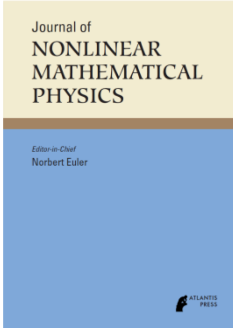

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A ROUTE TO ROUTH — THE CLASSICAL SETTING

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There is a well known principle in classical mechanic stating that a variational problem independent of a configuration space variable w (so called cyclic variable), but dependent on its velocity w' can be expressed without both w and w'. This principle is known as the Routh reduction.

In this paper, we start to develop a purely geometric approach to this reduction. We do not limit ourselves to rather special problems of mechanics and in a certain sense we are able to obtain explicit formulae for the reduced variational integral.

Keywords: Calculus of variations; Routh reduction; cyclic variables; Poincaré–Cartan form; Noether's theorem.

2000 Mathematics Subject Classification: 49S05, 49N99, 70H03

1. Introduction

We discuss a variational integral that admits a local group of symmetries. Our aim is to reduce the original variational problem to a simpler one.

Such idea of *reduction* of a variational problem is certainly not new — on the contrary the *reduction theory* has its origins in the classical works of Euler and Lagrange and in mechanics is known as the *Jacobi* or *Maupertuis principle*. The approach developed by Routh which is associated with systems having *cyclic variables* is especially important as a starting point for our purposes. Routh himself was able to apply his idea to stability theory [12] and the method is still used in classical mechanics [2, 3, 7, 11]. Nevertheless all mentioned works are concerned with a special choice either of the symmetry group or of the variational integral. Therefore they do not fully clarify the genuine mechanism of the reduction procedure.

The reduction of *variational problems* should not be confused with the reduction of *Hamiltonian systems*. The later reduction was started in the famous book [4] and is related to the *symplectic geometry*. This approach to the mechanic was developed even in the infinite-dimensional direction, see e.g. the works of Arnold [1], Smale [13] and the well-known Hamiltonian version of soliton theory. In the finite-dimensional setting important results have been obtained by Marsden and coworkers [8–10] however these results do not concern the reduction of variational integrals which are related to the *contact geometry* (or *jet space theory*). An interesting survey of various symmetry problems in the calculus of variations is contained in Krupková [6], but without any mention of the reduction procedure.

Our approach to the reduction theory is purely geometrical. The main tool is the Poincaré–Cartan form (sometimes in literature the Beltrami form, Lepagean equivalent or Hilbert invariant integral), the concept of which was introduced to the calculus of variations independently by Cartan [4] and Whittaker [14]. Because the concept of the Poincaré–Cartan forms can be introduced even for any general Lagrange variational problem, our approach is not limited to (rather) special problems of mechanics but admits far going generalizations. However, the most general setting of the reduction problem leads to the constrained variational integrals, rather nontrivial contact structures and nonclassical Poincaré–Cartan forms.

In this introductory article our aims are strongly limited and of very modest nature. We try to indicate the most important ingredients of the reduction problem as elementary as possible: the orbit space equipped with contact structure, the reduction to the subspace of orbits, the induced variational integral on the orbit space and the dominant role of the Poincaré–Cartan form. We deal only with the traditional Routh-setting of the problem, namely with the *independent variable preserving point symmetries* of the *first order* variational integral. For the convenience of the reader, this is made with the use of local coordinates and explicit calculations but the final result is expressed in coordinate-free terms. In order to make the exposition self-contained, we recall some fundamental concepts and results in the form which will be useful in subsequent generalization.

2. Motivation

In this section we give a concise overview of the classical Routh method of elimination of a cyclic variable z from the variational integral

$$\int f(x, y(x), z(x), \dot{y}(x), \dot{z}(x))\, dx, \quad x \in \mathbb{R}, \quad \dot{} := \frac{d}{dx}, \tag{2.1}$$

as it could be extracted e.g. from [5] or [11].

The extremals of (2.1) satisfy the Euler–Lagrange system

$$\frac{\partial f}{\partial y} = \frac{d}{dx}\frac{\partial f}{\partial \dot{y}}, \quad \frac{\partial f}{\partial z} = \frac{d}{dx}\frac{\partial f}{\partial \dot{z}}. \tag{2.2}$$

Because we moreover assume that the variable z is cyclic (which means $\partial f/\partial z = 0$), we have $f = f(t, y, \dot{y}, \dot{z})$, and the second Euler–Lagrange equation in (2.2) is

$$\frac{\partial f}{\partial \dot{z}} = c, \tag{2.3}$$

where $c \in \mathbb{R}$ is an arbitrary constant. Supposing the *normal case* defined by

$$\frac{\partial^2 f}{\partial \dot{z}^2} \neq 0, \tag{2.4}$$

Eq. (2.3) locally determines the function $\dot{z} = g(x, y, \dot{y}; c)$ as follows from the implicit function theorem.

Let us, (at this place), define the *Routh function* (*Routhian*)

$$\tilde{f}(x, y, \dot{y}; c) := f(x, y, \dot{y}, g(x, y, \dot{y}; c)) - cg(x, y, \dot{y}; c)$$

and the *Routh variational integral*

$$\int \tilde{f}(x, y(x), \dot{y}(x); c)\, dx, \tag{2.5}$$

depending on the parameter c. Now a direct computation gives the identity

$$\begin{aligned}\frac{\partial \tilde{f}}{\partial y} - \frac{d}{dx}\frac{\partial \tilde{f}}{\partial \dot{y}} &= \frac{\partial f}{\partial y} + \frac{\partial f}{\partial \dot{z}}\frac{\partial g}{\partial y} - c\frac{\partial g}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}} + \frac{\partial f}{\partial \dot{z}}\frac{\partial g}{\partial \dot{y}} - c\frac{\partial g}{\partial \dot{y}}\right)\\ &= \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}}\end{aligned}$$

which implies that the system

$$\frac{\partial \tilde{f}}{\partial y} = \frac{d}{dx}\frac{\partial \tilde{f}}{\partial \dot{y}}, \quad \frac{\partial f}{\partial \dot{z}} = c$$

is equivalent to the original Euler–Lagrange system (2.2).

Summarizing these facts, we have shown that *if* $\partial f/\partial z = 0$ *and* $\partial^2 f/\partial \dot{z}^2 \neq 0$, *then extremals of the Routh variational integral* (2.5) *are just the* y*-components of the original variational integral* (2.1).

The true sense of the Routh reduction can be clarified with the help of the Poincaré–Cartan form

$$\breve{\varphi} = f dx + \frac{\partial f}{\partial \dot{y}}(dy - \dot{y}\, dx) + \frac{\partial f}{\partial \dot{z}}(dz - \dot{z}\, dx).$$

Indeed, restriction of the form $\breve{\varphi}$ by (2.3) gives

$$\begin{aligned}\breve{\varphi}|_{\dot{z}=g} &= f|_{\dot{z}=g} dx + \left.\frac{\partial f}{\partial \dot{y}}\right|_{\dot{z}=g}(dy - \dot{y}\, dx) + c(dz - g dx)\\ &= \tilde{f} dx + \left.\frac{\partial f}{\partial \dot{y}}\right|_{\dot{z}=g}(dy - \dot{y}\, dx) + c\, dz\\ &=: \overline{\varphi} + c\, dz,\end{aligned}$$

where the Routh function $\tilde{f}$ naturally appears. Then the last term $c\, dz = d(cz)$ is an unimportant total differential and $\overline{\varphi}$ may be identified as a Poincaré–Cartan form of the restricted variational problem by applying a general Theorem 4.1 without any direct calculation. Therefore $\tilde{f}$, the coefficient of dx, is the Lagrange function of the restricted problem and the Routh reduction follows.

