. (4.7)–(4.9) and (5.14), under the conditions:

$$\bar{\alpha} = \left( \frac{2\pi}{\beta} + \frac{\beta}{2} \right), \quad \bar{\beta} = \left( \frac{2\pi}{\beta} - \frac{\beta}{2} \right), \quad \text{or: } e^\eta = \frac{2\sqrt{\pi}}{\beta} = \sqrt{1 + \frac{g}{\pi}}, \quad (5.15)$$

reproduces the bosonization relations (3.8), (3.9) and (4.1) as following:

$$\hat{J}_{(\Psi)}^\mu(x) \stackrel{w}{=} \frac{\beta}{2\sqrt{\pi}} \hat{J}_{(\chi)}^\mu(x) = -\frac{\beta}{2\pi} \epsilon^{\mu\nu} \partial_\nu \phi(x), \quad \text{for: } Z_{(\Psi)}(a) = (-\Lambda^2 a^2)^{-\bar{\beta}^2/4\pi}, \quad Z_{(\chi)}(a) = 1, \quad (5.16)$$

demonstrating self-consistency of all the above calculations. The last equality of Eq. (5.15) is easily recognized as the well-known Coleman identity [7]. The weak sense of bosonization rules (5.16), unlike (4.1), is directly manifested by the difference of renormalization constants  $Z_{(\Psi)}(a)$  and  $Z_{(\chi)}(a)$  defined by Eqs. (5.12) and (5.16) for the various fields  $\Psi(x)$  and  $\chi(x)$  respectively. At the same time, by making use of (4.1) and (4.5), for the Johnson commutators [2–6] of Heisenberg fields (5.13) and their currents (5.16):

$$[\hat{J}_{(\Psi)}^0(x), \Psi(y)]|_{x^0=y^0} = -\underline{a}\Psi(y)\delta(x^1 - y^1), \quad (5.17)$$

$$[\hat{J}_{(\Psi)}^1(x), \Psi(y)]|_{x^0=y^0} = -\bar{\alpha}\gamma^5\Psi(y)\delta(x^1 - y^1), \quad (5.18)$$

$$[\hat{J}_{(\Psi)}^0(x), \hat{J}_{(\Psi)}^1(y)]|_{x^0=y^0} = -ic_{(\Psi)}\partial_{x^1}\delta(x^1 - y^1), \quad (5.19)$$

upon the above accepted definitions and relations (5.15) one obtains:

$$\underline{a} = 1, \quad \bar{a} = \frac{\beta^2}{4\pi}, \quad c_{(\Psi)} = \frac{\beta^2}{4\pi^2}, \quad (5.20)$$

$$\text{and finds: } \underline{a}\bar{a} = \pi c_{(\Psi)}, \quad \underline{a} - \bar{a} = g c_{(\Psi)}, \quad (5.21)$$

in agreement with [3–5]. On the other hand, in accordance with [8, 24, 25], the algebra of the Heisenberg operator of the conserved fermionic charge, by virtue of (5.17) and (5.20), coincides with the algebra of the conserved fermionic charge  $O/\sqrt{\pi}$  from Eq. (4.3) for the free trial field  $\chi(x)$ . Note, that the use of relations (5.17) and (5.18) for calculation of the commutator in Eq. (2.8) violates the equations of motion (2.8) and (2.9), as well as the above-mentioned attempt to use the commutator (5.19) in equation (3.3).

## 6. Conclusion

We have shown here, that the Thirring model [1–5], as well as the Federbush one [24], is exactly solvable due to intrinsic hidden exact linearizability of its HEq, and that the bosonization rules make an operator sense only among the free fields operators. For the Heisenberg currents these rules are applicable only in a weak sense (5.16), that is naturally manifested also as various values of Schwinger’s terms (5.19) and (5.20) for the free and Heisenberg currents in inequivalent field representations (4.12), (5.10) and (5.13), respectively:

$$c_{(\chi)} = c_{(\lambda)} = \frac{1}{\pi}, \quad c_{(\Psi)} = \frac{1}{\pi + g}, \quad (6.1)$$

in agreement with [11]. Similarly to the solution [24] of Federbush model, the linear homogeneous HEq (5.1) does not define the normalization of HF (5.13) and (5.15), which, as well as for the free fields  $\chi(x), \lambda(x)$ , is fixed [13] only by the anticommutation relations (2.4). We want to point out, that unlike [17–19] the bosonization procedure of [7–16] is considered here as a particular case of dynamical mapping onto the “Schrödinger” physical field [22–24] defined at  $t = 0$ . From this view point the results of [12] and [13] look as DM of Thirring field onto the free massive scalar field  $\phi_m(x)$ , or free massive Dirac field  $\psi_M(x)$  respectively. The general form of solution (5.4) should give a possibility to describe all phases of the theory under consideration. We postpone the discussion of these features to subsequent works.

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