This is the idea of our approach which we wish to carry over more general variational problems in the future.

3. Hypotheses and Auxiliary Results

In order to avoid technical difficulties we will limit ourselves to the C^∞ smooth and local theory. This gives us the ability to use the existence theorem for ordinary differential equations and the implicit function theorem.

We will work in the *infinite order jet space* $\mathbb{M}(m)$ $(m > 1)$ with *jet coordinates*

$$x,\ w_r^i, \quad (i = 1, \dots, m;\ r = 0, 1, \dots),$$

equipped with the module $\Omega(m)$ of *contact forms*

$$\omega = \sum_{i=1}^{m} \sum_{r=0}^{r(\omega)} a_r^i \omega_r^i, \quad \text{where } \omega_r^i := dw_r^i - w_{r+1}^i dx. \tag{3.1}$$

The superscripts $r(\omega)$ in (3.1) are *finite* and we shall occasionally omit them for brevity. The coefficients

$$a_r^i = a_r^i(x, w_0^1, \dots, w_s^j, \dots)$$

are C^∞-smooth functions, each depending on a *finite* number of jet coordinates. Throughout this work we will always suppose that all considered functions are C^∞-smooth and that all functions, hence all contact forms, depend (only) on some *finite* number of variables (different from case to case). The space $\mathbb{M}(m)$ is *universal* in the sense that it can be used in the future study of all higher order integrals, as well.

In this setting a useful tool is the differential operator

$$D := \frac{\partial}{\partial x} + \sum_{i=1}^{m} \sum_{r=0}^{\infty} w_{r+1}^i \frac{\partial}{\partial w_r^i}, \tag{3.2}$$

which enables us to express the differential of any function f as a suitable combination of basic contact forms

$$df = Df\,dx + \sum_{i=1}^{m} \sum_{r=0} \frac{\partial f}{\partial w_r^i} \omega_r^i. \tag{3.3}$$

Let us consider the first order scalar variational integral

$$\int f(x, w_0^1, \dots, w_0^m, w_1^1, \dots, w_1^m)\, dx, \quad \left(w_r^i = \frac{d^r w^i}{dx^r},\ w^i = w^i(x) \right), \tag{3.4}$$

together with its *Poincaré–Cartan form*

$$\breve{\varphi} := f\,dx + \sum_{i=1}^{m} \frac{\partial f}{\partial w_1^i} \omega_0^i. \tag{3.5}$$

A direct computation gives

$$d\breve{\varphi} = \sum_{i=1}^{m} e_i \omega_0^i \wedge dx \quad \mathrm{mod}\ \Omega(m) \wedge \Omega(m), \tag{3.6}$$

where the coefficients e_i are

$$e_i = \frac{\partial f}{\partial w_0^i} - D\frac{\partial f}{\partial w_1^i} \quad (i = 1, \dots, m).$$

Definition 3.1. The equations $e_i = 0$, $(i = 1, \dots, m)$ are called the *Euler–Lagrange* equations of the variational integral (3.4), and any curve $\mathbb{P} : (a, b) \to \mathbb{M}(m)$, $\mathbb{P} : x \mapsto (x, w_0^1(x), \dots, w_r^i(x), \dots)$ such that

$$\mathbb{P}^*\omega_r^i = 0, \quad \mathbb{P}^*e_i = 0 \quad (i = 1, \dots, m;\ r = 0, 1, \dots) \tag{3.7}$$

is said to be an *extremal* of the variational integral (3.4).

Let us remark that this definition is in accordance with the traditional concepts. Indeed, (3.7_1) implies

$$0 = \mathbb{P}^*(dw_r^i - w_{r+1}^i dx) = dw_r^i(x) - w_{r+1}^i(x)dx,$$

whence $w_{r+1}^i = dw_r^i(x)/dx$ which, together with (3.7_2), gives

$$0 = \mathbb{P}^*e_i = \mathbb{P}^*\frac{\partial f}{\partial w_0^i} - \frac{d}{dx}\mathbb{P}^*\frac{\partial f}{\partial w_1^i} \quad \text{for } i = 1, \dots, m,$$

hence the equations (3.7_2) are really just the traditional Euler–Lagrange equations.

Lemma 3.1. *If a curve $\mathbb{P}$ satisfies the condition (3.7_1), then the system of equations (3.7_2) is equivalent to any of the following two identities*

$$\mathbb{P}^*(Z \lrcorner\, d\breve{\varphi}) = 0 \quad \textit{for all vector fields } Z \tag{3.8}$$

or

$$D \lrcorner\, d\breve{\varphi} = 0 \quad \textit{along } \mathbb{P}, \tag{3.9}$$

where "along $\mathbb{P}$" means at any point of the subset $\mathbb{P}((a, b)) \subset \mathbb{M}(m)$.

Proof. Both assertions follow from direct computation. □

3.1. *Infinitesimal symmetries, Noether's theorem*

Definition 3.2. A vector field Z locally defined on $\mathbb{M}(m)$ by

$$Z := z\frac{\partial}{\partial x} + \sum_{i=1}^{m}\sum_{r=0}^{\infty} z_r^i \frac{\partial}{\partial w_r^i}, \tag{3.10}$$

where the coefficients $z = z(x, w_0^1, \dots, w_s^j, \dots)$ and $z_r^i = z_r^i(x, w_0^1, \dots, w_s^j, \dots)$ are smooth functions is called (*generalized, Lie-Bäcklund*) *infinitesimal symmetry* of the variational problem (3.4) if

$$L_Z\Omega(m) \subset \Omega(m), \quad \text{and} \quad L_Z\breve{\varphi} \in \Omega(m), \tag{3.11}$$

where L_Z is the usual Lie-derivative with respect to the vector field Z.

In this article we will deal with more special vector field Z, however the proofs of Theorem 3.1 and Corollary 3.1 do not simplify.

There is a useful characteristic of fields fulfilling the first inclusion (3.11_1). Since (3.11_1) is equivalent to the inclusions $L_Z\omega_r^i \in \Omega(m)$, the congruences

$$\begin{aligned} L_Z(dw_r^i - w_{r+1}^i dx) &= dz_r^i - z_{r+1}^i dx - w_{r+1}^i dz \\ &= Dz_r^i dx - z_{r+1}^i dx - w_{r+1}^i Dz\, dx \quad \mathrm{mod}\ \Omega(m) \end{aligned}$$

imply that (3.11_1) is expressed by the recurrences

$$z_{r+1}^i = Dz_r^i - w_{r+1}^i Dz \tag{3.12}$$

for coefficients z_r^i of the vector field Z in (3.10).

On the contrary the second condition (3.11_2) reads

$$Z \lrcorner\, d\breve{\varphi} + d(Z \lrcorner\, \breve{\varphi}) \in \Omega(m). \tag{3.13}$$

and gives a close connection between fields and Euler–Lagrange equations.

In order to sketch a link to the usual approach, let us note that the condition (3.11) is equivalent to

$$L_Z\Omega(m) \subset \Omega(m), \quad \text{and} \quad L_Z(f dx) \in \Omega(m). \tag{3.14}$$

Now

$$L_Z(f dx) = Zf dx + f dz \sim (Zf + fDz)dx \quad \mathrm{mod}\ \Omega(m),$$

and (3.14) imply the common symmetry requirement

$$Zf + fDz = 0,$$

which is useful in particular examples.

Theorem 3.1. *Let Z be a vector field satisfying (3.11_2). Then the function $\breve{\varphi}(Z)$ is constant on every extremal of the variational integral (3.4).*

Proof. If $\mathbb{P}$ is an extremal of (3.4), then in view of (3.7_1), (3.8), and (3.11_2)

$$0 = -\mathbb{P}^*(Z \lrcorner\, d\breve{\varphi}) = \mathbb{P}^*(d(Z \lrcorner\, \breve{\varphi})) = d\mathbb{P}^*(Z \lrcorner\, \breve{\varphi}) = d\mathbb{P}^*\breve{\varphi}(Z).$$

□

As a simple corollary we obtain the celebrated *Noether's theorem*.

Corollary 3.1. *Let Z be an infinitesimal symmetry of the variational integral (3.4). Then the function*

$$G := \breve{\varphi}(Z) = zf + \sum_{i=1}^{m} (z_0^i - w_1^i z) \frac{\partial f}{\partial w_1^i} \tag{3.15}$$

is a first integral of the Euler–Lagrange system of (3.4).

3.2. *x-preserving pointwise symmetries, the orbit space* $\mathbb{M}(m)_{orb}$

Probably the most surprising and at the same time the most annoying property of vector fields on $\mathbb{M}(m)$ is that in general they do not need to generate any flow. For this and other, mainly technical reasons we shall henceforth deal only with rather special *x-preserving pointwise infinitesimal symmetries* that is with the vector fields

$$Z := \sum_{i=1}^{m} \sum_{r=0}^{\infty} z_r^i \frac{\partial}{\partial w_r^i}, \quad z_0^i = z_0^i(x, w_0^1, \dots, w_0^m) \tag{3.16}$$

fulfilling (3.11).

When Z is an x-preserving pointwise infinitesimal symmetry, the recurrences (3.12) simplify to

$$z_{r+1}^i = D z_r^i. \tag{3.17}$$

This implies that the Lie-brackets

$$[Z, D] = ZD - DZ = \sum_{i=1}^{m} \sum_{r=0}^{\infty} Z w_{r+1}^i \frac{\partial}{\partial w_r^i} - \sum_{i=1}^{m} \sum_{r=0}^{\infty} D z_r^i \frac{\partial}{\partial w_r^i} = 0 \tag{3.18}$$

vanish identically since $Z w_{r+1}^i = z_{r+1}^i = D z_r^i$.

But the most important consequences are that any pointwise infinitesimal symmetry (3.16) locally generates certain *flow*

$$\mathcal{F}_Z^\tau : \mathbb{M}(m) \times (-\varepsilon, \varepsilon) \to \mathbb{M}(m), \quad \varepsilon > 0,$$

with very special properties due to the identity (3.18), and that locally any point $P \in \mathbb{M}(m)$ is contained in an *orbit*

$$\{\mathcal{F}_Z^\tau(P) : -\varepsilon < \tau < \varepsilon\}.$$

In our setting any two orbits are either disjoint or identical, which gives us the possibility to work with the factorspace $\mathbb{M}(m)_{orb}$ defined in such a way that every (local) orbit of $\mathbb{M}(m)$ is represented by a point of this *orbit space* $\mathbb{M}(m)_{orb}$.

3.3. *Coordinates on* $\mathbb{M}(m)$, $\mathbb{M}(m)_{orb}$

We shall introduce some suitable (local) coordinates in $\mathbb{M}(m)_{orb}$. To that end let us remind that any function W locally defined on $\mathbb{M}(m)$ and constant on any orbit of the vector field Z,

$$W(\mathcal{F}_Z^\tau(P)) \equiv \text{const}(P), \quad P \in \mathbb{M}(m), \tag{3.19}$$

is called a *first integral* of Z. Clearly instead of (3.19) we could equivalently use the condition

$$\frac{d}{d\tau} W(\mathcal{F}_Z^\tau(P)) = 0 \quad \text{for } -\varepsilon < \tau < \varepsilon, \quad P \in \mathbb{M}(m).$$

But for any $P \in \mathbb{M}(m)$ and $-\varepsilon < \tau < \varepsilon$

$$\frac{d}{d\tau} \mathcal{F}_Z^\tau(P) = Z(\mathcal{F}_Z^\tau(P)), \tag{3.20}$$

by the definition of the Z-flow, hence

$$\frac{d}{d\tau}W(\mathcal{F}_Z^\tau(P)) = dW(\mathcal{F}_Z^\tau(P))\frac{d}{d\tau}\mathcal{F}_Z^\tau(P) = dW(\mathcal{F}_Z^\tau(P))Z(\mathcal{F}_Z^\tau(P)) = ZW(\mathcal{F}_Z^\tau(P)),$$

and it follows that $ZW \equiv 0$ whenever W is a first integral.

The differential equation (3.20) written in terms of the jet coordinates x, w_r^i is

$$\begin{aligned} &\frac{d}{d\tau}x \circ \mathcal{F}_Z^\tau = 0, \\ &\frac{d}{d\tau}w_r^i \circ \mathcal{F}_Z^\tau = z_r^i(x \circ \mathcal{F}_Z^\tau, w_0^1 \circ \mathcal{F}_Z^\tau, \dots, w_0^m \circ \mathcal{F}_Z^\tau, w_1^1 \circ \mathcal{F}_Z^\tau, \dots, w_r^m \circ \mathcal{F}_Z^\tau), \\ &\qquad \text{for } i = 1, \dots, m; \ r = 0, 1, \dots . \end{aligned} \tag{3.21}$$

Because for $r = 0$; $i = 1, \dots, m$ this system is closed and $(m+1)$-dimensional, standard theory of ordinary differential equations grants the (local) existence of m first integrals of (3.20)

$$x \quad \text{and} \quad W_0^k = W_0^k(x, w_0^1, \dots, w_0^m) \quad \text{for } k = 1, \dots, m-1, \tag{3.22}$$

which are functionally independent, especially

$$\operatorname{rank}\left[\frac{\partial W_0^k}{\partial w_0^i}\right]_{i=1,\dots,m;\ k=1,\dots,m-1} = m - 1. \tag{3.23}$$

Therefore

$$ZW_0^k = 0 \quad \text{for } k = 1, \dots, m-1,$$

and, in view of the commutativity of Z and D given by (3.18), all functions

$$W_r^k = W_r^k(x, w_0^1, \dots, w_0^m, \dots, w_r^1, \dots, w_r^m) := D^r W_0^k, \quad (k = 1, \dots, m-1;\ r = 0, 1, \dots),$$

are first integrals of Eq. (3.20) as well.

Moreover, let $W_0^m = W_0^m(x, w_0^1, \dots, w_0^m)$ be a local solution of the partial differential equation

$$ZW_0^m = \sum_{i=1}^m z_0^i \frac{\partial W_0^m}{\partial w_0^i} = 1. \tag{3.24}$$

Then, in view of (3.18), all functions

$$W_r^m = W_r^m(x, w_0^1, \dots, w_0^m, \dots, w_r^1, \dots, w_r^m) := D^r W_0^m, \quad (r = 0, 1, \dots),$$

are first integrals of Eq. (3.20).

It is easy to see that due to (3.23) and (3.24) all here considered first integrals are functionally independent. Therefore the following proposition holds.

Proposition 3.1. *Let a vector field (3.16) be an x-preserving pointwise infinitesimal symmetry of the variational problem (3.4) and W_r^i be the functions defined above.*

Then the functions

$$\begin{array}{llll} x, & W_0^1, \ldots, W_0^{m-1}, & W_0^m, & \\ & W_1^1, \ldots, W_1^{m-1}, & W_1^m, & \\ & \vdots \qquad\qquad \vdots & \vdots & (r = 0, 1, \ldots) \\ & W_r^1, \ldots, W_r^{m-1}, & W_r^m, & \\ & \vdots \qquad\qquad \vdots & \vdots & \end{array} \tag{3.25}$$

are local coordinates on $\mathbb{M}(m)$, *and the functions*

$$\begin{array}{llll} x, & W_0^1, \ldots, W_0^{m-1}, & & \\ & W_1^1, \ldots, W_1^{m-1}, & W_1^m, & \\ & \vdots \qquad\qquad \vdots & \vdots & (r = 0, 1, \ldots) \\ & W_r^1, \ldots, W_r^{m-1}, & W_r^m, & \\ & \vdots \qquad\qquad \vdots & \vdots & \end{array} \tag{3.26}$$

are local coordinates on $\mathbb{M}(m)_{orb}$.

The last part of Proposition 3.1 follows from the definition of the space $\mathbb{M}(m)_{orb}$ as the factor space of the total space $\mathbb{M}(m)$.

Remark 3.1. Following the common conventions, we identify any function V defined on $\mathbb{M}(m)_{orb}$ with its pullback W on $\mathbb{M}(m)$. Alternatively saying any (local) first integral W on $\mathbb{M}(m)$ can be represented as a pullback of a function V on $\mathbb{M}(m)_{orb}$. We omit the pullback notation occasionally and identify $W \equiv V$.

3.4. *The normal case*

We have obtained coordinates on $\mathbb{M}(m)$ and on $\mathbb{M}(m)_{orb}$. But these coordinates are not optimal for our purposes because they are not related to the function G introduced in (3.15).

To remedy this situation, we define the sequence of functions

$$G_1 := G - c, \quad G_{r+1} := D^r G_1, \quad (r = 1, 2, \ldots),$$

where $c \in \mathbb{R}$ is an arbitrary but fixed constant (parameter). As in the previous cases $G_r = G_r(x, w_0^1, \ldots, w_0^m, \ldots, w_r^1, \ldots, w_r^m)$.

Definition 3.3. The situation when the above defined functions

$$\begin{array}{llll} x, & W_0^1, \ldots, W_0^{m-1}, & W_0^m, & \\ & W_1^1, \ldots, W_1^{m-1}, & G_1, & \\ & \vdots \qquad\qquad \vdots & \vdots & (r = 1, 2, \ldots) \\ & W_r^1, \ldots, W_r^{m-1}, & G_r, & \\ & \vdots \qquad\qquad \vdots & \vdots & \end{array} \tag{3.27}$$

determine (local) coordinates on $\mathbb{M}(m)$ will be called the *normal case*.

Just as in Sec. 2 the assumption of the normal case is crucial for our theory. Therefore it is of paramount importance to have effective criteria of the normal case. This is seemingly impossible since our (new) definition of the normal case is based on the explicit knowledge of a complete set of first integrals of the differential equation (3.20). However, and this is a real surprise, there is one.

Lemma 3.2. *For the variational integral* (3.4) *and the x-preserving pointwise symmetry* (3.16) *we have the normal case if and only if*

$$\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\partial^2 f}{\partial w_1^i \partial w_1^j} z_0^i z_0^j \neq 0. \tag{3.28}$$

Proof. We just have to prove that the system (3.27) is functionally independent. Due to the structure of its Jacobi matrix this is equivalent to

$$\operatorname{rank}\begin{bmatrix} \dfrac{\partial W_0^1}{\partial w_0^1}, & \dots, & \dfrac{\partial W_0^1}{\partial w_0^m} \\ \vdots & & \vdots \\ \dfrac{\partial W_0^{m-1}}{\partial w_0^1}, & \dots, & \dfrac{\partial W_0^{m-1}}{\partial w_0^m} \\ \dfrac{\partial G}{\partial w_1^1}, & \dots, & \dfrac{\partial G}{\partial w_1^m} \end{bmatrix} = m. \tag{3.29}$$

Since $ZW_0^k = 0$ for $k = 1, \dots, m-1$ the requirement (3.29) is equivalent to the inequality

$$\sum_{j=1}^{m}\frac{\partial G}{\partial w_1^j} z_0^j \neq 0.$$

But in view of (3.15) and (3.16) we have

$$\frac{\partial G}{\partial w_1^j} = \sum_{i=1}^{m}\frac{\partial^2 f}{\partial w_1^i \partial w_1^j} z_0^i \quad \text{for } j = 1, \dots, m,$$

which concludes the proof. □

Let us note that the normal case is trivially satisfied for non-degenerated variational problems that is for the problems fulfilling

$$\det\left[\frac{\partial^2 f}{\partial w_1^i \partial w_1^j}\right] \neq 0.$$

Example 3.1. Let us try out what is the normal case for the original Routh problem solved in Sec. 2. Here $Z = \partial/\partial z$, so

$$G = \breve{\varphi}(Z) = \left(f\,dx + \frac{\partial f}{\partial \dot y}(dy - \dot y\,dx) + \frac{\partial f}{\partial \dot z}(dz - \dot z\,dx)\right)\left(0\frac{\partial}{\partial y} + 1\frac{\partial}{\partial z}\right) = \frac{\partial f}{\partial \dot z},$$

and the normal case condition (3.28) reads

$$\frac{\partial^2 f}{\partial \dot{z}^2} \neq 0,$$

which is in full agreement with the traditional concept (2.4).

Now, similarly to Proposition 3.1, we would like to extend Definition 3.3 to $\mathbb{M}(m)_{orb}$. To this end we need to refine (3.11_2) as follows.

Lemma 3.3. *Let Z be an x-preserving pointwise infinitesimal symmetry* (3.10). *Then $L_Z\breve{\varphi} = 0$.*

Proof. On the one hand in view of (3.11) we have

$$L_Z\breve{\varphi} = \sum_{i=1}^{m}\sum_{r=0} a_r^i \omega_r^i,$$

hence

$$\begin{aligned} dL_Z\breve{\varphi} &= \sum_{i=1}^{m}\sum_{r=0} da_r^i \wedge \omega_r^i + \sum_{i=1}^{m}\sum_{r=0} a_r^i dx \wedge \omega_{r+1}^i \\ &= \sum_{i=1}^{m}\sum_{r=0}\left(Da_r^i dx + \sum_{j=1}^{m}\sum_{s=0}\frac{\partial a_r^i}{\partial w_s^j}\omega_s^j\right)\wedge\omega_r^i + \sum_{i=1}^{m}\sum_{r=0} a_r^i dx\wedge\omega_{r+1}^i \\ &\sim \sum_{i=1}^{m}\sum_{r=0}\left(Da_r^i dx\wedge\omega_r^i + a_r^i dx\wedge\omega_{r+1}^i\right) \quad \text{mod all } \omega_r^i\wedge\omega_s^j. \end{aligned} \tag{3.30}$$

On the other hand in view of (3.6) we have

$$\begin{aligned} dL_Z\breve{\varphi} &= L_Z d\breve{\varphi} \sim \sum_{i=1}^{m} e^i \left(L_Z\omega_0^i\right)\wedge dx \sim \sum_{i=1}^{m} e^i (dz_0^i - w_1^i dz)\wedge dx \\ &= \sum_{i=1}^{m} e^i \left[Dz_0^i dx + \sum_{j=1}^{m}\frac{\partial z_0^i}{\partial w_0^j}\omega_0^j - w_1^i\left(Dz dx + \sum_{j=1}^{m}\frac{\partial z}{\partial w_0^j}\omega_0^j\right)\right]\wedge dx \sim 0. \end{aligned} \tag{3.31}$$

Now if we compare (3.30) with (3.31), and use the fact that all a_r^i depend on a finite number of jet coordinates, we obtain the system

$$\sum_{r=1}^{r(i)}(Da_r^i + a_{r-1}^i)\omega_r^i + a_{r(i)}^i\omega_{r(i)+1}^i = 0 \quad i = 1,\dots,m,$$

where $r(i)$ are such integers that $a_r^i = 0$ for $r > r(i)$. The backward induction gives $a_r^i = 0$ identically for $i = 1,\dots,m$ and $0 \le r \le r(i)$, therefore $L_Z\breve{\varphi} = 0$. □

Lemma 3.4. *Let Z be an x-preserving infinitesimal symmetry* (3.16) *of the variational problem* (3.4) *and W_r^i, G_r are the functions defined above.*

Then in the normal case the functions

$$\begin{array}{llll} x,\, W_0^1, \dots, W_0^{m-1}, & & & \\ \quad W_1^1, \dots, W_1^{m-1}, G_1, & & & \\ \quad \vdots \qquad\quad \vdots \quad\;\; \vdots & (r = 1, 2, \dots) & & \\ \quad W_r^1, \dots, W_r^{m-1}, G_r, & & & \\ \quad \vdots \qquad\quad \vdots \quad\;\; \vdots & & & \end{array} \tag{3.32}$$

determine (local) coordinates on $\mathbb{M}(m)_{orb}$.

Proof. By virtue of (3.13) and (3.15)

$$ZG_1 = dG(Z) = d(\breve{\varphi}(Z))(Z) = -d\breve{\varphi}(Z, Z) = 0,$$

hence G_1 is a first integral of the equation (3.20) and by (3.18) the same is true for the whole sequence $\{G_r\}_{r=1}^{\infty}$. According to Remark 3.1 we can regard all functions x, W_r^i, and G_r in (3.32) as functions on the orbit space $\mathbb{M}(m)_{orb}$. In the normal case the functions W_1^m, $W_2^m \dots$ in (3.26) may be replaced by functions $G_1, G_2, \dots$, hence the system (3.32) determines (local) coordinates on $\mathbb{M}(m)_{orb}$. □

Remark 3.2. In terms of alternative coordinates (3.25) on $\mathbb{M}(m)$, we have

$$D = \frac{\partial}{\partial x} + \sum_{i=1}^{m} \sum_{r=0}^{\infty} W_{r+1}^i \frac{\partial}{\partial W_r^i}, \tag{3.33}$$

whence the module $\Omega(m)$ consists exactly of all forms

$$\omega = \sum_{i=1}^{m} \sum_{r=0}^{r(i)} A_r^i \Omega_r^i, \tag{3.34}$$

where

$$\Omega_r^i := dW_r^i - W_{r+1}^i dx. \tag{3.35}$$

The vector field D has a good sense even on the space $\mathbb{M}(m)_{orb}$, namely it may be applied to any coordinate (3.26) on $\mathbb{M}(m)_{orb}$ and therefore to any function on $\mathbb{M}(m)_{orb}$. These results expressed in alternative coordinates (3.27) or (3.32) should be regarded as lucky accidents — they cannot be directly generalized to x-destroying groups.

Especially in terms of coordinates (3.26) the field D and the contact forms ω are represented as in (3.33) and (3.34), however, the terms $W_1^m \partial/\partial W_0^m$ and $A_0^m \Omega_0^m$ should be removed.

4. Main Results

Before we state our main results concerning a purely geometric formulation of the Routh reduction we have to prove a fundamental characteristic of the Poincaré–Cartan forms. It will save us a substantial amount of work in our proofs.

Theorem 4.1. *A one-form*

$$\psi = f dx + \sum_{i=1}^{m} \sum_{r=0} a_r^i \omega_r^i,$$

is the Poincaré–Cartan form of the variational integral (3.4) *iff*

$$d\psi = 0 \quad \text{mod all } \omega_0^{\mathrm{i}}, \quad \omega_{\mathrm{r}}^{\mathrm{i}} \wedge \omega_{\mathrm{s}}^{\mathrm{j}}. \tag{4.1}$$

Proof. The necessity of (4.1) follows immediately from the form of $d\breve{\varphi}$ (3.6). In order to prove its sufficiency let us note that

$$\begin{aligned}
d\psi &= df \wedge dx + \sum_{i=1}^{m} \sum_{r=0} (da_r^i \wedge \omega_r^i - a_r^i \omega_{r+1}^i \wedge dx) \\
&= \left(D f dx + \sum_{i=1}^{m} \frac{\partial f}{\partial w_0^i} \omega_0^i + \sum_{i=1}^{m} \frac{\partial f}{\partial w_1^i} \omega_1^i \right) dx \\
&\quad + \sum_{i=1}^{m} \sum_{r=0} \left[\left(D a_r^i dx + \sum_{j=1}^{m} \sum_{s=0} \frac{\partial a_r^i}{\partial w_s^j} \omega_s^j \right) \wedge \omega_r^i - a_r^i \omega_{r+1}^i \wedge dx \right] \\
&= \sum_{i=1}^{m} \left[\frac{\partial f}{\partial w_1^i} \omega_1^i - \sum_{r=1} D a_r^i \omega_r^i - \sum_{r=0} a_r^i \omega_{r+1}^i \right] \wedge dx \quad \text{mod } \omega_0^i, \quad \omega_r^i \wedge \omega_s^j.
\end{aligned}$$

Because all sums in the last term are finite, there exist nonnegative integers $r(i)$, $i = 1, \ldots, n$, such that $a_r^i = 0$ for all $r > r(i)$. The equations

$$\frac{\partial f}{\partial w_1^i} \omega_1^i - \sum_{r=1}^{r(i)} D a_r^i \omega_r^i - \sum_{r=0}^{r(i)} a_r^i \omega_{r+1}^i = 0, \quad (i = 1, \ldots, m)$$

give $a_{r(i)}^i = 0$. Hence by the backward induction we obtain $a_r^i = 0$ for $r > 0$, $i = 1, \ldots, m$ and the equations

$$\frac{\partial f}{\partial w_1^i} \omega_1^i - a_0^i \omega_1^i = 0 \quad \text{for } i = 1, \ldots, m.$$

Therefore ψ is the Poincaré–Cartan form of (3.4). □

Our main theorem not only gives a generalization of the classical Routh reduction to pointwise symmetries, but it yields an explicit representation of the Routh function.

In its proof, we introduce a subspace $\overline{\mathbb{M}}(m) \subset \mathbb{M}(m)$ defined in terms of coordinates (3.25) by a sequence of equations, and depending on a parameter $c \in \mathbb{R}$. The same sequence of equations, this time in coordinates (3.26), will define the subspace $\overline{\mathbb{M}}(m)_{orb} \subset \mathbb{M}(m)_{orb}$.

Arguments analogous to those in Remark 3.2 shows that the vector field D and the module of contact forms $\Omega(m)$ make good sense even for the subspaces $\overline{\mathbb{M}}(m)$ and $\overline{\mathbb{M}}(m)_{orb}$, and that they may be expressed by (3.33) and (3.34), where all summands $W_{r+1}^m \partial / \partial W_0^m$ and $A_r^m \Omega_r^m$ $(r = 0, 1, \ldots)$ should be removed.

Theorem 4.2. *Let* $Z := \sum_{i=1}^{m}\sum_{r=0}^{\infty} z_r^i \partial/\partial w_r^i$ *be an* x*-preserving pointwise infinitesimal symmetry of the variational integral* (3.4), *where the integrand* f *fulfills the normality condition*

$$\sum_{i=1}^{m}\sum_{j=1}^{m} \frac{\partial^2 f}{\partial w_1^i \partial w_1^j} z_0^i z_0^j \neq 0.$$

Then for any constant $c \in \mathbb{R}$ *and for any solution* $W_0^m = W_0^m(x, w_0^1, \ldots, w_0^m)$ *of the equation*

$$ZW_0^m = 1$$

the restriction $\tilde{f}$ *of the function* $f - cDW_0^m$ *to the subspace*

$$\overline{\mathbb{M}}(m) := \{D^r(\breve{\varphi}(Z) - c) = 0;\ r = 0, 1, \ldots\} \subset \mathbb{M}(m)$$

is a Routhian *of the original variational integral* (3.4). *More exactly* $\tilde{f}$ *can be represented as a composed function* $\tilde{f} = \tilde{f}(x, W_0^1, \ldots, W_0^{m-1}, W_1^1, \ldots, W_1^{m-1}; c)$ *depending on the parameter* c, *and regarded as a function on* $\overline{\mathbb{M}}(m)_{orb}$.

Moreover the extremals of the Routh variational integral

$$\int \tilde{f}(x, W_0^1, \ldots, W_0^{m-1}, W_1^1, \ldots, W_1^{m-1}, c)\, dx \quad \left(W_r^i = \frac{d^r W^i}{dx^r},\ W^i = W^i(x)\right) \tag{4.2}$$

on the orbit space $\overline{\mathbb{M}}(m)_{orb}$ *are just natural projections of the extremals of the original variational integral* (3.4) *lying in the subspace* $\overline{\mathbb{M}}(m)$.

Proof. We will work in coordinates (3.25) on $\mathbb{M}(m)$. Then the variational integral (3.4) looks like

$$\int F(x, W_0^1, \ldots, W_0^m, W_1^1, \ldots, W_1^m)\, dx, \tag{4.3}$$

where $W_r^i = d^r W^i/dx^r$, $W^i = W^i(x)$ and the smooth function F is defined by

$$F(x, W_0^1, \ldots, W_0^m, W_1^1, \ldots, W_1^m) := f(x, w_0^1, \ldots, w_0^m, w_1^1, \ldots, w_1^m).$$

Next, for $i = 1, \ldots, m$ consider the coordinate functions $W_0^i = W_0^i(x, w_0^1, \ldots, w_0^m)$. If we express the differential dW_0^i in coordinates (3.25) and in original coordinates x, w_r^i, we obtain

$$W_1^i dx + \Omega_0^i \quad \text{and} \quad DW_0^i dx + \sum_{j=1}^{m} \frac{\partial W_0^i}{\partial w_0^j}\omega_0^j.$$

Therefore

$$\Omega_0^i = \sum_{j=1}^{m} \frac{\partial W_0^i}{\partial w_0^j}\omega_0^j \quad \text{for } i = 1, \ldots, m.$$

According to Proposition 3.1 the matrix

$$\left[\frac{\partial W^i}{\partial w_0^j}\right]_{i,j=1,\ldots,m}$$

is invertible, hence

$$\omega_0^i = \sum_{j=1}^{m} a_j^i \Omega_0^j \quad \text{for } i = 1, \dots, m \tag{4.4}$$

for suitable smooth functions $a_j^i = a_j^i(x, W_0^1, \dots, W_0^m, W_1^1, \dots, W_1^m)$.

Next, the Poincaré–Cartan form of the original variational integral (3.4) looks in coordinates (3.25) like

$$\begin{aligned}\breve{\varphi} &= F dx + \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{\partial F}{\partial W_1^j} \frac{\partial W_1^j}{\partial w_1^i} \sum_{k=1}^{m} a_k^i \Omega_0^k \\ &= F dx + \sum_{i=1}^{m} A_i \Omega_0^i,\end{aligned}$$

where $A_i = A_i(x, W_0^1, \dots, W_0^m, W_1^1, \dots, W_1^m)$. The easy part of Theorem 4.1 tells us that

$$d\breve{\varphi} = 0 \quad \text{mod all } \omega_0^i,$$

and this together with (4.4) gives

$$d\breve{\varphi} = 0 \quad \text{mod all } \Omega_0^i.$$

Now the hard part of Theorem 4.1 implies that $\breve{\varphi}$ is the Poincaré–Cartan form even for the transformed variational integral (4.3), therefore

$$\breve{\varphi} = F dx + \sum_{i=1}^{m} \frac{\partial F}{\partial W_1^i} \Omega_0^i.$$

In other words the concept of Poincaré–Cartan form has a good geometric sense.

Since Z is an x-preserving pointwise infinitesimal symmetry of (3.4), the same is true in the new coordinates (3.25), moreover it is easy to see that

$$Z = \frac{\partial}{\partial W_0^m},$$

so

$$\begin{aligned}0 = L_Z \breve{\varphi} &= (L_Z F) dx + F(L_Z dx) + \sum_{i=1}^{m} \left(L_Z \frac{\partial F}{\partial W_1^i} \right) \Omega_0^i + \sum_{i=1}^{m} \frac{\partial F}{\partial W_1^i} \left(L_Z \Omega_0^i \right) \\ &= \frac{\partial F}{\partial W_0^m} dx + \sum_{i=1}^{m} \left(L_Z \frac{\partial F}{\partial W_1^i} \right) \Omega_0^i,\end{aligned}$$

hence $\partial F / \partial W_0^m = 0$. Consequently

$$F = F(x, W_0^1, \dots, W_0^{m-1}, W_1^1, \dots, W_1^m) \tag{4.5}$$

is independent of W_0^m.

Next, the direct computation gives $G = \breve{\varphi}(Z) = \partial F/\partial W_1^m$ and we see that in the new coordinates (3.25) the normal case is expressed by

$$\frac{\partial^2 F}{\partial W_1^{m2}} \neq 0.$$

The implicit function theorem thus ensures that we can uniquely solve the equation $\breve{\varphi}(Z) = c$ for

$$W_1^m = g(x, W_0^1, \ldots, W_0^{m-1}, W_1^1, \ldots, W_1^{m-1}, c).$$

At this moment all assertions of the theorem can be obtained analogously as the classical results in Sec. 2. In view of (4.5) the Euler–Lagrange equations of (3.4) are

$$\frac{\partial F}{\partial W_0^i} - D\frac{\partial F}{\partial W_1^i} = 0 \quad \text{for } i = 1, \ldots, m-1, \tag{4.6a}$$

$$\frac{\partial F}{\partial W_1^m} = c. \tag{4.6b}$$

The function $f - cDW_0^m$ restricted to the space $\overline{\mathbb{M}}(m)$ is

$$\tilde{f} = (F - cW_1^m)\,|_{W_1^m = g} = F(x, W_0^1, \ldots, W_0^{m-1}, W_1^1, \ldots, W_1^{m-1}, g(\cdot)) - cg(\cdot),$$

and we can easily compute

$$\frac{\partial \tilde{f}}{\partial W_0^i} - D\frac{\partial \tilde{f}}{\partial W_1^i} = \frac{\partial F}{\partial W_0^i} - D\frac{\partial F}{\partial W_1^i}\bigg|_{W_1^m = g} \qquad \text{for } i = 1, \ldots, m-1,$$

whence the system

$$\frac{\partial \tilde{f}}{\partial W_0^i} - D\frac{\partial \tilde{f}}{\partial W_1^i} = 0 \quad \text{for } i = 1, \ldots, m-1,$$

$$\frac{\partial F}{\partial W_1^m} = c$$

is equivalent to the system (4.6).

Because the equations (4.6a) may be identified with the Euler–Lagrange system of the variational integral (4.2) on the space $\overline{\mathbb{M}}(m)_{orb}$, the proof is completed. □

Corollary 4.1. *Let $\breve{\varphi}$ and $\tilde{\varphi}$ be the Poincaré–Cartan forms of the variational integrals* (3.4) *and* (4.2) *respectively. Then*

$$i^*\breve{\varphi} = p^*\tilde{\varphi} + c dW_0^m, \tag{4.7}$$

where

$$p : \overline{\mathbb{M}}(m) \to \overline{\mathbb{M}}(m)_{orb}$$

is the natural projection, and

$$i : \overline{\mathbb{M}}(m) \to \mathbb{M}(m)$$

is the inclusion mapping.

Proof. The identity (4.7) follows at once from the formula

$$
\begin{aligned}
i^*\breve{\varphi} &= F|_{W_1^m=g}dx + \sum_{i=1}^{m-1} \left.\frac{\partial F}{\partial W_1^m}\right|_{W_1^m=g} \left(dW_0^i - W_1^i dx\right) + c\left(dW_0^m - W_1^m dx\right) \\
&= \tilde{f}dx + \sum_{i=1}^{m-1} \left.\frac{\partial F}{\partial W_1^m}\right|_{W_1^m=g} \left(dW_0^i - W_1^i dx\right) + c dW_0^m
\end{aligned}
$$

for the restriction to the space $\overline{\mathbb{M}}(m)$. □

5. Simple Applications

The following examples transparently illustrates the sense of various choices of function W_0^m in Theorem 4.2. The resulting Routh integrals differ by a total differential.

Example 5.1. Let us consider the variational integral (2.1) fulfilling the normality condition (2.4). The Poincaré–Cartan form of (2.1) is

$$\breve{\varphi} = f dx + \frac{\partial f}{\partial \dot{y}}\omega_y + \frac{\partial f}{\partial \dot{z}}\omega_z,$$

where $\omega_\alpha := d\alpha - \dot{\alpha}dx$ for $\alpha \in \{y, z\}$.
Consider the vector field

$$Z := \frac{\partial}{\partial z},$$

then

$$\breve{\varphi}(Z) = \frac{\partial f}{\partial \dot{z}}, \quad L_Z\omega_y = L_Z\omega_z = 0,$$

and due to cyclicity of z

$$L_Z\breve{\varphi} = \left(L_Z\frac{\partial f}{\partial \dot{y}}\right)\omega_y + \left(L_Z\frac{\partial f}{\partial \dot{z}}\right)\omega_z \in \Omega(2),$$

whence Z is an x-preserving pointwise infinitesimal symmetry of (2.1) and all assumptions of Theorem 4.2 are fulfilled.

Next any smooth function $W(x, y, z) := z + C(x, y)$ is a solution of the partial differential equation

$$1 = ZW\left(= \frac{\partial W}{\partial z}\right),$$

and

$$f - cDW = f - c\left(C_x\dot{x} + C_y\dot{y} + \dot{z}\right), \quad c \in \mathbb{R}.$$

Now

$$\overline{\mathbb{M}}(2) := \left\{(x, y, z, \dot{z}, \dot{z}, \ldots) : \frac{\partial f}{\partial \dot{z}} = c, \text{ and } D^r\frac{\partial f}{\partial \dot{z}} = 0 \text{ for } r = 1, 2, \ldots\right\},$$

hence

$$(f - cDW)|_{\overline{\mathbb{M}}(2)} = \tilde{f}(x, y, \dot{y}; c) - c(C_x\dot{x} + C_y\dot{y}),$$

where $\tilde{f}$ is the classical Routhian. The Routh variational integral is

$$\int \tilde{f}(x, y, \dot{y}; c)\, dx - c \int (C_x\dot{x} + C_y\dot{y})\; dx = \int \tilde{f}\, dx - c \int dC,$$

where $C = C(x, y)$ is an arbitrary function.

Example 5.2. Let us consider the first order scalar variational integral

$$\int f(x, w_0^1, \dots, w_0^m, w_1^1, \dots, w_1^m)\, dx, \quad \left(w_r^i = \frac{d^r w^i}{dx^r},\; w^i = w^i(x) \right), \tag{5.1}$$

and the infinitesimal symmetry $Z := \sum_{i=1}^m a^i \partial/\partial w_0^i, (a^i \in \mathbb{R})$. Let the normal case (3.28) condition

$$\sum_{i=1}^m \sum_{j=1}^m a^i a^j \frac{\partial^2 f}{\partial w_1^i \partial w_1^j} \neq 0$$

be fulfilled. Equation $ZW_0^m = 1$ has the solution of the kind

$$W_0^m = \sum_{i=1}^m b^i w_0^i + C(x, \dots, a^i w_0^j - a^j w_0^i, \dots),$$

where $\sum_{i=1}^m a^i b^i = 1$ and C is a general solution of the homogeneous equation $ZC = 0$.

Now, we have the Routh function

$$R = f - c \left(\sum_{i=1}^m b^i w_1^i + \frac{\partial C}{\partial x} + \sum_{i=1}^m w_1^i \frac{\partial C}{\partial w_0^i} \right)$$

which should be restricted to the subspace

$$\breve{\varphi}(Z) = \sum_{i=1}^m a^i \frac{\partial f}{\partial w_1^i} = c.$$

Then the Routh variational integral becomes a variational integral on the space of orbits defined by $a^i w_0^j - a^j w_0^i = const$ $(i, j = 1, \dots, m)$, and it is possible to obtain explicit formulae for particular choice of function f.

The infinitesimal invariance is ensured if

$$f = F(x, \dots, a^i w_0^j - a^j w_0^i, \dots, w_1^1, \dots, w_1^m)$$

which also clarifies the interrelation to the orbit space. The particular choice $a^1 = b^1 = 1,\; a^i = b^i = 0,\; (i = 2, \dots, m),\; C = 0$ provides the classical multidimensional Routh theorem.

Example 5.3. Let us consider again the variational integral (5.1), but this time together with the infinitesimal symmetry

$$Z := \sum_{i=1}^{m} a^i w_0^i \frac{\partial}{\partial w_0^i} + \sum_{i=1}^{m} a^i w_1^i \frac{\partial}{\partial w_1^i} + \cdots, \quad (a^i \in \mathbb{R}).$$

Let the normal case condition

$$\sum_{i=1}^{m} \sum_{j=1}^{m} a^i a^j w_0^i w_0^j \frac{\partial^2 f}{\partial w_1^i \partial w_1^j} \neq 0$$

be fulfilled. Equation $ZW_0^m = 1$ has a solution

$$W_0^m = \sum_{i=1}^{m} b^i \log w_0^i + C(x, \ldots, a^i \log w_0^j - a^j \log w_0^i, \ldots),$$

where $\sum_{i=1}^{m} a^i b^i = 1$ and C is a general solution of the homogeneous equation $ZC = 0$. Now, we have the Routh function

$$R = f - c \sum_{i=1}^{m} \left(b^i \frac{w_1^i}{w_0^i} + \frac{\partial C}{\partial x} + \sum_{j=1}^{m} w_1^j \frac{\partial C}{\partial w_0^j} \right),$$

which should be restricted to the subspace

$$\breve{\varphi}(Z) = \sum_{i=1}^{m} a^i w_0^i \frac{\partial f}{\partial w_1^k} = c.$$

Functions

$$a^i \log w_0^j - a^j \log w_0^i, \quad a^i \log w_1^j - a^j \log w_0^i, \quad \text{and} \quad a^i \log w_1^j - a^j \log w_1^i, \quad (i, j = 1, \ldots, m)$$

are constant on orbits and the kernel function f is composition of the coordinate x and these functions.

Example 5.4. Let us consider the Lagrange variational integral

$$\int \left(\frac{1}{2} \sum_{i,j=1}^{m} a^{ij} \dot{q}^i \dot{q}^j - V \right) dt, \tag{5.2}$$

where a^{ij} and V are functions of configuration variables $q^1, \ldots, q^m$ but independent of the time t. It is well-known, that the integrand in (5.2) admits the infinitesimal symmetry $\partial/\partial t$. In order to apply the Routh theorem, let us use the substitution

$$t = w_0^0(x), \quad w_0^i(x) = q^i(w_0^0(x)) \quad (i = 1, \ldots, m),$$

whence

$$dt = w_1^0(x)dx, \quad w_1^i(x) = \dot{q}^i(w_0^0(x))w_1^0(x)dx \quad (i = 1, \dots, m),$$

in terms of jet coordinates. Then the original integral turns into

$$\int \left(\frac{1}{2} \sum_{i,j=1}^{m} a^{ij} w_1^i w_1^j \frac{1}{w_1^0} - V w_1^0 \right) dx,$$

which admits the infinitesimal symmetry $Z = \partial/\partial w_0^0$. The classical Routh theorem can be applied (with a slight change in the upper indices). Clearly

$$\breve{\varphi}(Z) = -\frac{1}{2} \sum_{i,j=1}^{m} a^{ij} w_1^i w_1^j \frac{1}{(w_1^0)^2} - V = -\frac{1}{2} \sum_{i,j=1}^{m} a^{ij} \dot{q}^i \dot{q}^j - V,$$

and we obtain the Routh variational integral

$$\int \left(\frac{1}{2} \sum_{i,j=1}^{m} a^{ij} w_1^i w_1^j \frac{1}{w_1^0} - V w_1^0 - c w_1^0 \right) dx = \int \left(\frac{1}{2} \sum_{i,j=1}^{m} a^{ij} \dot{q}^i \dot{q}^j - V - c \right) dt,$$

on the level set $\breve{\varphi}(Z) = c$. The final result reads

$$\int \sum_{i,j=1}^{m} a^{ij} \dot{q}^i \dot{q}^j dt = \int (2c - V) dt = \pm \int \sqrt{\sum_{i,j=1}^{m} a^{ij} \dot{q}^i \dot{q}^j} \sqrt{\pm 2c \mp V} dt,$$

in accordance with the well-known Jacobi–Maupertuis variational principles, see also [3].

Example 5.5. The variational integral

$$\int F(x, y^2 + z^2, yy' + zz', y'^2 + z'^2)\, dx$$

admits the rotation group, hence the infinitesimal transformation

$$Z = -z \frac{\partial}{\partial y} + y \frac{\partial}{\partial z} - z' \frac{\partial}{\partial y'} + y' \frac{\partial}{\partial z'} - \cdots .$$

Function $W = \arctan(z/y)$ satisfies $ZW = 1$. Moreover

$$\breve{\varphi}(Z) = -(yF_v + 2y'F_w)z + (zF_v + 2z'F_w)y = 2(yz' - y'z)F_w.$$

We omit the rather complicated normality requirement. The Routh function

$$\tilde{f} = F - c \left(y' \frac{\partial}{\partial y} + z' \frac{\partial}{\partial z} \right) \arctan \frac{z}{y} = F - c \frac{yz' - y'z}{y^2 + z^2}$$

should be considered at the space of orbits (circles $x^2 + y^2 = \text{Const.}$, $x = \text{Const.}$) under the condition $\breve{\varphi}(Z) = c$.

For example, let us choose $F := y'^2 + z'^2$. Then

$$\tilde{f} = y'^2 + z'^2 - c \frac{yz' - y'z}{y^2 + z^2}$$

is considered under the condition $2(yz'-y'z)=c$. Let us introduce coordinates x, $u=y^2+z^2$, $u'=2(yy'+zz'),\ldots$ on the orbit space. Then, after simple calculations

$$y'^2+z'^2=\frac{u'^2+c^2}{4u},$$

and

$$\tilde{f}=\frac{u'^2+c^2}{4u}-\frac{c^2}{2u}.$$

Therefore the Routh function $\tilde{f}$ is indeed defined on the orbit space. The normality requirement $u\neq 0$ can be easily found.

Analogously the Lagrange variational invariant with respect to the rotation can be reduced to the orbit space of certain circles with constant regular momentum and this result exactly correspond to the reduction to the level set $\breve{\varphi}=$ const. of constant energy mentioned in Example 5.4.

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